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WIDE-BAND QUADRUPOLE FOCUSING SYSTEM FOR NEUTRINO AND MUON BEAMS

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Quadrupole focusing system is inherently narrow-band unless one uses the AG quadrupole train only as a containment device and depends on adiabatic tapering of the train to adiabatically reduce the divergence of the beam as suggested by A. Maschke. We consider the train having a repetitive cell structure but each cell is scaled up in length and aperture and scaled down in field gradient from the previous cell.

Formalism

To take care of the scaling we write the displacement-divergence vector of a particle in the beam at the entrance to the first cell as

$$\begin{pmatrix} \dfrac{x_o}{\sqrt{\ell_o}} \\ x_o'\sqrt{\ell_o} \end{pmatrix}$$

where ℓ_o is the scaling length of cell 1 and can be taken as the length of a typical quadrupole in cell 1. The typical transfer matrix for such a quadrupole, then, looks like

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$$\begin{pmatrix} \cos\phi & \frac{1}{\phi}\sin\phi \\ -\phi\sin\phi & \cos\phi \end{pmatrix}$$

where $\phi \equiv \sqrt{\frac{eB'_o}{cp}}\,\ell_o$. We assume that in successive cells B' scales as $\frac{1}{\ell^2}$, hence ϕ is the same for all cells. Writing the transfer matrix for cell 1 as

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

at the end of cell 1 we, then, have

$$\begin{pmatrix} \frac{x_1}{\sqrt{\ell_o}} \\ x'_1\sqrt{\ell_o} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \frac{x_o}{\sqrt{\ell_o}} \\ x'_o\sqrt{\ell_o} \end{pmatrix}$$

or

$$\begin{pmatrix} \frac{x_1}{\sqrt{\ell_1}} \\ x'_1\sqrt{\ell_1} \end{pmatrix} = \begin{pmatrix} \sqrt{\frac{\ell_o}{\ell_1}} & 0 \\ 0 & \sqrt{\frac{\ell_1}{\ell_o}} \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \frac{x_o}{\sqrt{\ell_o}} \\ x'_o\sqrt{\ell_o} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{a}{r} & \frac{b}{r} \\ rc & rd \end{pmatrix} \begin{pmatrix} \frac{x_o}{\sqrt{\ell_o}} \\ x'_o\sqrt{\ell_o} \end{pmatrix}$$

where $r^2 \equiv \frac{\ell_1}{\ell_o}$. We further assume that successive cells are

identically scaled, namely

$$\frac{\ell_n}{\ell_{n-1}} = \frac{\ell_{n-1}}{\ell_{n-2}} = \ldots = \frac{\ell_2}{\ell_1} = \frac{\ell_1}{\ell_o} \equiv r^2$$

then

$$\begin{pmatrix} \dfrac{x_n}{\sqrt{\ell_n}} \\ \\ x_n'\sqrt{\ell_n} \end{pmatrix} = \begin{pmatrix} \dfrac{a}{r} & \dfrac{b}{r} \\ \\ rc & rd \end{pmatrix}^n \begin{pmatrix} \dfrac{x_o}{\sqrt{\ell_o}} \\ \\ x_o'\sqrt{\ell_o} \end{pmatrix}$$

if, as usual, the cell transfer matrix is parametrized as

$$\begin{pmatrix} \dfrac{a}{r} & \dfrac{b}{r} \\ \\ rc & rd \end{pmatrix} = \cos\mu + \begin{pmatrix} \alpha & \beta \\ \\ -\gamma & -\alpha \end{pmatrix} \sin\mu$$

where $\gamma = \dfrac{1 + \alpha^2}{\beta}$, then, the usual conclusions follow, namely

(1) For stable containment we must have

$$\left|\frac{1}{2}\left(\frac{a}{r} + rd\right)\right| = \left|\cos\mu\right| < 1$$

(2) The quadratic form

$$I = \gamma \frac{x_n^2}{\ell_n} + 2\alpha x_n x_n' + \beta \ell_n x_n'^2$$

is an invariant and represents an ellipse in the x' vs. x phase plane for stable containment. Therefore we see that

x is scaled by r^n and

x' is scaled by $\frac{1}{r^n}$

in going through n cells.

(3) As $p \to \infty$ we have $a \to 1$, $d \to 1$, and

$$\left|\frac{1}{2}\left(\frac{a}{r} + rd\right)\right| \to \left|\frac{1}{2}\left(\frac{1}{r} + r\right)\right| \geq 1$$

If we, now, equate this to cosh μ we get $\mu = \ln r$ which simply says that the "apparent" instability causes an "apparent" exponential increase of the beam divergence at the same rate as the adiabatic reduction so that the "true" beam divergence is not affected by the quadrupole train and that the train acts only as a drift space. Thus, we see that although the containment condition given above does not give the limits of "real" stability it does give the limit within which the adiabatic reduction is effective. In the following we shall nevertheless refer to this as the "containment condition."

(4) The scaling of x implies that the apertures of the quadrupoles should be scaled by r^n. Since B' is scaled by $\frac{1}{r^{2n}}$ together this means that the pole field is scaled by $\frac{1}{r^n}$.

(5) The acceptance is proportional to $\frac{1}{\beta}$ and, hence, roughly proportional to $\frac{1}{p}$ which matches well the $\frac{1}{p}$ dependence of the mean production cone angle of pions or kaons. The details of the acceptance have to be investigated with a computer.

(6) If L_o is the length of cell 1 the total length of n cells is

$$L = L_o(1 + r^2 + r^4 + \ldots + r^{2(n-1)})$$

$$= L_o \frac{r^{2n} - 1}{r^2 - 1}.$$

For the same reduction of the beam divergence, namely the same r^n, the total length of the quadrupole train is shorter with larger r. On the other hand as will be seen later the larger is r the smaller is the momentum range with stable containment. The design must, therefore, be an appropriate compromise.

Examples

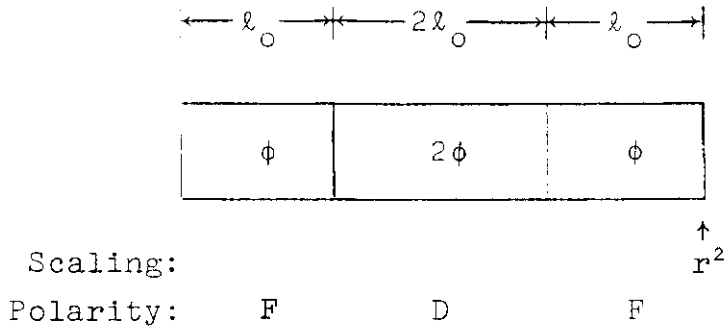
Let us assume that we would like to reduce the divergence of the beam x' by about a factor of 10, that the total length of the quadrupole train should not be much longer than 100 m, and that the momentum range desired is as large as possible from 20 BeV/c on up.

It can be seen easily that for an asymmetric system in which the containment conditions in the x and the y planes are grossly different the momentum range is rather restricted. We are, thus, led to the following symmetric systems where the containment conditions in the two planes are identical (namely $\mu_x = \mu_y$).

(1) Symmetric triplet cells

With the parameters of cell 1 defined as in the

diagram



The containment condition works out to be

$$\cos \mu = \frac{1}{2}\left(r + \frac{1}{r}\right) \cosh 2\phi \cos 2\phi \begin{matrix} < +1 \\ > -1 \end{matrix}$$

This condition is plotted in Figure 1 where the shaded area gives the region of stable containment.

(a) We can choose $r = 3$ and have two cells ($n = 2$). The reduction in divergence (X') is, then, $\frac{1}{r^n} = \frac{1}{9}$. The range of ϕ for stable containment from Figure 1 is 0.63 to 0.88. If $\phi = 0.88$ corresponds to 20 BeV/c the momentum range is, then, 20 - 39 BeV/c. If we make $\ell_o = 1.5$ m we get $B_o' = 230$ kG/m and the total length for two cells = 60 m. In summary we have

First triplet $\begin{cases} \ell_o = 1.5 \text{ m} & L_o = 6 \text{ m} \\ B_o' = \pm 230 \text{ kG/m} \end{cases}$

Second triplet $\begin{cases} \ell_1 = 13.5 \text{ m} & L_1 = 54 \text{ m} \\ B_1' = \pm 2.84 \text{ kG/m} \end{cases}$

Total length $L = 60$ m

Momentum range 20 - 39 BeV/c
(ϕ range 0.88 - 0.63)

Reduction in divergence 1/9

(b) We can choose r = 2 and have three cells (n = 3). The ϕ range is 0.53 to 0.91 corresponding to a momentum range of 20 - 59 BeV/c. If we make ℓ_o = 1.5 m we get B_o' = 245 kG/m. The total length for three cells is 126 m. To summarize

First triplet $\begin{cases} \ell_o = 1.5 \text{ m} & L_o = 6 \text{ m} \\ B_o' = \pm 245 \text{ kG/m} \end{cases}$

Second triplet $\begin{cases} \ell_1 = 6.0 \text{ m} & L_1 = 24 \text{ m} \\ B_1' = \pm 15.3 \text{ kG/m} \end{cases}$

Third triplet $\begin{cases} \ell_2 = 24 \text{ m} & L_2 = 96 \text{ m} \\ B_2' = \pm 0.96 \text{ kG/m} \end{cases}$

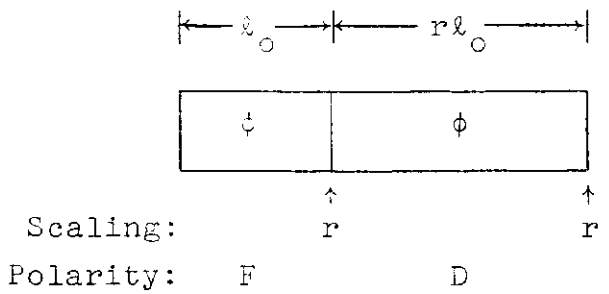
Total length L = 126 m

Momentum range 20 - 59 BeV/c
(ϕ range 0.91 - 0.53)

Reduction in divergence 1/8

(2) Doublet cells with half scaling between the two quadrupoles

The cell looks like



The containment condition works out to be

$$\cos \mu = \frac{1}{2}\left(r + \frac{1}{r}\right) \cosh \phi \cos \phi \begin{matrix} < +1 \\ > -1 \end{matrix}$$

The same plot in Figure 1 can be used if we multiply the numbers on the ϕ axis by 2. The same two cases as in the triplet system now become

(a) $r = 3$ and two cells (4 quadrupoles)

cell	quad	ℓ(m)	B'(kG/m)
1st	F	3	230
	D	9	-25.6
2nd	F	27	2.84
	D	81	-0.316

Total length L = 120 m

Momentum range 20 - 39 BeV/c

(ϕ range 1.76 - 1.26)

Reduction in divergence 1/9

(b) r = 2 and three cells (6 quadrupoles)

cell	quad	ℓ(m)	B'(kG/m)
1st	F	3	245
	D	6	-61.3
2nd	F	12	15.3
	D	24	-3.83
3rd	F	48	0.957
	D	96	-0.239

Total length L = 189 m

Momentum range 20 - 59 BeV/c

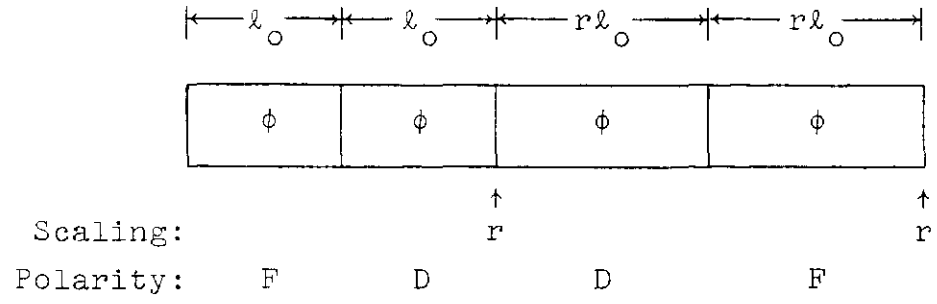
(ϕ range 1.82 - 1.06)

Reduction in divergence 1/8

We see that, in general, this arrangement has no advantage over the triplet cells.

(3) Triplet cells with half scaling in middle

The cell, now, looks like



The containment condition works out to be

$$\cos \mu = \frac{1}{4}\left(r + 2 + \frac{1}{r}\right)\left(\cosh 2\phi \cos 2\phi + 1\right) - 1 \begin{matrix} < +1 \\ > -1 \end{matrix}$$

This is plotted in Figure 2.

(a) r = 3 and two cells (8 quadrupoles)

cell	quad	ℓ(m)	B'(kG/m)
1st	F	1.5	260
	D	1.5	-260
	D	4.5	-28.9
	F	4.5	28.9
2nd	F	13.5	3.21
	D	13.5	-3.21
	D	40.5	-0.357
	F	40.5	0.357

Total length L = 120 m

Momentum range 20 - 40 BeV/c

(ϕ range 0.937 - 0.66)

Reduction in divergence 1/9

(b) r = 2 and three cells (12 quadrupoles)

cell	quad	ℓ(m)	B'(kG/m)
1st	F	1.5	260
	D	1.5	-260
	D	3.0	-65.1
	F	3.0	65.1
2nd	F	6.0	16.3
	D	6.0	-16.3
	D	12.0	-4.07
	F	12.0	4.07
3rd	F	24.0	1.02
	D	24.0	-1.02
	D	48.0	-0.254
	F	48.0	0.254

Total length L = 189 m

Momentum range 20 - 60 BeV/c

(ϕ range 0.937 - 0.54)

Reduction in divergence 1/8

We see that, in general, this arrangement again has no advantage over the triplet cells.

Discussions

(1) The "containment condition" even when properly interpreted as the condition for effective adiabatic reduction of beam divergence is, strictly speaking, exact only for an infinitely long quadrupole train. For a short train with only a few cells this condition is too restrictive. The practical effective momentum range is generally broader than that indicated.

(2) Within the effective range given by the containment condition the reduction of divergence is uniformly $\frac{1}{r^n}$. At higher momentum, because of the $\frac{1}{p}$ dependence of the production cone angle of pions and kaons the required reduction of beam divergence is less. This further extends the practical effective momentum range at the high momentum end. All this indicates that it is more difficult to construct a wide-band system at the low momentum end.

(3) For the very long and weak (low B') quadrupoles toward the end of the train it is more reasonable and practical to substitute shorter and stronger ones. At the ϕ values indicated above thin lens approximation is good enough.

(4) There is no point in studying the acceptance analytically in detail. This can be made only for an infinitely long train which is a poor approximation for 2 or 3

cells. Also to investigate the practical overall effectiveness of the system one should include the finite target size, the distributions of the pions and kaons at production, their decays, the neutrino distributions from the decays, the geometry of the decay channel and the detector etc., in addition to the acceptance of the train. All these can be folded into the calculation only by the use of a computer program.

(5) The adiabatically scaled quadrupole train studied here also makes a good transition section to match to a muon-channel. Actually in a μ-channel with such a scaled quadrupole matching section the neutrino beam will come "for free."

(6) Compared to the conventional coaxial "horn" the quadrupole system has the disadvantage (or advantage, depending on the application) of having no charge discrimination. For particles of opposite charge the focusing actions of the quadrupoles are only such as having the x and the y planes interchanged.

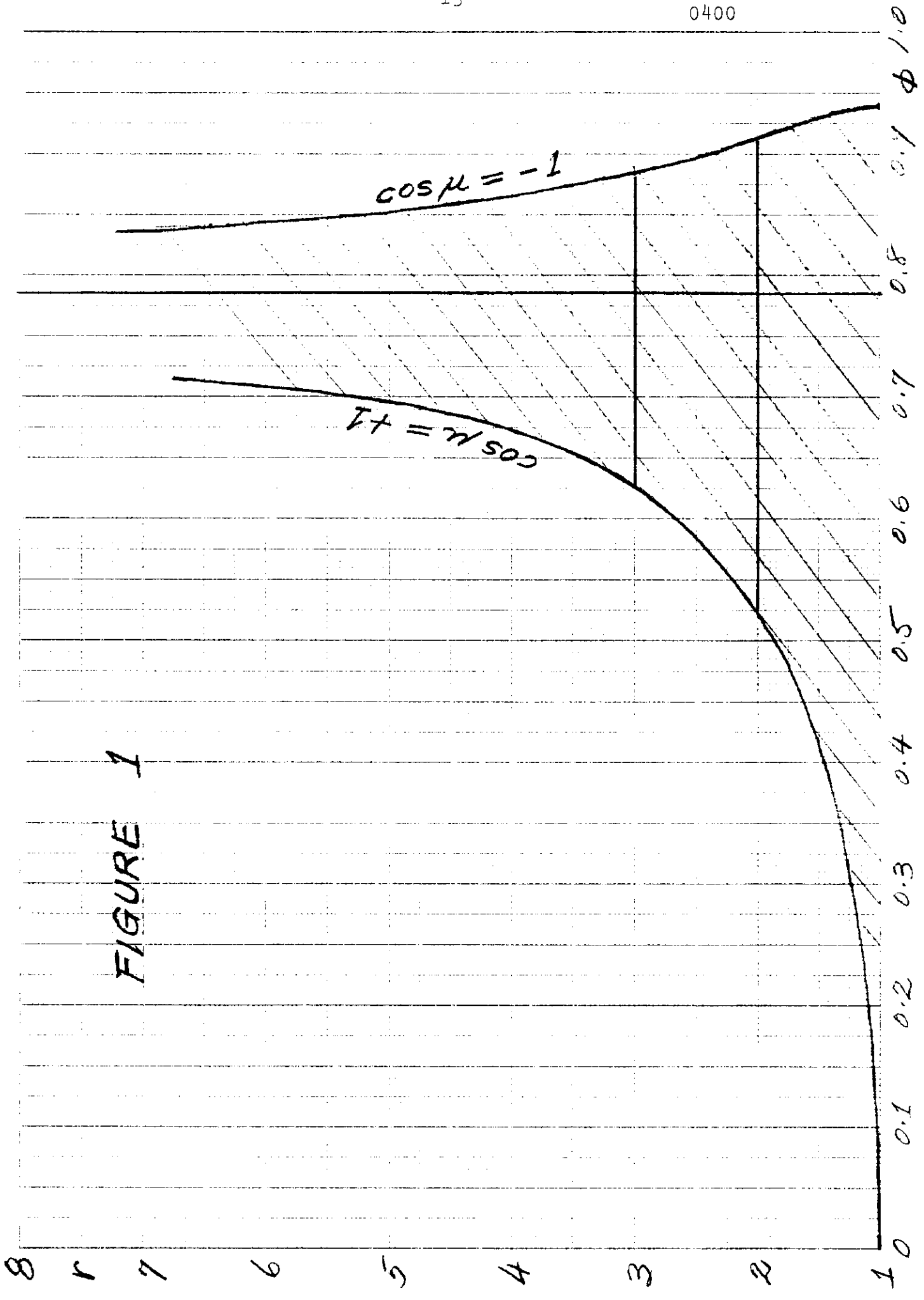


FIGURE 1

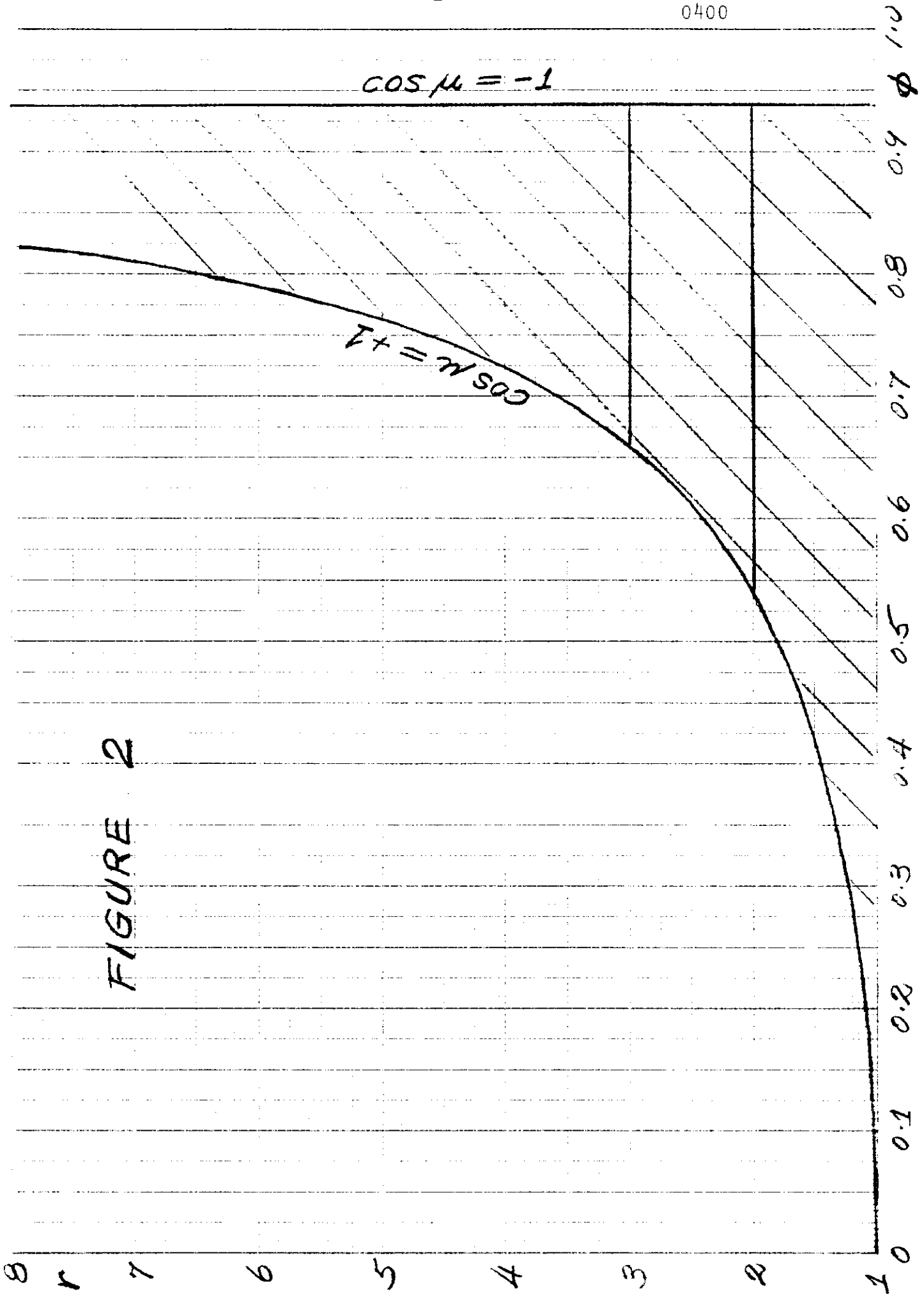


FIGURE 2