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NONLINEAR BETATRON MOTION IN IDEAL BOOSTER LATTICE

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PURPOSE

To determine the characteristics of nonlinear betatron motion for an ideal magnetic confining field possessing median plane symmetry. Application is made to the properties of the booster using third order effects in the hamiltonian. Fourth order effects are estimated.

HAMILTONIAN

The hamiltonian in which the generalized momenta, π_x and π_y, are expressed in units of the particle momentum is[1]

$$H = -(1+\frac{x}{\rho})\sqrt{1-(\pi_x-\frac{e}{p}A_x)^2-(\pi_y-\frac{e}{p}A_y)^2}-\frac{e}{p}(1+\frac{x}{\rho})A_s, \text{ (emu)} \tag{1}$$

where the curvilinear coordinates (x,y,s) are taken to be the orthogonal set in which s measures the distance along a curve in the median plane, y is normal to the median plane, and x is in the direction of the outward normal to the curve.

POTENTIALS

A scalar potential Φ such that $\vec{B} = \nabla\Phi$ may be written as[2]

$$\Phi = \sum_{m=0} \sum_{n=0} \frac{A_{mn}(s)}{m!n!} x^m y^n , \tag{2}$$

where, in order to satisfy Laplace's equation,

$$\begin{aligned} &A_{m+2,n}+(3m+1)kA_{m+1,n}+m(3m-1)k^2A_{mn}+m(m-1)^2k^3A_{m-1,n} \\ &+A_{m,n+2}+3mkA_{m-1,n+2}+3m(m-1)k^2A_{m-2,n+2} \\ &+m(m-1)(m-2)k^3A_{m-3,n+2}-mk'A'_{m-1,n}+A''_{mn}+mkA''_{m-1,n} = 0 \end{aligned} \tag{3}$$

Here $k = 1/\rho$.

The vector potential is gauge dependent. Choose the gauge for which $xA_x + yA_y = 0$. Then[2]

$$A_x = -\frac{1}{1+kx}\Sigma\frac{A'_{mn}}{m!(n+1)!}x^m y^{n+1} + \frac{1}{1+kx}\Sigma\frac{C'_m}{m!}x^m + \Sigma\frac{D_{mn}}{(m-1)!n!}x^{m-1}y^n \quad (4)$$

$$A_y = \Sigma\frac{D_{mn}}{m!(n-1)!}x^m y^{n-1} \quad (5)$$

$$A_s = \Sigma\frac{A_{mn}}{(m-1)!(n+1)!}x^{m-1}y^{n+1} - \Sigma\frac{C_m}{(m-1)!}x^{m-1} + \frac{1}{1+kx}\Sigma\frac{D'_{mn}}{m!n!}x^m y^n, \quad (6)$$

where

$$A_{02} = -A_{20} + kA_{10} - A''_{00} \quad (7)$$

$$A_{12} = -A_{30} - kA_{20} + k^2A_{10} + k'A'_{00} - A''_{10} + 2kA''_{00} \quad (8)$$

$$A_{03} = -A_{21} - kA_{11} - A''_{01} \quad (9)$$

$$C_o = C_1 = 0 \quad (10)$$

$$C_2 = A_{01} \quad (11)$$

$$C_3 = A_{11} - kA_{01} \quad (12)$$

$$C_4 = A_{21} - kA_{11} + 3k^2A_{01} - A''_{01} \quad (13)$$

$$D_{00} = D_{10} = D_{20} = D_{01} = D_{02} = D_{03} = 0 \quad (14)$$

$$D_{30} = -A'_{01} \quad (15)$$

$$D_{40} = -A'_{11} + k'A_{01} + 4kA'_{01} \quad (16)$$

$$D_{11} = \frac{1}{2}A'_{00} \quad (17)$$

$$D_{21} = \frac{2}{3}A'_{10} - \frac{2}{3}kA'_{00} \quad (18)$$

$$D_{12} = \frac{1}{3}A'_{01} \quad (19)$$

APPROXIMATE HAMILTONIAN

Expanding the hamiltonian of Eq. (1) in powers of π_x, π_y, x, and y, one finds (dropping the -1)

$$H = -kx+\frac{1}{2}(1+kx)(\pi_x{}^2-2\frac{e}{p}\pi_x A_x+\frac{e^2}{p^2}A_x{}^2+\pi_y^2-2\frac{e}{p}\pi_y A_y+\frac{e^2}{p^2}A_y^2)$$
$$- \frac{e}{p}(1+kx)A_s + \ldots\ldots , \tag{20}$$

where a suitably restricted number of terms is to be included in the vector potential expansions to give say a third order expansion for the hamiltonian. Thus

$$(1+kx)A_x= - \frac{1}{2}A'_{00}y -(\frac{1}{3}A'_{10}+ \frac{1}{6}kA'_{00})xy - \frac{1}{3}A'_{01}y^2 + \ldots \tag{21}$$

$$(1+kx)A_y = \frac{1}{2}A'_{00}x + (\frac{1}{3}A'_{10}+ \frac{1}{6}kA'_{00})x^2+ \frac{1}{3}A'_{01}xy + \ldots\ldots \tag{22}$$

$$(1+kx)A_s= -A_{01}x + A_{10}y- \frac{1}{2}(A_{11}+kA_{01})x^2+\frac{1}{2}(2A_{20}+2kA_{10}+A'_{00})xy$$
$$+ \frac{1}{2}A_{11}y^2 - \frac{1}{6}(A_{21}+2kA_{11})x^3+ \frac{1}{2}(A_{30}+2kA_{20}+\frac{2}{3}A''_{10}-\frac{2}{3}k'A'_{00}$$
$$-\frac{2}{3}kA''_{00})x^2y +\frac{1}{2}(A_{21}+kA_{11}+\frac{1}{3}A''_{01})xy^2-\frac{1}{6}(A_{30}+kA_{20}-k^2A_{10}$$
$$-k'A'_{00}+A''_{10}-2kA''_{00})y^3+ \ldots\ldots \tag{23}$$

$$(1+kx)A_x^2= \frac{1}{4}A'^2_{00}y^2+A'_{00}(\frac{1}{3}A'_{10}- \frac{1}{12}kA'_{00})xy^2+\frac{1}{3}A'_{00}A'_{01}y^3 + \ldots \tag{24}$$

$$(1+kx)A_y^2= \frac{1}{4}A'^2_{00}x^2+A'_{00}(\frac{1}{3}A'_{10}-\frac{1}{12}kA'_{00})x^3+\frac{1}{3}A'_{00}A'_{01}x^2y + \ldots \tag{25}$$

If the hamiltonian is arranged according to orders of the expansion variables[3]

$$H = H^{(1)}+ H^{(2)}+ H^{(3)} + \ldots , \tag{26}$$

then

$$H^{(1)} = (\frac{e}{p}A_{01}-k)x - \frac{e}{p}A_{10}y . \tag{27}$$

$$H^{(2)} = \frac{1}{2}(\pi_x{}^2+\pi_y{}^2)+ \frac{1}{2}\frac{e}{p} A'_{00}(y\pi_x- x\pi_y)$$
$$+ \frac{1}{2}\frac{e}{p} (A_{11}+ kA_{01}+\frac{1}{4}\frac{e}{p}A'^2_{00})x^2- \frac{1}{2}\frac{e}{p}(2A_{20}+2kA_{10}+A'_{00})xy$$
$$- \frac{1}{2}\frac{e}{p} (A_{11} -\frac{1}{4}\frac{e}{p}A'^2_{00})\ y^2. \tag{28}$$

$$H^{(3)} = \frac{1}{2}kx(\pi_x{}^2+\pi_y{}^2)+ \frac{1}{3}\frac{e}{p}\left[(A'_{10}+ \frac{1}{2}kA'_{00})x+A'_{01}y\right](y\pi_x-x\pi_y)$$
$$+ \frac{1}{6}\frac{e}{p}\left[A_{21}+2kA_{11}+ \frac{e}{p} A'_{00}(A'_{10}- \frac{1}{4}kA'_{00})\right] x^3$$
$$- \frac{1}{6}\frac{e}{p}\left[3A_{30}+6kA_{20}+2A''_{10}-2k'A'_{00}-2kA''_{00}- \frac{e}{p} A'_{00}A'_{01}\right] x^2y$$
$$- \frac{1}{6}\frac{e}{p}\left[3A_{21}+3kA_{11}+A''_{01} - \frac{e}{p}A'_{00}(A'_{10}- \frac{1}{4}kA'_{00})\right] xy^2$$
$$+ \frac{1}{6}\frac{e}{p}\left[A_{30}+kA_{20}-k^2A_{10}-k'A'_{00}+A''_{10}-2kA''_{00}+ \frac{e}{p}A'_{00}A'_{01}\right] y^3. \tag{29}$$

MEDIAN PLANE SYMMETRY

For the ideal guide field one invokes median plane symmetry. In this case the coefficients in the expansion of the scalar potential Φ have the property

$$A_{mn}(s) = 0 \quad (\text{even } n) \ . \tag{30}$$

Equations (27-29) reduce to

$$H^{(1)} = (\frac{e}{p}A_{01}-k)\ x \tag{31}$$

$$H^{(2)} = \frac{1}{2}(\pi_x{}^2+\pi_y{}^2) + \frac{1}{2}\frac{e}{p}(A_{11}+kA_{01})x^2- \frac{1}{2}\frac{e}{p} A_{11}y^2 \tag{32}$$

$$H^{(3)} = \frac{1}{2}kx(\pi_x{}^2+\pi_y{}^2)+ \frac{1}{3}\frac{e}{p} A'_{01}y(y\pi_x-x\pi_y)$$
$$+ \frac{1}{6}\frac{e}{p} (A_{21}+ 2kA_{11})x^3$$
$$- \frac{1}{6}\frac{e}{p} (3A_{21}+ 3kA_{11}+A''_{01})xy^2 \tag{33}$$

EQUILIBRIUM ORBIT

The reference curve of the curvilinear coordinate system along which distance is measured by the coordinate s is to be chosen such that $H^{(1)} = 0$ in Eq.(31). Thus

$$\frac{e}{p}A_{01} = k \ . \tag{34}$$

ACTION ANGLE VARIABLES

A contact transformation is made from the variables (x, π_x, y, π_y) to $(\phi_x, \rho_x, \phi_y, \rho_y)$ using the generator[4]

$$F_1(x,\phi_x,y,\phi_y;s) = \frac{x^2}{2\beta_x}(\cot\psi_x + \tfrac{1}{2}\beta') + \frac{y^2}{2\beta_y}(\cot\psi_y + \tfrac{1}{2}\beta_y') \ . \tag{35}$$

The new hamiltonian $K = H + \frac{\partial F}{\partial s}$ becomes

$$K^{(2)} = \frac{\nu_x}{R}\rho_x + \frac{\nu_y}{R}\rho_y \ , \tag{36}$$

$$\begin{aligned} K^{(3)} = {} & k\sqrt{2\beta_x\rho_x}\sin\psi_x\left\{\frac{\rho_x}{\beta_x}(\cos\psi_x + \frac{\beta_x'}{2}\sin\psi_x)^2 + \frac{\rho_y}{\beta_y}(\cos\psi_y + \frac{\beta_y'}{2}\sin\psi_y)^2\right\} \\ & + \frac{1}{3}\frac{e}{p}A_{01}'\sqrt{2\beta_y\rho_y}\sin\psi_y\left\{\sqrt{\frac{\beta_y\rho_x\rho_y}{\beta_x}}\sin\psi_y(\cos\psi_x + \frac{\beta_x'}{2}\sin\psi_x)\right. \\ & \qquad \left. - \sqrt{\frac{\beta_x\rho_x\rho_y}{\beta_y}}\sin\psi_x(\cos\psi_y + \frac{\beta_y'}{2}\sin\psi_y)\right\} \\ & + \frac{1}{6}\frac{e}{p}(A_{21} + 2kA_{11})2^{3/2}\beta_x^{3/2}\rho_x^{3/2}\sin^3\psi_x \\ & - \frac{1}{6}\frac{e}{p}(3A_{21} + 3kA_{11} + A_{01}'')2^{3/2}\beta_x^{1/2}\beta_y\rho_x^{1/2}\rho_y\sin\psi_x\sin^2\psi_y \ , \end{aligned} \tag{37}$$

where

$$\psi_x = \phi_x - \int\left(\frac{\nu_x}{R} - \frac{1}{\beta_x}\right)ds \ ; \ \psi_y = \phi_y - \int\left(\frac{\nu_y}{R} - \frac{1}{\beta_y}\right)ds \ . \tag{38}$$

Note that the s-dependence of the hamiltonian, K, is only in the third and higher orders.

Expanding $K^{(3)}$ one has

$$K^{(3)} = k\left(\frac{2}{\beta_x}\right)^{1/2} \rho_x^{3/2} \sin\psi_x \cos^2\psi_x + k\,\frac{(2\beta_x)^{1/2}}{\beta_y}\,\rho_x^{1/2}\rho_y \sin\psi_x \cos^2\psi_y$$

$$+ k\left(\frac{2}{\beta_x}\right)^{1/2} \beta'_x \rho_x^{3/2} \cos\psi_x \sin^2\psi_x + \frac{1}{3}\,\frac{e}{p} A'_{01} \left(\frac{2}{\beta_x}\right)^{1/2} \beta_y \rho_x^{1/2} \rho_y \cdot$$

$$\cos\psi_x \sin^2\psi_y$$

$$+ \left[k(2\beta_x)^{1/2} \frac{\beta'_y}{\beta_y}\,\rho_x^{1/2}\rho_y - \frac{1}{3}\,\frac{e}{p}\,A'_{01}(2\beta_x)^{1/2}\rho_x^{1/2}\rho_y\right] \sin\psi_x \cdot$$

$$\sin\psi_y \cos\psi_y$$

$$+ \left[\frac{1}{4}k(2\beta_x)^{1/2}\frac{\beta'^2_y}{\beta_y}\,\rho_x^{1/2}\rho_y + \frac{1}{6}\,\frac{e}{p}\,A'_{01}\left(\beta'_x\beta_y\left(\frac{2}{\beta_x}\right)^{1/2} - \beta'_y(2\beta_x)^{1/2}\right)\cdot\right.$$

$$\rho_x^{1/2}\rho_y - \frac{1}{6}\,\frac{e}{p}(3A_{21}+3kA_{11}+A''_{01})\,2^{3/2}\beta_x^{1/2}\beta_y$$

$$\left.\rho_x^{1/2}\rho_y\right] \sin\psi_x \sin^2\psi_y$$

$$+ \left[\frac{1}{4}k\left(\frac{2}{\beta_x}\right)^{1/2}\beta'^2_x\rho_x^{3/2} + \frac{1}{6}\,\frac{e}{p}\,(A_{21}+2kA_{11})(2\beta_x)^{3/2}\rho_x^{3/2}\right] \cdot$$

$$\sin^3\psi_x \qquad (39)$$

After expanding the trignometric forms in $K^{(3)}$ the resulting expression may be rearranged according to coefficients of the independent trigonometric forms. Thus

Coefficient of $\sin\psi_x$:

$$\left[\frac{1}{4}k\left(\frac{2}{\beta_x}\right)^{1/2}\left(1+\frac{3}{4}\beta'^2_x\right) + \frac{1}{8}\,\frac{e}{p}(A_{21}+2kA_{11})(2\beta_x)^{3/2}\right]\rho_x^{3/2}$$

$$+ \left[\frac{1}{2}k\,\frac{(2\beta_x)^{1/2}}{\beta_y}\cdot\left(1+\frac{1}{4}\beta'^2_y\right) + \frac{1}{12}\,\frac{e}{p}\,A'_{01}\left(\beta'_x\beta_y\left(\frac{2}{\beta_x}\right)^{1/2} - \beta'_y(2\beta_x)^{1/2}\right)\right.$$

$$\left. - \frac{1}{6}\,\frac{e}{p}\,(3A_{21}+3kA_{11}+A''_{01})(2\beta_x)^{1/2}\beta_y\right]\rho_x^{1/2}\rho_y \qquad (40)$$

Coefficient of $\cos\psi_x$:

$$\frac{1}{4}k\left(\frac{2}{\beta_x}\right)^{1/2}\beta_x'\,\rho_x^{3/2} + \frac{1}{6}\,\frac{e}{p}\,A_{01}'\left(\frac{2}{\beta_x}\right)^{1/2}\beta_y\,\rho_x^{1/2}\rho_y \tag{41}$$

Coefficient of $\sin 3\psi_x$:

$$\left[\frac{1}{4}k\left(\frac{2}{\beta_x}\right)^{1/2}\left(1 - \frac{1}{4}\beta_x'^2\right) - \frac{1}{24}\,\frac{e}{p}(A_{21}+2kA_{11})(2\beta_x)^{3/2}\right]\rho_x^{3/2} \tag{42}$$

Coefficient of $\cos 3\psi_x$:

$$-\frac{1}{4}k\left(\frac{2}{\beta_x}\right)^{1/2}\beta_x'\,\rho_x^{3/2} \tag{43}$$

Coefficient of $\sin(2\psi_y+\psi_x)$:

$$\left[\frac{1}{4}k\frac{(2\beta_x)^{1/2}}{\beta_y}\cdot\left(1 - \frac{1}{4}\beta_y'^2\right) - \frac{1}{24}\,\frac{e}{p}\,A_{01}'\left(\beta_y'\beta_y\left(\frac{2}{\beta_x}\right)^{1/2} \quad \beta_y'(2\beta_x)^{1/2}\right)\right.$$

$$\left. + \frac{1}{12}\,\frac{e}{p}(3A_{21}+3kA_{11}+A_{01}'')(2\beta_x)^{1/2}\beta_y\right]\rho_x^{1/2}\rho_y \tag{44}$$

Coefficient of $\cos(2\psi_y+\psi_x)$:

$$\left[-\frac{1}{4}k(2\beta_x)^{1/2}\,\frac{\beta_y'}{\beta_y} + \frac{1}{12}\,\frac{e}{p}\,A_{01}'\left(\frac{2}{\beta_x}\right)^{1/2}\cdot(\beta_x-\beta_y)\right]\rho_x^{1/2}\rho_y \tag{45}$$

Coefficient of $\sin(2\psi_y-\psi_x)$:

$$\left[-\frac{1}{4}k\,\frac{(2\beta_x)^{1/2}}{\beta_y}\cdot(1-\frac{1}{4}\beta_y'^2) + \frac{1}{24}\,\frac{e}{p}\,A_{01}'\left(\beta_x'\beta_y\left(\frac{2}{\beta_x}\right)^{1/2} - \beta_y'(2\beta_x)^{1/2}\right)\right.$$

$$\left. - \frac{1}{12}\,\frac{e}{p}(3A_{21}+3kA_{11}+A_{01}'')(2\beta_x)^{1/2}\beta_y\right]\rho_x^{1/2}\rho_y \tag{46}$$

Coefficient of $\cos(2\psi_y-\psi_x)$:

$$\left[\frac{1}{4}k(2\beta_x)^{1/2}\frac{\beta_y'}{\beta_y} - \frac{1}{12}\,\frac{e}{p}\,A_{01}'\left(\frac{2}{\beta_x}\right)^{1/2}\cdot(\beta_x+\beta_y)\right]\rho_x^{1/2}\rho_y \tag{47}$$

FURTHER APPROXIMATIONS

At this stage several cases will be discussed. For each case one seeks a transformation to a rotating coordinate system for which the new hamiltonian becomes stationary[5,6,7,8]. Consider first the terms in $K^{(3)}$ that vary as $\sin\psi_x$ and $\cos\psi_x$. A generator

$$F_2(\phi_x, J_x; s) = (\phi_x - mN\frac{s}{R})\ J_x \tag{48}$$

transforms a hamiltonian containing $K^{(2)}$ and the restricted $K^{(3)}$ to

$$\begin{aligned} W = {} & \frac{\nu_y}{R}\rho_y + (\nu_x - mN)\frac{J_x}{R} + \left[A(s)J_x^{3/2} + B(s)J_x^{1/2}\rho_y\right]\cdot \\ & \sin\left(\gamma_x - \int\left(\frac{\nu_x - mN}{R} - \frac{1}{\beta_x}\right)ds\right) + \left[C(s)J_x^{3/2} + D(s)J_x^{1/2}\rho_y\right]\cdot \\ & \cos\left(\gamma_x - \int\left(\frac{\nu_x - mN}{R} - \frac{1}{\beta_x}\right)ds\right) \end{aligned} \tag{49}$$

in the variables $(\gamma_x\ J_x, \phi_y, J_y; s)$. Since the coefficients A, B, etc. are periodic with a period of the circumference/N, and since $\int\left(\frac{\nu_x}{R} - \frac{1}{\beta}\right)ds$ has zero average value, terms independent of s will arise from the cross combination of the terms in the Fourier expansion of the coefficients with the Fourier expansion of the phase modulated trigonometric terms. The lowest order stationary terms will arise from setting m = 1. All other terms will oscillate rapidly with respect to s and may be considered to have zero average. Thus resonance effects are expected for $\nu_x = N$. Since this condition has been avoided in the booster design, all effects from the terms that vary as $\sin\psi_x$ and $\cos\psi_x$ will be small. Hence these terms will be dropped from the hamiltonian.

EFFECT OF TERMS IN $\sin 3\psi_x$ and $\cos 3\psi_x$

In this case let the coefficients be ($\theta = s/R$):

$$A(\theta) = \left[\frac{1}{4}k\left(\frac{2}{\beta_x}\right)^{1/2}\left(1 - \frac{1}{4}\beta_x'^2\right) - \frac{1}{24}\frac{e}{p}(A_{21}+2kA_{11})(2\beta_x)^{3/2}\right]R^{3/2} \tag{50}$$

and

$$B(\theta) = -\frac{1}{4}k\left(\frac{2}{\beta_x}\right)^{1/2}\beta_x' R^{3/2} . \tag{51}$$

The hamiltonian is now taken to be

$$K' = \nu_x\rho_x + \nu_y\rho_y + \left[A(\theta)\sin 3\psi_x + B(\theta)\cos 3\psi_x\right]\rho_x^{3/2} , \tag{52}$$

where ρ_x and ρ_y are measured in units of R.

Transform to a rotating coordinate system $(\gamma_x, J_x; \gamma_y, J_y)$ using the generator

$$F_2(\phi_x, J_x, \phi_y, J_y; \theta) = (\phi_x - \frac{m}{3}N\theta)J_x + \phi_y J_y \tag{53}$$

and consider only the lowest order (m = 1).

$$K = (\nu_x - \frac{N}{3})J_x + \nu_y J_y + \left[A(\theta)\sin 3\psi_x + B(\theta)\cos 3\psi_x\right]J_x^{3/2}, \tag{54}$$

where, for convenience

$$\psi_x = \gamma_x + \frac{N}{3}\theta - \int\left(\frac{\nu_x}{R} - \frac{1}{\beta_x}\right) ds . \tag{55}$$

Fourier analyse $A(\theta)$, $B(\theta)$, and $\int\left(\frac{\nu_x}{R} - \frac{1}{\beta_x}\right)ds$ remembering that the average value of the integral is zero. Thus, let

$$A(\theta) = A_o + A_1 \cos N\theta + A_2 \sin N\theta + \ldots \tag{56}$$

$$B(\theta) = B_o + B_1 \cos N\theta + B_2 \sin N\theta + \ldots \tag{57}$$

$$\int\left(\frac{\nu_x}{R} - \frac{1}{\beta_x}\right) ds = P_1 \sin N\theta + \text{(sine series only by choice of origin)} \tag{58}$$

By retaining only the lowest order terms, the hamiltonian becomes

$$K = \left(\nu_x - \frac{N}{3}\right)J_x + \nu_y J_y + \Big[(A_o + A_1 \cos N\theta + A_2 \sin N\theta)\sin(3\gamma_x + N\theta - 3P_1 \sin N\theta) + (B_o + B_1 \cos N\theta + B_2 \sin N\theta)\cos(3\gamma_x + N\theta - 3P_1 \sin N\theta)\Big] J_x^{3/2} \tag{59}$$

Expand out to the two lowest orders in the phase flutter P_1, cross multiply and retain only the stationary terms. Then, if

$$A^2 = \left[A_o J_1(3P_1) + \tfrac{1}{2}(A_1 - B_2)J_o(3P_1) + \tfrac{1}{2}(A_1 + B_2)J_2(3P_1)\right]^2 + \left[B_o J_1(3P_1) + \tfrac{1}{2}(A_2 + B_1)J_o(3P_1) - \tfrac{1}{2}(A_2 - B_1)J_2(3P_1)\right]^2 \tag{60}$$

and a corresponding phase α are constructed where J_o, J_1, and J_2 are Bessel functions, the hamiltonian becomes

$$K = (\nu_x - \tfrac{N}{3})J_x + \nu_y J_y - AJ_x^{3/2}\sin(3\gamma_x + \alpha) \tag{61}$$

Since the hamiltonian does not contain γ_y, the conjugate variable J_y is a constant of the motion. Hence, let

$W = K - \nu_y J_y$. Then

$$W = \left(\nu_x - \frac{N}{3}\right)J_x - AJ_x^{3/2}\sin(3\gamma_x + \alpha) \tag{62}$$

Since $\nu_x \cong 6.75$ and $N = 24$ for the booster, the interest in the above hamiltonian is not in resonant growth of amplitude but rather in determining the amplitude variation of the betatron frequency.

To this end one further contact transformation is of value. If the hamiltonian can be made to contain only the action variables then the betatron frequencies are readily obtainable. Consider a transformation[6] from (γ_x, J_x) to $(\underline{\gamma_x}, \underline{J_x})$ using the generator

$$F_2(\gamma_x, \underline{J_x}) = \gamma_x \underline{J_x} - t\, \underline{J_x}^{3/2} \cos(3\gamma_x + \alpha) \tag{63}$$

Then, if the term in t is considered small one has

$$\gamma_x = \underline{\gamma_x} + \frac{3}{2}\, t\, \underline{J_x}^{1/2} \cos(3\underline{\gamma_x} + \alpha) + \ldots \tag{64}$$

$$J_x = \underline{J_x} + 3t\, \underline{J_x}^{3/2} \sin(3\underline{\gamma_x} + \alpha) + \frac{27}{2}\, t^2 \underline{J_x}^2 \cos^2(3\underline{\gamma_x} + \alpha) + \ldots \tag{65}$$

Let

$$\varepsilon_x = \nu_x - \frac{N}{3} \tag{66}$$

and, to eliminate the term in $\underline{J_x}^{3/2}$, choose

$$t = \frac{A}{3\varepsilon_x} . \tag{67}$$

Then

$$W = \varepsilon_x \underline{J_x} - \frac{3A^2}{4\varepsilon_x} \underline{J_x}^2 + \ldots \tag{68}$$

Of course the coefficient of $\underline{J_x}^2$ will be changed by the presence of terms of this order in the original hamiltonian. This effect is estimated later. Thus an estimate of the variation of the betatron frequency with amplitude is

$$\Delta\nu_x = \frac{\partial W}{\partial \underline{J_x}} = \varepsilon_x - \frac{3A^2}{2\varepsilon_x} \underline{J_x} \tag{69}$$

Note that $\underline{J_x}$ may be found from the beam emittance

$$E_x = \iint d\pi_x dx = R\iint d\underline{J_x} d\underline{\gamma_x} = 2\pi R\ \underline{J_x}\ . \tag{70}$$

EFFECT OF TERMS IN $\sin(2\psi_y + \psi_x)$ AND $\cos(2\psi_y + \psi_x)$

As in the previous case let the coefficients be

$$A(\theta) = \left[\frac{1}{4}k\ \frac{(2\beta_x)^{1/2}}{\beta_y}\cdot\left(1 - \frac{1}{4}\beta_y'^2\right) - \frac{1}{24}\ \frac{e}{p}\ A_{01}'\left(\frac{2}{\beta_x}\right)^{\frac{1}{2}}(\beta_x'\beta_y - \beta_y'\beta_x)\right.$$

$$\left. + \frac{1}{12}\ \frac{e}{p}(3A_{21}+3kA_{11}+A_{01}'')(2\beta_x)^{1/2}\beta_y\right] R^{3/2}, \tag{71}$$

$$B(\theta) = \left[-\ \frac{1}{4}k(2\beta_x)^{1/2}\ \frac{\beta_y'}{\beta_y} + \frac{1}{12}\ \frac{e}{p}\ A_{01}'\left(\frac{2}{\beta_x}\right)^{1/2}\cdot(\beta_x - \beta_y)\right] R^{3/2}. \tag{72}$$

The hamiltonian is now taken to be

$$K' = \nu_x\rho_x + \nu_y\rho_y + \left[A(\theta)\sin(2\psi_y+\psi_x) + B(\theta)\cos(2\psi_y+\psi_x)\right]\rho_x^{1/2}\rho_y, \tag{73}$$

where again ρ_x and ρ_y have been made dimensionless by measuring in units of R.

Transform to a rotating coordinate system $(\gamma_x, J_x; \gamma_y J_y)$ using the generator

$$F_2(\phi_x, J_x, \phi_y, J_y; \theta) = (\phi_x - aN\theta)J_x + (\phi_y - bN\theta)J_y. \tag{74}$$

Then

$$K = (\nu_x - aN)J_x + (\nu_y - bN)J_y + \left[A(\theta)\sin(2\psi_y+\psi_x)\right.$$

$$\left. + B(\theta)\cos(2\psi_y+\psi_x)\right]J_x^{1/2}J_y, \tag{75}$$

where, for convenience

$$\psi_x = \gamma_x + aN\theta - \int\left(\frac{\nu_x}{R} - \frac{1}{\beta_x}\right)ds\ ;\ \psi_y = \gamma_y + bN\theta - \int\left(\frac{\nu_y}{R} - \frac{1}{\beta_y}\right)ds\ . \tag{76}$$

In order to make stationary terms possible in the cross combination of the Fourier analyzed $A(\theta)$ and $B(\theta)$ with the trignometric terms

$$a + 2b = p \text{ (an integer)} . \tag{77}$$

Consider only the lowest order term (p=1). As before Fourier analyze

$$A(\theta) = A_o + A_1 \cos N\theta + A_2 \sin N\theta + \ldots \tag{78}$$

$$B(\theta) = B_o + B_1 \cos N\theta + B_2 \sin N\theta + \ldots \tag{79}$$

$$\int \left(\frac{\nu_x}{R} - \frac{1}{\beta_x} \right) ds = P_1 \sin N\theta + \ldots \tag{80}$$

$$\int \left(\frac{\nu_y}{R} - \frac{1}{\beta_y} \right) ds = -Q_1 \sin N\theta + \ldots \tag{81}$$

The average value of each integral is zero and, in addition, the lattice is assumed to have the symmetry implied by the series expansion.

By retaining only the lowest order terms, the hamiltonian becomes

$$\begin{aligned} K = {} & (\nu_x - aN) J_x + (\nu_y - bN) J_y \\ & + \Big\{ (A_o + A_1 \cos N\theta + A_2 \sin N\theta) \sin \left[2\gamma_y + \gamma_x + N\theta - (P_1 - 2Q_1) \sin N\theta \right] \\ & \quad + (B_o + B_1 \cos N\theta + B_2 \sin N\theta) \cos \left[2\gamma_y + \gamma_x + N\theta - (P_1 - 2Q_1) \sin N\theta \right] \Big\} J_x^{1/2} J_y \end{aligned} \tag{82}$$

Expand out to the two lowest orders in the phase flutters P_1 and Q_1, cross multiply and retain only the stationary terms. Then, if

$$\begin{aligned} A^2 = {} & \left[A_o J_1(P_1 - 2Q_1) + \tfrac{1}{2}(A_1 - B_2) J_o(P_1 - 2Q_1) + \tfrac{1}{2}(A_1 + B_2) J_2(P_1 - 2Q_1) \right]^2 \\ & + \left[B_o J_1(P_1 - 2Q_1) + \tfrac{1}{2}(A_2 + B_1) J_o(P_1 - 2Q_1) - \tfrac{1}{2}(A_2 - B_1) J_2(P_1 - 2Q_1) \right]^2 \end{aligned} \tag{83}$$

and the corresponding phase α are formed, the hamiltonian becomes

$$K = (\nu_x - aN)J_x + (\nu_y - bN)J_y - A\, J_x^{1/2} J_y \sin(2\gamma_y + \gamma_x + \alpha) \tag{84}$$

Another constant of the motion may be constructed by noticing that in the variables $(\xi, J_\xi; \eta, J_\eta)$ the generator

$$F_2(\gamma_x, J_\xi; \gamma_y J_\eta) = (2\gamma_y + \gamma_x)J_\xi + (2\gamma_y - \gamma_x)J_\eta \tag{85}$$

transforms the hamiltonian to

$$K = (\nu_x - aN)(J_\xi - J_\eta) + 2(\nu_y - bN)(J_\xi + J_\eta) - 2A(J_\xi - J_\eta)^{1/2}(J_\xi + J_\eta)\sin(\xi + \alpha) . \tag{86}$$

Since this hamiltonian is independent of η, the corresponding conjugate variable J_η is a constant of the motion. Hence

$$J_y - 2J_x = 4J_\eta = \text{constant} \tag{87}$$

Then, remembering that $a + 2b = 1$

$$K = \left[2\nu_y - \nu_x - (2b - a)N\right] J_\eta + (2\nu_y + \nu_x - N)J_\xi - 2A(J_\xi - J_\eta)^{1/2}(J_\xi + J_\eta)\sin(\xi + \alpha) . \tag{88}$$

In order to determine the amplitude variation of the betatron tunes, transform to the variables $\left(\underline{\xi}, \underline{J_\xi}\right)$ using the generator

$$F_2\left(\xi, \underline{J_\xi}\right) = \xi \underline{J_\xi} - t\left(\underline{J_\xi} - J_\eta\right)^{1/2}\left(\underline{J_\xi} + J_\eta\right)\cos(\xi + \alpha) \tag{89}$$

which is used to eliminate the $\underline{\xi}$ variable in the transformed hamiltonian to the order of accuracy being considered. The new hamiltonian becomes

$$K = \left[2\nu_y - \nu_x - (2b - a)N\right] J_\eta + \varepsilon \underline{J_\xi} - \frac{A^2}{\varepsilon}(3\underline{J_\xi} - J_\eta)(\underline{J_\xi} + J_\eta)\left(1 - \cos\left[2(\underline{\xi} + \alpha)\right]\right) , \tag{90}$$

where

$$\varepsilon = 2\nu_y + \nu_x - N, \tag{91}$$

and the parameter t was chosen so that

$$\varepsilon t = 2A \tag{92}$$

One further transformation would be necessary to remove the term in $\cos\left[2(\underline{\xi}+\alpha)\right]$. However, since this term is already of higher order its average value is a sufficient approximation. Thus

$$K = \left[2\nu_y - \nu_x - (2b-a)N\right] J_\eta + \varepsilon J_{\underline{\xi}} - \frac{A^2}{\varepsilon}\,(2J_{\underline{\xi}} - J_\eta)\,(J_{\underline{\xi}} + J_\eta) \tag{93}$$

Since only the action variables are contained in the hamiltonian the betatron tunes are:

$$\Delta\nu_{\underline{\xi}} = \frac{\partial K}{\partial J_{\underline{\xi}}} = \varepsilon - 2\,\frac{A^2}{\varepsilon}(3J_{\underline{\xi}} + J_\eta) = 2\Delta\nu_y + \Delta\nu_x \tag{94}$$

$$\Delta\nu_{\underline{\eta}} = \frac{\partial K}{\partial J_\eta} = 2\nu_y - \nu_x - (2b-a)N - 2\,\frac{A^2}{\varepsilon}(J_{\underline{\xi}} - J_\eta) = 2\Delta\nu_y - \Delta\nu_x \tag{95}$$

or rewriting such that the original linear tunes are (ν_{xo}, ν_{yo}) then

$$\nu_x = \nu_{xo} - 2\,\frac{A^2}{\varepsilon}\,(J_{\underline{\xi}} + J_\eta) \tag{96}$$

$$\nu_y = \nu_{yo} - 2\,\frac{A^2}{\varepsilon}\,J_{\underline{\xi}}\ , \tag{97}$$

where now

$$\varepsilon = 2\nu_{yo} + \nu_{xo} - N \tag{98}$$

Note that a Poincaré invariant yields

$$E_x + E_y = \iint d\pi_x dx + \iint d\pi_y dy = R\iint dJ_\xi d_\xi + R\iint dJ_\eta d_\eta = 2\pi R(J_\xi + J_\eta) \tag{99}$$

and that the previous invariant

$$4J_\eta = J_y - 2J_x = \frac{E_y}{2\pi R} - \frac{E_x}{\pi R} \tag{100}$$

Thus

$$\nu_x = \nu_{xo} - \frac{A^2}{\pi\varepsilon R}(E_x + E_y) \tag{101}$$

$$\nu_y = \nu_{yo} - \frac{A^2}{\pi\varepsilon R}(\tfrac{1}{2}E_x + \tfrac{5}{4}E_y) \tag{102}$$

EFFECT OF TERMS IN $\sin(2\psi_y - \psi_x)$ AND $\cos(2\psi_y - \psi_x)$

The effectiveness of these terms depends on the magnitude of $\varepsilon = 2\nu_y - \nu_x - N$ which for the booster amounts to -16.9. This value is too large to produce much amplitude shift and, therefore, this case will not be considered.

NUMERICAL RESULTS FOR BOOSTER

Numerical input comes from the linear orbit program SYNCH which in turn utilizes the basic booster parameters. This input is tabulated in Tables (1 - 3).

Harmonic analysis of $\nu_x - R/\beta_x$ and $\nu_y - R/\beta_y$ was carried out using the betatron functions graphically interpolated to values at regular intervals. The program HANAC then yielded $P_1 = .2212$ and $Q_1 = .2020$.

Secondly, the values of the coefficients A and B in Equations (50 - 51) and Equations (71 - 72) were obtained at the longitudinal locations indicated in Table 3. To do this the hard edge effects were omitted with the intention of handling them separately. The program COEF evaluated these functions using the input from Tables (1 - 3).

The hard edge effects arose from the terms containing A'_{01}, A''_{01}; and from A_{11}, A_{21} when finite edge angles were employed to yield parallel end faces of the magnets. The terms in A'_{01} have a delta function contribution at the magnet ends. Similarly the term in A''_{01} has a delta function derivative contribution at the magnet ends. To obtain the hard edge effects due to finite edge angles one replaces A_{11} in Eq. (50) and (71) by

$$A_{11} \rightarrow A_{11}S - A_{01}\tan\alpha\delta(N\theta)\cdot\frac{N}{R}, \tag{103}$$

where $S(\frac{R}{N}\cdot N\theta - x\tan\alpha)$ is a step function. For A_{21} the replacement is

$$A_{21} \rightarrow A_{21}S - 2\frac{N}{R}A_{11}\tan\alpha\delta(N\theta) + A_{01}\frac{N^2}{R^2}\tan^2\alpha\delta'(N\theta). \tag{104}$$

Notice that the longitudinal variable used is $N\theta$ which increases 2π/sector. The edge angle is different for the

F and D magnets because of the different magnetic radii and is of opposite sign on each side of the magnet. The program COEF gives the contributions to the A and B coefficients that are proportional to the δ-function and to the δ'-function.

The program HANAB provides a fourier analysis of the discontinuous functions A and B in which the δ-function and δ'-function contributions have been removed. Since the COEF program provides the strengths of the δ-function and δ'-function contributions at each end of the four magnets in a sector, it is a relatively simple matter to find the harmonic coefficients of the sum of these contributions. In this manner the net harmonic coefficients as defined in Egs. (56 - 57) and Egs. (78 - 79) are determined.

Having determined the harmonic coefficients of lowest order, P_1, Q_1, A_0, A_1, A_2, B_0, B_1, B_2, the amplitudes A in Eg. (60) and in Eg. (83) may be found. These amplitudes together with the measure of the distance from the resonance ε and the beam emittances give the tune shifts. Table 4 gives the harmonic coefficients as outlined and Table 5 gives the tune shift results. Table 5 presents the third order tune shift due to the proximity of the operating point to the $3\nu_x = 24$ resonance and the $\nu_x + 2\nu_y = 24$ resonance.

NONRESONANT FOURTH ORDER TUNE SHIFT

As is already evident from the third order effects, it would be a rather tedious exercise to evaluate the fourth order effects with the same degree of detail as the third order. The dominant terms in the fourth order hamiltonian are

$$H^{(4)} = \frac{1}{24}\frac{e}{p}A_{31}(x^4 - 6x^2y^2 + y^4). \tag{105}$$

After the action-angle transformation of Eg. (34), this becomes

$$K^{(4)} = \frac{kk_3}{6}(\beta_x^2\rho_x^2\sin^4\psi_x - 6\beta_x\beta_y\rho_x\rho_y\sin^2\psi_x\sin^2\psi_y$$

$$+ \beta_y^2\rho_y^2\sin^4\psi_y), \tag{106}$$

where Eg. (38) gives the expressions for ψ_x and ψ_y.

The fourth order hamiltonian has a nonresonant contribution given principally by the following average

$$K^{(4)} = \frac{1}{16}\left\langle kk_3\beta_x^2R^2\right\rangle_{Av.} J_x^2 - \frac{1}{4}\left\langle kk_3\beta_x\beta_yR^2\right\rangle_{Av.} J_xJ_y$$

$$+ \frac{1}{16}\left\langle kk_3\beta_y^2R^2\right\rangle_{Av.} J_y^2 \tag{107}$$

Letting

$$B = \frac{1}{16}\left\langle kk_3\beta_x^2R^2\right\rangle \tag{108}$$

$$C = -\frac{1}{4}\left\langle kk_3\beta_x\beta_yR^2\right\rangle \tag{109}$$

$$D = \frac{1}{16}\left\langle kk_3\beta_y^2R\right\rangle, \tag{110}$$

the nonresonant tune shift becomes

$$\delta\nu_x = 2BJ_x + CJ_y \tag{111}$$

$$\delta\nu_y = + CJ_x + 2DJ_y \tag{112}$$

The program COEF evaluates the hamiltonian coefficients in Eq. (106) and the program HANAB produces the azimuthal average. Using

$$E_x = 2\pi RJ_x \qquad E_y = 2\pi RJ_y \tag{113}$$

to evaluate J_x and J_y from the horizontal and vertical beam

emittances, one may evaluate the tune shifts. Table 6 gives these results. As may be seen the fourth order non-resonant effects are larger than the third order resonant effects transformed to fourth order.

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TABLE 1. GENERAL BOOSTER PARAMETERS

Average Radius (m)	75.4717
Number of Sectors	24
Radial Emittance of Beam (mm-mrad)	50π
Vertical Emittance of Beam (mm-mrad)	20π
Radial Betatron Oscillations per Revolution	6.7
Vertical Betatron Oscillations per Revolution	6.8

TABLE 2. BOOSTER MAGNET PARAMETERS

	F	D
Magnetic Radius (m)	40.8469	48.0341
Magnet Length (m)	2.8896	2.8896
Normalized Gradient (m^{-1})	2.2147	-2.7719
Normalized Second Gradient (m^{-2})	.6079	-1.2560
Normalized Third Gradient (m^{-3})	-21.63	6.45
Magnet Location (radians, 2π/sector)		
Entrance (first magnet)	.1908	1.2687
Exit (first magnet)	1.1097	2.1876
Entrance (second magnet)	5.1735	4.0957
Exit (second magnet)	6.0924	5.0146
Edge Angles (radians)		
Entrance (first magnet)	.03536	.03007
Exit (first magnet)	-.03536	-.03007
Entrance (second magnet)	.03536	.03007
Exit (second magnet)	-.03536	-.03007

TABLE 3. ORBIT FUNCTIONS FROM SYNCH

Longitudinal Position (radians, 2π/sector)

.0000	.0954	.1908	.1908	.4205
.6502	.8800	1.1097	1.1097	1.1892
1.2687	1.2687	1.4984	1.7281	1.9579
2.1876	2.1876	2.6646	3.1416	3.6186
4.0957	4.0957	4.3253	4.5550	4.7848
5.0146	5.0146	5.0940	5.1735	5.1735
5.4032	5.6329	5.8627	6.0924	6.0924
6.1878	6.2832			

BETAX (m)

33.6626	33.6653	33.6733	33.6733	32.8011
30.1501	26.0209	20.8814	20.8814	19.0473
17.3038	17.3038	13.2161	10.4388	8.6365
7.5917	7.5917	6.4890	6.1214	6.4890
7.5917	7.5917	8.6365	10.4388	13.2161
17.3038	17.3038	19.0473	20.8814	20.8814
26.0209	30.1501	32.8011	33.6733	33.6733
33.6653	33.6626			

ALPHAX

.0000	-.0089	-.0178	-.0470	1.2428
2.3918	3.2696	3.7768	3.7588	3.5776
3.3965	3.3857	2.3291	1.5537	.9660
.4948	.4901	.2450	.0000	-.2450
-.4901	-.4948	-.9660	-1.5537	-2.3291
-3.3857	-3.3965	-3.5776	-3.7588	-3.7768
-3.2696	-2.3918	-1.2428	.0470	.0178
.0089	.0000			

TABLE 3. ORBIT FUNCTIONS FROM SYNCH (cont.)

BETAY (m)

5.2734	5.2905	5.3417	5.3417	5.7548
6.7198	8.3470	10.8223	10.8223	11.8806
12.9992	12.9992	16.1015	18.5549	20.0667
20.4567	20.4567	20.1193	20.0068	20.1193
20.4567	20.4567	20.0667	18.5549	16.1015
12.9992	12.9992	11.8806	10.8223	10.8223
8.3470	6.7198	5.7548	5.3417	5.3417
5.2905	5.2734			

ALPHAY

.0000	-.0569	-.1138	-.1092	-.4680
-.8804	-1.3933	-2.0655	-2.0561	-2.1768
-2.2976	-2.2895	-1.9618	-1.4002	-.6716
.1371	.1500	.0750	.0000	-.0750
-.1500	-.1371	-.6716	1.4002	1.9618
2.2895	2.2976	2.1768	2.0561	2.0655
1.3933	.8804	.4680	.1092	.1138
.0569	.0000			

TABLE 4. HARMONIC COEFFICIENTS OF $A(\theta)$ and $B(\theta)$

Resonance	$(3\nu_x = 24)$	$(\nu_x + 2\nu_y = 24)$
Radial Phase Flutter (P_1)	.2212	.2212
Vertical Phase Flutter (Q_1)	.2020	.2020
Average Coefficient (A_0)	-47.44	59.25
Coefficient of $\cos N\theta$ (A_1)	-119.41	-94.18
Coefficient of $\sin N\theta$ (A_2)	.00	.00
Average Coefficient (B_0)	.00	.00
Coefficient of $\cos N\theta$ (B_1)	.00	.00
Coefficient of $\sin N\theta$ (B_2)	4.65	-3.19

TABLE 5. THIRD ORDER RESONANT TUNE SHIFT

	$(3\nu_x = 24)$	$(\nu_x + 2\nu_y = 24)$
Amplitude (A)	70.80	50.70
Distance from Resonance (ε)	-1.3	-3.7
Radial Tune Shift	.00113	.00100
Vertical Tune Shift		.00071

TABLE 6. FOURTH ORDER NONRESONANT TUNE SHIFT

Amplitude (B)	-46139
Amplitude (C)	33866
Amplitude (D)	1726
Radial Tune Shift	-0.0261
Vertical Tune Shift	0.0117