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Universal weak coupling limit in (2+1)–dimensional SU(2) lattice gauge theory

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Abstract:
Mass gaps and wavefunctions in (2+1) dimensional SU(2) lattice gauge theory (no quarks) are investigated. Starting with lattice Hamiltonians possessing exactly known ground states and the correct naive continuum limit, it is possible to reach the very deep weak coupling region. Using variational approximation and rescaling all parameters with the help of the dimensionful coupling constant g^2, we gain a formulation that is independent of the special choice of the Hamiltonian in the weak coupling limit. The mass gap can be calculated and a kind of wave function for excited states to accesible.

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1. Introduction

A Yang–Mills model based on a gauge group SU(2) in three space time dimensions is the simplest nontrivial non–Abelian gauge theory. Considering a theory without quarks leaves the problem of a finite mass for the glueball. Feynman [1] argued that gauge invariance of the theory does not allow any long range correlations. Any excitations above the ground state should have finite energy.

Lattice gauge theory (LGT) is a method to get information about nonperturbativ aspects of such a theory. By virtue of different methods, namely in the Euclidean version Monte–Carlo (MC) [2,3], and in the Hamiltonian version cluster expansion methods [4] and again variational calculations [5,9,10], scaling behaviour of physical quantities according to their physical dimension is established. The mass gap M and the string tension σ of (2+1)–dimensional SU(2) LGT show the following weak coupling behaviour

$$M = c_n \cdot g^2 \tag{1.1a}$$

and

$$\sigma = c_\sigma \cdot g^4. \tag{1.1b}$$

Here g^2 is the physical coupling (charge), which can be expressed in the space–time dimensions by the bare (lattice) coupling $\tilde{g}^2$ and the lattice spacing a by

$$g^2 = \tilde{g}^2/a \tag{1.2}$$

Most lattice methods are strong coupling methods, while the continuum limit arises in the weak coupling region. Weak coupling expansions on a finite lattice [6] (L lattice units in each direction) show difficulties if the IR–limes $L \to \infty$ is performed.

The Wilson–LGT–action [7] is constructed in a way that it reproduces the continuum Yang–Mills–action in the formal limit $a \to 0$ by a replacement of a SU(2)–valued link variable

$$U_\ell = \exp(iagA_\mu(\vec{x})) \tag{1.3}$$

The Yang–Mills–vector–potential $A_\mu(\vec{x})$ is an element of the SU(2)–algebra. A link ℓ is specified by a lattice point $\vec{n}$ and a direction μ, the continuum vector $\vec{x}$ is

$$\vec{x} = a \cdot \vec{n}\,. \tag{1.4}$$

By carrying out the continuum limit in one (time)direction the Hamiltonian formulation [8] is obtained. Guo and Zheng [9] proposed a series of Hamiltonians which differ from the usual Kogut–Susskind Hamiltonian

$$H_{KS} = \frac{\tilde{g}^2}{2a} \sum_{\ell,\alpha} E^{\alpha}_{\ell} E^{\alpha}_{\ell} - \frac{1}{a\tilde{g}^2} \sum_{p} \mathrm{tr}(U_{\partial p} + U^{+}_{\partial p}) \tag{1.5}$$

by a term ΔH which is Hamiltonian and vanishes in the continuum limit $a \to 0$, so that the same continuum limit is achieved. This new form of lattice Hamiltonian

$$H_{GZ} = \frac{\tilde{g}^2}{2a} \sum_{\ell,\alpha} \exp[-R]\, E^{\alpha}_{\ell} \exp[2R]\, E^{\alpha}_{\ell} \exp[-R]$$

$$= \frac{\tilde{g}^2}{2a} \sum_{\ell,\alpha} \left\{ E^{\alpha}_{\ell} E^{\alpha}_{\ell} - \left[E^{\alpha}_{\ell}, \left[E^{\alpha}_{\ell}, R\right]\right] - \left[E^{\alpha}_{\ell}, R\right]\left[E^{\alpha}_{\ell}, R\right] \right\} \tag{1.6}$$

has an exactly known ground state.

The ground state is

$$|\psi_0\rangle = \exp(R)|0\rangle \tag{1.7}$$

where $|0\rangle$ is the state defined by $E^{\alpha}_{\ell}|0\rangle = 0$, the ground state of the strong coupling limit of the Kogut–Susskind Hamiltonian. If one generalizes the ansatz of Guo and Zheng for R to

$$R = \sum_{pl.} \frac{2N}{\tilde{g}^4} \sum_{n} d_n \left[\frac{A_p}{2N}\right]^n \tag{1.8}$$

with

$$A_p = \mathrm{tr}\left[U_{\partial P} + U^{+}_{\partial P}\right] \tag{1.9}$$

the condition $\lim_{a \to 0} (H_{KS} - H_{GZ}) = 0$

demands

$$\sum_{n} \frac{4}{2N} d_n\, n\, (nN^2 - 2 + n) = 2 \tag{1.10a}$$

and

$$\sum_{n} d_n \cdot n = 0 \tag{1.10b}$$

Eqs. (1.8) – (1.10a/b) are valid for a general gauge group SU(N). In the following we will restrict to N = 2.

Knowing the exact ground state is important for calculating a mass gap by a variational ansatz. So it is an upper bound to an exact mass gap.

II. The variational ansatz according to Guo and Zheng.

The variational states are chosen in a form

$$|\psi\rangle = \sum_n p_n |\psi_n\rangle. \tag{2.1}$$

The p_n will be variational parameters. The trial functions $|\psi_n\rangle$ are

$$|\psi_n\rangle = \left[\frac{1}{\mathcal{N}^{\frac{1}{2}}} \sum_x \phi_n(x) - v_n\right] |\psi_0\rangle , \tag{2.2}$$

with

$$v_n = \frac{1}{\mathcal{N}^{\frac{1}{2}}} \langle\psi_0|\sum_x \phi_n(x)|\psi_0\rangle / \langle\psi_0|\psi_0\rangle . \tag{2.3}$$

They are orthogonal to $|\psi_0\rangle$. The factor $1/\mathcal{N}^{\frac{1}{2}}$ is introduced to keep matrix elements finite:

$$\mathcal{N} = \sum_x 1 = \text{Number of lattice sites.} \tag{2.4}$$

The trial functions $|\psi_n\rangle$ are translation invariant and have to be build up by gauge invariant operators. Normalized expectation values $\langle Op\rangle_0$ are

$$\langle Op\rangle_0 = \langle\psi_0|Op|\psi_0\rangle / \langle\psi_0|\psi_0\rangle. \tag{2.5}$$

We only use trial functions with

$$\left[R, \left[E^{\alpha}_{\ell'}, \phi_n(x)\right]\right] = 0 . \tag{2.6}$$

The lattice has to be large enough, so boundary effects need not to be taken care of. So matrix elements can be calculated:

$$C_{mn} = \langle\psi_m|H \cdot \frac{2a}{\tilde{g}^2}|\psi_n\rangle = -\langle\psi_0| \sum_{\ell,\alpha,x} \left[E^{\alpha}_{\ell'}\phi_n(x)\right] \left[E^{\alpha}_{\ell'}\phi_n(0)\right]|\psi_0\rangle \quad (2.7)$$

and

$$D_{mn} = \langle\psi_m|\psi_n\rangle = \sum_x \left[\langle\psi_0|\phi_m(0)\phi_n(x)|\psi_0\rangle - v_n v_m \langle\psi_0|\psi_0\rangle\right] \quad (2.8)$$

The mass gap m is determined by

$$\det|C_{mn} - \lambda D_{mn}| = 0 \quad (2.9)$$

with

$$\lambda = 2a/\tilde{g}^2 \cdot m \quad (2.10)$$

In (2.8) normalisation of C_{mn} and D_{mn} is irrelevant and we may therefore use instead

$$\hat{C}_{mn} = -\sum_{\ell,\alpha,x} \left\langle\left[E^{\alpha}_{\ell'}\phi_m(0)\right] \left[E^{\alpha}_{\ell'}\phi_n(x)\right]\right\rangle_0 \quad (2.7')$$

and

$$\hat{D}_{mn} = \sum_x \left[\langle\phi_0(0)\phi_n(x)\rangle_0 - \langle\phi_m(0)\rangle_0 \langle\phi_n(x)\rangle_0\right] \quad (2.8')$$

This variational formulation is more or less a strong coupling access. In the strong coupling region, that means large $\tilde{g}^2 >> 1$, the Hamiltonians H_{KS} (1.5) and H_{GZ} (1.6) coincide in leading order in $1/\tilde{g}^2$, and so does the spectrum. The low lying excited states are build up of small objects. The first excited state corresponds to a trial function $\phi_1(x) = \mathrm{Tr}U_{\partial P}$ describing a one–plaquette Wilson loop with a strong coupling mass gap $\lambda_{SC} = 3$.

Contributions to matrix elements $\hat{C}_{mn}$ stem from configurations having common links. Configurations without intersection contribute neither to $\hat{C}_{mn}$, nor to $\hat{D}_{mn}$. Low lying states will be dominated by objects having a small circumference compared with the covered area.

If we use as trial functions rectangular Wolson–loops of $n_1 \times n_2$ lattice units, all matrix elements can be expressed by

$$U = < \frac{1}{2} \mathrm{tr}\, U_{\partial P} >_0 \tag{2.11a}$$

and

$$A = < \frac{1}{4} (\mathrm{tr}\, U_{\partial P})^2 >_0 \tag{2.11b}$$

In (2+1)–dimensional lattice gauge theory, after a gauge fixing we can change the link integration variables $[dU_\ell]$ into plaquette integration variables $[dU_{\partial P}]$. The Jacobian of this transformation is 1 and the plaquette integrations become uncorrelated. The expectation values

$$B_i = < (\mathrm{tr}\, U_1 U_2 \ldots U_i))^2 >_0 \tag{2.12}$$

can be expressed by A [9].

$$B_i = \frac{4}{3} (1-A) + \frac{4A-1}{3} B_{i-1} \tag{2.13}$$

and

$$B_1 = 4A \tag{2.14}$$

which is solved by

$$B_i = 1 + 3 \left[\frac{4A-1}{3}\right]^i \tag{2.15}$$

In eq. (2.6) and (2.7) the indices m,n are replaced by multiindices (m_1,m_2) (n_1,n_2) denoting the Wilson loops. So the matrix elements $C_{(m_1,m_2)(n_1,n_2)}$ and $D_{(m_1,m_2)(n_1,n_2)}$ can be evaluated:
With

$$\min_1 = \min(m_1,n_1), \qquad \max_1 = \max(m_1,n_1) \tag{2.16}$$

$\min_2$, $\max_2$ analogously

one gets:

$$C_{(m_1,m_2)(n_1,n_2)} = U^{m_1m_2+n_1n_2} \cdot$$

$$\Bigg[\sum_{i=1}^{\min_1} (2 + (\max_1 - \min_1 - 1)\, \delta_{i,\min_1}) \cdot$$

$$2i\, U^{-2i\cdot\min_2} (1 - \tfrac{1}{4} B_{i\cdot\min_2})$$

$$+ \sum_{i=1}^{\min_2} (2 + (\max_2 - \min_2 - 1)\, \delta_{i,\min_2}) \cdot$$

$$2i\, U^{-2i\cdot\min_1} (1 - \tfrac{1}{4} B_{i\cdot\min_1}) \Bigg] \tag{2.17}$$

and

$$D_{(m_1,m_2)(n_1,n_2)} = U^{m_1m_2+n_1n_2} \cdot \Bigg[\sum_{i=1}^{\min_1} (2 + (\max_1 - \min_1 - 1)\, \delta_{i,\min_1})$$

$$\cdot \sum_{j=1}^{\min_2} (2 + (\max_2 - \min_2 - 1)\, \delta_{j,\min_2})$$

$$\cdot (U^{-2ij} B_{ij} - 4) \Bigg] \tag{2.18}$$

This corresponds to the result of Guo and Zheng [9] for quadratic Wilson loops.

III. The weak coupling region

If we look at smaller values of $\tilde{g}^2$ larger and larger objects determine the low lying states. Guo and Zheng [9] claim that quadratic Wilson–loops dominate the weak coupling region. To reach a value of $1/\tilde{g}^2 \sim 7$ they need loops covering 130×130 lattice units. To recover the naive continuum limit the vacuum state function $e^R|0>$ should concentrate at the group unit 1. If we have

$$R = \sum_{\text{pl.}} \frac{4}{\tilde{g}^4} \sum_n d_n \left[\frac{A_p}{4}\right]^n \tag{1.8}$$

with

$$A_p = \mathrm{tr}(U_{\partial P} + U^+_{\partial P}) = 4 \cos \varphi_p \tag{1.9}$$

φ_p is the central angle of the SU(2)–valued plaquette variable $U_{\partial P}$, then R should have a maximum at $\varphi_p = 0$. This demands

$$\sum_n n\, d_n > 0 \tag{3.1a}$$

in contradiction to (1.10b) or, if $\sum_n n\, d_n = 0$ the condition is

$$\sum_n n^2\, d_n < 0 \tag{3.1b}$$

in contradiction to (1.10a).

If the conditions (1.10a) and (1.10b) are fulfilled $\varphi_p = 0$ is even a local minimum of R. The Hamiltonian does not have the naive continuum limit. But we can get information about the matrix elementx C_{mn} and D_{mn} and about the scaled mapp gap λ in the limit $\tilde{g}^2 \to 0$.

If we want to be sure that the limit $\tilde{g}^2 \to 0$ happens at $\varphi_p = 0$ we make a choice of coefficients $\{d_n\}$ fulfilling conditions

$$\sum_n d_n\, n\,(5n - 2) = 0 \tag{1.10a'}$$

and

$$K = 4 \sum_n n\, d_n > 0 \tag{3.2}$$

We have to be aware that the naive continuum limit of such a Hamiltonian has contributions of the kind $K^2(\partial_\mu F_{\nu\mu})^2$ [9].
$\cos\varphi_p = 1$ is now a saddle point of e^R.

Integrals of the type

$$I_\ell = \int d\mu(U_{\partial P}) \cos^\ell \varphi_p \, e^{2R(\cos\varphi_p)} \tag{3.3}$$

can be evaluated in the limit $\tilde{g}^2 \to 0$:
The result is

$$I_\ell = \sqrt{\frac{2}{\pi}}\frac{\tilde{g}^2}{2K} e^{\left[\frac{8}{\tilde{g}^4}\sum_n d_n\right]} \left\{1 - \tilde{g}^4 \left\{\frac{3}{2K}\left[\frac{1}{3} + \frac{\ell}{2}\right] + 2 \cdot \frac{4}{(2K)^2} \cdot 15 \cdot \sum_n d_n \frac{n(3n-2)}{24}\right\} + 0(\tilde{g}^8)\right\} \tag{3.4}$$

The expectation values U and A can be evaluated:

$$U = \langle \tfrac{1}{4}\mathrm{Tr}U_{\partial P}\rangle_0 = I_1/I_0 = \left[1 - \tilde{g}^4 \frac{3}{4K} + 0(\tilde{g}^8)\right] \tag{3.5}$$

and

$$A = \langle \tfrac{1}{4}\mathrm{Tr}U_{\partial P}\rangle_0 = I_2/I_0 = \left[1 - 2\tilde{g}^4 \frac{3}{4K} + 0(\tilde{g}^8)\right] \tag{3.6}$$

How can we get from a strong coupling theory into the weak coupling region: all sizes should scale according to their physical dimensions.

Smaller $\tilde{g}^2$ demand larger sizes n on a lattice (in lattice units). Let

$$\nu = \tilde{g}^2 n \cdot \sqrt{\frac{3}{2K}} \tag{3.7}$$

fixed, so that

$$U^{n^2} = \left[1 - \frac{\nu^2}{2n^2} + \frac{0(\tilde{g}^4\nu^2)}{n^2}\right]^{n^2} = e^{-\nu^2/2}\, e^{0(\tilde{g}^4\nu^2)} \tag{3.8}$$

and

$$B_{n^2} = 1 + 3\left[1 - \frac{4}{3}\frac{\nu^2}{n^2} + \frac{0(\nu^2\tilde{g}^4)}{n^2}\right]^{n^2} = 1 + 3e^{-\frac{4}{3}\nu^2}\, e^{0(\tilde{g}^4\nu^2)} \tag{3.9}$$

The combination

$$U^{-2n^2}(4 - B_{n^2}) = e^{\nu^2}(3 - 3e^{-\frac{4}{3}\nu^2})\, e^{0(\tilde{g}^4)} \tag{3.10}$$

and

$$(U^{-2n^2}B_{n^2} - 4) = e^{\nu^2}(1 + 3e^{-\frac{4}{3}\nu^2})\, e^{0(\tilde{g}^4)} - 4 \tag{3.11}$$

occur in the matrix elements C_{mn} and D_{mn}.

We consider matrix elements depending on scaled coefficients (μ_1,μ_2), (ν_1,ν_2) and obtain to leading oder in $\tilde{g}^2$

$$\begin{aligned}
C(\mu_1,\mu_2,\nu_1,\nu_2) = {} & \left[\frac{2K}{3\tilde{g}^4}\right] e^{-\frac{1}{2}(\mu_1\mu_2+\nu_1\nu_2)} \cdot \\
& \Big\{\frac{3}{2}\Big[2\int_0^{\min(\mu_1,\nu_1)} dx \cdot x \left[e^{x\cdot\min(\mu_2,\nu_2)} - e^{-\frac{1}{3}x\cdot\min(\mu_2,\nu_2)}\right] \\
& + |\mu_1-\nu_1|\, \min(\mu_1,\nu_1) \cdot \left[e^{\min(\mu_1,\nu_1)\min(\mu_2,\nu_2)} - e^{-\frac{1}{3}\min(\mu_1,\nu_1)\cdot\min(\mu_2,\nu_2)}\right] \\
& + \frac{3}{2}\Big[2\int_0^{\min(\mu_2,\nu_2)} dy\; y \left[e^{\min(\mu_1,\nu_1)\cdot y} - e^{-\frac{1}{3}\min(\mu_1,\nu_1)\cdot y}\right] \\
& + |\mu_2-\nu_2|\, \min(\mu_2,\nu_2) \cdot \left[e^{\min(\mu_1,\nu_1)\min(\mu_2,\nu_2)} - e^{-\frac{1}{3}\min(\mu_1,\nu_1)\cdot\min(\mu_2,\nu_2)}\right]\Big]\Big\}
\end{aligned} \tag{3.12}$$

$$D(\mu_1,\mu_2)(\nu_1,\nu_2) = \left[\frac{2K}{3\tilde{g}^4}\right] e^{-\frac{1}{2}(\mu_1\mu_2+\nu_1\nu_2)}$$

$$\cdot\left\{ \int_0^{\min(\mu_1,\nu_1)} dx\,(2 + \delta(x-\min(\mu_1,\nu_1))\cdot|\mu_1-\nu_1|)\right.$$

$$\cdot \int_0^{\min(\mu_2,\nu_2)} dy\,(2 + \delta(y-\min(\mu_2,\nu_2))\cdot|\mu_2-\nu_2|)$$

$$\left.\cdot\,(e^{xy} + 3\,e^{-\frac{1}{3}xy} - 4)\right\} \tag{3.13}$$

The common constant factors have no influence on the value of λ or the coefficient function $f(\nu_1,\nu_2)$. So they may be dropped.

The coefficient function $f(\nu_1,\nu_2)$ are obtained from the functions $p(n_1,n_2)$ eq. (2.1) by scaling the arguements. The new variables ν_i are dimensionless, according to (3.7) they are combinations of the physical length $n\cdot a$ and the physical coupling $g^2 = \tilde{g}^2/a$. The scale factor $\sqrt{\frac{3}{2K}}$ is the only influence of the special choice of the initial Hamiltonian.

$$\nu_i = \sqrt{\frac{3}{2K}}\,(n_i\cdot a)\cdot(\tilde{g}^2/a) \tag{3.7'}$$

We can speak of a kind of universality. All Hamiltonians (1.6) with $K > 0$ (3.2) will show the same behaviour in the weak coupling region.

Now we can let go K to zero to establish the naive continuum limit. Arisue [10] obtains a wavefunction depending on scaled variables according eq. (3.7′) in the weak coupling limit even without taking care of conditions (1.10a) and (1.10b)

The new problem is to minimize

$$\lambda = \min_{f(\vec{\mu})} \frac{\int_0^\infty d\mu_1 \int_0^\infty d\mu_2 \int_0^\infty d\nu_1 \int_0^\infty d\nu_2\; C(\vec{\mu},\vec{\nu})\; f(\vec{\mu})\; f(\vec{\nu})}{\int_0^\infty d\mu_1 \int_0^\infty d\mu_2 \int_0^\infty d\nu_1 \int_0^\infty d\nu_2\; D(\vec{\mu},\vec{\nu})\; f(\vec{\mu})\; f(\vec{\nu})} \tag{3.14}$$

to get the mass gap λ.

We have

$$C(\vec{\mu},\vec{\nu}) = \frac{3}{4}\, e^{-(\mu_1\mu_2+\nu_1\nu_2)/2} \Big[2(|\mu_1-\nu_1|\ \epsilon_1+|\mu_2-\nu_2|\,\epsilon_2)\ I(\epsilon) + 4\Big[{}^{\epsilon}1/\epsilon_2 + {}^{\epsilon}2/\epsilon_1\Big]\ (H(\epsilon) - G(\epsilon))\Big] \tag{3.15}$$

and

$$D(\vec{\mu},\vec{\nu}) = e^{-(\mu_1\mu_2+\nu_1\nu_2)/2} \Big[|\mu_1-\nu_1|\,|\mu_2-\nu_2|\ H(\epsilon) + 2(|\mu_1-\nu_1|\ \epsilon_2+|\mu_2-\nu_2|\,\epsilon_1)\ G(\epsilon) + 4F(\epsilon)\Big] \tag{3.16}$$

with $\epsilon_1 = \min(\mu_1,\nu_1)$, $\epsilon_2 = \min(\mu_2,\nu_2)$ and (3.17a,b)

$$\epsilon^2 = \epsilon_1 \cdot \epsilon_2 \tag{3.17c}$$

Further

$$I(\epsilon) = e^{\epsilon^2} - e^{-\frac{1}{3}\epsilon^2} = \frac{d}{d(\epsilon^2)}\, H(\epsilon) \tag{3.18a}$$

$$H(\epsilon) = e^{\epsilon^2} + 3e^{-\frac{1}{3}\epsilon^2} - 4 = \frac{d}{d\epsilon^2}\,(\epsilon^2 G(\epsilon)) \tag{3.18b}$$

$$G(\epsilon) = \frac{1}{\epsilon^2}\Big[e^{\epsilon^2} - 9e^{-\frac{1}{3}\epsilon^2} + 8 - 4\epsilon^2\Big] = \frac{d}{d\epsilon^2}\, F(\epsilon) \tag{3.18c}$$

$$F(\epsilon) = E(\epsilon^2) - 9E\Big[-\frac{1}{3}\epsilon^2\Big] - 4\epsilon^2 \tag{3.18d}$$

with

$$E(x) = \int_0^x \frac{e^y-1}{y}\, dy = \sum_{n=1}^{\infty} \frac{1}{n}\frac{x^n}{n!} \tag{3.18e}$$

A variation over $f(\vec{\nu})$ gives

$$\int_0^\infty d\mu_1 \int_0^\infty d\mu_2\ f(\vec{\mu})\ (C(\vec{\mu},\vec{\nu}) - \lambda D(\vec{\mu},\vec{\nu})) = 0 \qquad \forall_{\vec{\nu}} \tag{3.19}$$

for the minimizing function $f(\vec{\mu})$ and the mass gap λ.

IV. Results

If one takes eq. (3.19) and looks at the region of large $\nu_i >> 1$ one can get information about the asymptotic leading orders of $f(\vec{\mu})$. The only well defined form for $f(\vec{\mu})$ preserving the existence of the integrals in (3.19) is

$$f(\mu_1,\mu_2) = e^{-\frac{1}{4}\mu_1^2-\frac{1}{4}\mu_2^2}(\mu_1+\mu_2)^{\lambda/6}\, e^{-\frac{1}{2}\left[1+\frac{\lambda}{6}\right]\left[\frac{\mu_1-\mu_2}{\mu_1+\mu_2}\right]^2}\left[1-\left[\frac{\mu_1-\mu_2}{\mu_1+\mu_2}\right]^2\right]^{\lambda/18}\left[1+0\left[\frac{1}{\mu^2}\right]\right] \tag{4.1}$$

Taylor–expansion of eq. (3.19) in ν_i gives a series of equations of integrals over $f(\vec{\mu})$.

Restricting to functions representing only quadratic Wilson–loop trial functions one can get more information about excited states. But one has to be aware that every restriction in the trial functions leads us more and more away from a true excited state. Denote

$$\tilde{C}(\mu,\nu) = C(\mu,\mu,\nu,\nu), \tag{4.2}$$

$\tilde{D}(\mu,\nu)$ correspondingly, and the solution of the restricted equation (3.19) by $\tilde{f}(\mu)$.

$$\tilde{C}(\mu,\nu) = \frac{3}{4}e^{-(\mu^2+\nu^2)/2}[4|\mu-\nu|\,\epsilon\, I(\epsilon) + 8(H(\epsilon) - G(\epsilon))] \tag{4.3}$$

and

$$\tilde{D}(\mu,\nu) = e^{-(\mu^2+\nu^2)/2}[(\mu-\nu)^2\, H(\epsilon) + 4|\mu-\nu|\,\epsilon\, G(\epsilon) + 4F(\epsilon)] \tag{4.4}$$

with

$$\epsilon = \min(\mu,\nu) \tag{4.5}$$

The equation for the minimizing function $\tilde{f}(\mu)$ is

$$\int_0^\infty d\mu\, \tilde{f}(\mu)\,(\tilde{C}(\mu,\nu) - \lambda\tilde{D}(\mu,\nu)) = 0 \tag{4.6}$$

for all values of ν.

Looking at the asymptotic leading order $e^{\mu^2/2}$ of eq. (4.6) with the ansatz

$$\tilde{f}(\mu) = e^{-\beta\mu^2} f_R(\mu) \tag{4.8}$$

and $f_R(\mu)$ polynomially bounded gives a value of

$$\beta = \sqrt{3}/2 \tag{4.8}$$

The next order gives $f_R(\mu) = \mu^\gamma\left[1 + 0\left[\frac{1}{\mu^2}\right]\right]$ with

$$\gamma = -\frac{2}{9}\beta\left[\lambda + \frac{27}{2}\right] + 1 \tag{4.9}$$

The value of β is slightly larger as the corresponding value of $f(\vec{\mu})$ in eq. (4.1) for $\mu_1 = \mu_2$ due to contributions from $\mu_1 \neq \mu_1$.

Taylor expansion of eq. (4.6) in ν gives

$$0(\nu^3): \qquad \int_0^\infty d\mu\, e^{-\mu^2/2}\, \mu\, \tilde{f}(\mu) = 0 \tag{4.10a}$$

$$0(\nu^4): \qquad \int_0^\infty d\mu\, e^{-\mu^2/2} \left[\frac{8}{3} + \frac{4}{3}\lambda\mu^2\right] \tilde{f}(\mu) = 0 \tag{4.10b}$$

Even orders in ν give conditions for a_{2k+1} in a Taylor–expansion of $\tilde{f}(\mu)$. They depend on λ and

$$I = \int_0^\infty d\mu\, e^{-\mu^2/2}\, \tilde{f}(\mu) \tag{4.11}$$

Vanishing of the odd orders in ν and eq. (4.6) implies that $\tilde{f}(\mu)$ does not possess even orders in μ. But $\tilde{f}(\mu)$ may have an expansion

$$\tilde{f}(\mu) = \left[\sum_{k=0}^{\infty} b_{2k}\, \mu^{2k+\alpha} + I \cdot \sum_{k=0}^{\infty} a_{2k+1}\, \mu^{2k+1}\right] e^{-\beta\mu^2} \tag{4.12}$$

where necessarily

$$\alpha = \sqrt{10} - 3 \tag{4.13}$$

Now orders $\nu^{5+2k+\alpha}$ of eq. (4.6) give conditions for b_{2k}, in which b_0 can be set equal to one.

The first 100 coefficients b_{2k} and a_{2k+1} have been evaluated for $\lambda = 4.54$. This choice is motivated by the results described later, and is consistent with the results of other authors [2–5,19]. Both series show a pole in μ^2 at

$$\mu_0^2 = (3.279 \pm 0.01)\, e^{\pm i(0.52760 \pm 0.001)}$$

which limits the radius of convergence of the series. The coefficients b_{2k} and a_{2k+1} depend on λ. I is only defined implicitely by eq. (4.11) and (4.12). There is a more direct way to get information about λ and $f(\mu)$:
Again we make an ansatz for $f(\mu)$ using the information we got above

$$\tilde{f}(\mu) = \sum_{n=0}^{N} a_n f_n(\mu) \tag{4.14a}$$

with

$$f_n(\mu) = \mu^{n+\alpha}\, e^{-\beta\mu^2} \qquad n \text{ even} \tag{4.14b}$$

$$= \mu^{n}\, e^{-\beta\mu^2} \qquad n \text{ odd} \tag{4.14c}$$

(N+1) is the number of basic functions we need.

If we evaluate numerically the integrals

$$C_{mn} = \int_0^\infty d\mu \int_0^\infty d\nu\, \tilde{C}(\mu,\nu)\, \tilde{f}_m(\mu)\, \tilde{f}_n(\nu) \tag{4.15}$$

and

$$D_{mn} = \int_0^\infty d\mu \int_0^\infty d\nu\, \tilde{D}(\mu,\nu)\, \tilde{f}_m(\mu)\, \tilde{f}_n(\nu) \tag{4.16}$$

we have a condition

$$\det \| C_{mn} - \lambda\, D_{mn} \| = 0 \tag{4.17}$$

according eq. (2.8). The zeros λ_i of eq. (4.17) do not only give the mass gap, but also the higher energies in that sector. A precision of 0.01 for λ_0 is reached at $N = 4$, so we have to deal only with small matrices. Knowing λ_i it is possible to compute a_n with the help of eq. (4.6), which reads now

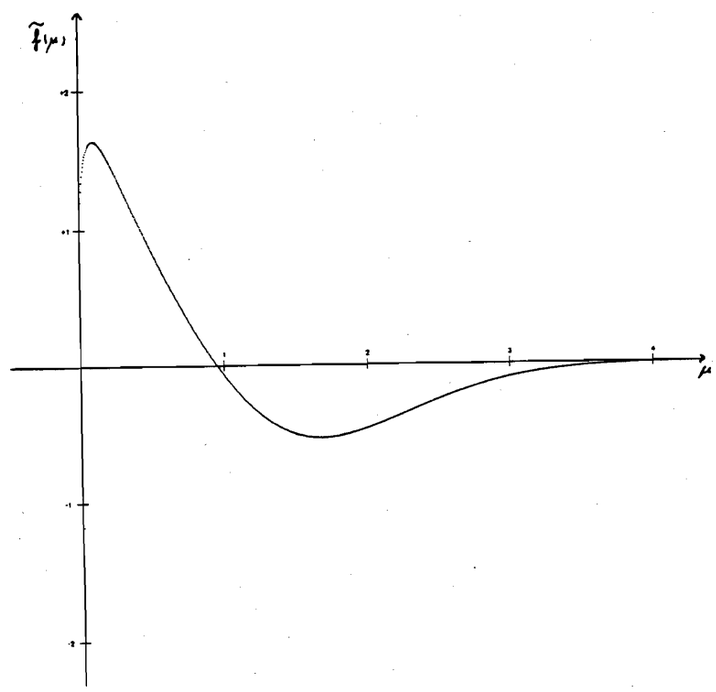
$$(C_{mn} - \lambda_i D_{mn})\, a_n = 0 \qquad (4.6')$$

At N = 10 we have a value of $\lambda_0 = 4{,}537637 \pm 2{,}5 \cdot 10^{-4}$.

The precision is determined by the numerical computation of the eq. (4.15), (4.16). The change from N = 9 to N = 10 had only lowered λ_0 by $2 \cdot 10^{-6}$. The (N = 10)–value for the next eigenvalue is $\lambda_1 = 8{,}6035 \pm 8 \cdot 10^{-4}$. The precision of the numerical computation and the improvement obtained going from N = 9 to N = 10 are of the same magnitude.

Evaluating eq. (4.6′) for λ_0 gives the following picture for the function $\tilde{f}(\mu)$ corresponding to a kind of wave–function for the first excited state.

Fig. 1:
Glueball wave function $\tilde{f}(\mu)$ depending on a scaled variable μ.
The scale of the y–axis is arbitrary.



Conclusion

We gained a weak coupling formulation of a SU(2) LGT in 2+1 dimensions. Starting with a series of Hamiltonians possessing an exactly known ground state we used a variational ansatz to get information about excited states [9]. Although a direct naive continuum limit of the Hamiltonian is impossible we could use the matrix elements of the variational ansatz to get a leading order formulation in $\tilde{g}^2$ after expressing all dimensionful sizes by their scaled counterparts. After restricting to test functions representing quadratic Wilson loops the eigenvalue equation (4.6) could be solved numerically. The minimizing function represents a translation invariant distribution of Wilson loops. Their lengths in lattice units would diverge as $\tilde{g} \to 0$ and so would exceed every finite lattice size L. This reflects the difficulties other authors have had working on a finite lattice.

The value for the mass gap $\lambda = 4.5376$ has to be understood as an upper bound to a real mass gap. Expressed by $\tilde{g}^2$ and a the value is

$$m = \lambda \cdot \frac{\tilde{g}^2}{2a} = 2.2688 \frac{\tilde{g}^2}{a} \tag{5.1}$$

This is consistent with the weak coupling limits of other authors. Arisue [10] has gained a lower value for the mass gap $m = (2.056 \pm 0.001)\, g^2$ working with rectangular Wilson–loops. Work on string tension of a static $q\bar{q}$–pair is in progress.

The same scaling procedure could be made for a U(1) LGT in 2+1 dimensions. In the O^{+-}–sector of such a theory (trial functions $\phi_n(x) = \mathrm{Tr}(U_{n\times n}(x) - U^{+}_{n\times n}(x))$) there are contributions to the functions $C(\mu,\nu)$ by configurations touching each other from outside. In addition is the function $I(0) \neq 0$. The values of $C(\mu,\nu)$ are lowered decisively so that the corresponding mass gap vanishes caused by Wilson loops that are small in the scaled variables $\nu = g^2 n$.

This is consistent with the result of [11] which claimes that the mass gap vanishes as $\tilde{g} \to 0$.

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