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Geometric interpretation of electromagnetism in a gravitational theory with space-time torsion

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Abstract

General relativity is extended to include electromagnetism by considering two aspects of torsion: its relation to spin established in Einstein–Cartan theory and the possible interpretation of the torsion trace as the electromagnetic potential. Starting with a Lagrangian built of Dirac spinors, orthonormal tetrads, and a complex rather than a real linear connection we define an extended spinor derivative by which we obtain not only a natural geometrization of electromagnetism, but can also clarify the underlying fibre bundle structure. Thereby the relation between torsion and electromagnetism turns out to be only a formal remnant of this geometrical background. Also, a new type of contact interaction between spinors emerges, which differs from the usual one in Einstein–Cartan theory. The splitting of the linear connection into a metric and an electromagnetic part together with a characteristic length scale in the theory suggest that gravity and electromagnetism can be embedded into a common geometric framework.

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I. INTRODUCTION

In general relativity the metric $g_{\mu\nu}$ completely determines the linear connection $\Gamma^{\alpha}{}_{\mu\beta}$, which becomes simply the Levi–Civita connection

$$\Gamma^{\alpha}{}_{\mu\beta} = \{^{\alpha}{}_{\mu\beta}\} := \frac{1}{2} g^{\alpha\epsilon} \left(\partial_\mu g_{\epsilon\beta} + \partial_\beta g_{\epsilon\mu} - \partial_\epsilon g_{\mu\beta} \right) . \tag{1}$$

The space-time geometry is influenced only by mass-energy, which causes curvature via the Einstein equation, and remains unaffected by spin.

In Einstein–Cartan theory[1] the connection is only required to be metric, $\nabla_\alpha g_{\mu\nu} = 0$, and is allowed to have non-vanishing torsion $T^{\alpha}{}_{\mu\beta} = \Gamma^{\alpha}{}_{\mu\beta} - \Gamma^{\alpha}{}_{\beta\mu}$ contrary to the torsion-less Levi–Civita connection. The structure of the connection now becomes[1]

$$\Gamma^{\alpha}{}_{\mu\beta} = \{^{\alpha}{}_{\mu\beta}\} + \frac{1}{2}(T_\mu{}^{\alpha}{}_{\beta} + T_\beta{}^{\alpha}{}_{\mu} + T^{\alpha}{}_{\mu\beta}) . \tag{2}$$

This generalization enables the space-time geometry to respond not only to mass but also to spin, where spinning matter produces torsion. For Dirac particles the torsion is totally antisymmetric in its indices and creates a cubic self-interaction term in the spinor equation[2]

$$i\gamma^\mu \nabla^*_\mu \psi - \frac{mc}{\hbar}\psi + \frac{3}{8} l_0{}^2 (\bar{\psi}\gamma^5\gamma^\delta\psi)\gamma^5\gamma_\delta\psi = 0 , \tag{3}$$

where ∇^*_μ is the covariant spinor derivative with respect to the Levi–Civita connection, see (22), and l_0 is the Planck length.

Besides this well-known aspect of torsion another physical role for it has been suggested in several works on the unification of gravitation and electromagnetism. The idea of such a geometrical unification is to omit any restrictions on the real linear connection $\Gamma^{\alpha}{}_{\mu\beta}$ and to incorporate the electromagnetic phenomena into this extended space-time geometry. More precisely, the electromagnetic vector potential A_μ is identified with the torsion trace $T_\mu = T^{\alpha}{}_{\mu\alpha}$. In the so-called non-symmetric unified field theory,[3] Einstein has considered a general linear connection, but his aim was to incorporate electromagnetism into the metric. He introduced a non-symmetric metric $\tilde{g}_{\mu\nu} (\neq \tilde{g}_{\nu\mu})$ and identified its antisymmetric part with the dual of the electromagnetic field strength. His theory was unsuccessful because it could

not account for the equation of motion.[4] To remedy this and various other shortcomings of Einstein's theory several authors have suggested to make the above mentioned identification $T_\mu \sim A_\mu$ in an ad hoc manner,[5] still using a non-symmetric metric. In subsequent works of McKellar[6] and Jakubiec and Kijowski[7] this is achieved without ad hoc assumptions, using the usual symmetric metric. McKellar starts with the metric $g_{\mu\nu}$ and a general linear connection $\Gamma^{\alpha}{}_{\mu\beta}$ and obtains

$$\Gamma^{\alpha}{}_{\mu\beta} = \{^{\alpha}{}_{\mu\beta}\} + \frac{1}{3}\delta^{\alpha}{}_{\beta} \cdot T_\mu \tag{4}$$

as the solution of the field equation, with the convention $\nabla_\mu X^\alpha = \partial_\mu X^\alpha + \Gamma^{\alpha}{}_{\mu\beta} X^\beta$ for the covariant derivative of a vector field X^α. His field equations taken together resemble the source-free Einstein–Maxwell equations, provided that $T_\mu \sim A_\mu$ holds. Ferraris and Kijowski[8] start not with a metric but with $\Gamma^{\alpha}{}_{\mu\beta}$ alone and arrive at (4) but, contrary to McKellar, they regard $\Gamma^{\alpha}{}_{\mu\alpha}$, which is not a vector, as the electromagnetic potential and deduce a theory of electromagnetism differing from the usual U(1) gauge theory, whereas Jakubiec and Kijowski[7] return to the identity $T_\mu \sim A_\mu$ and include Dirac spinors in the unification. Unfortunately, the employed spinor derivative requires two connections $\Gamma^{\alpha}{}_{\mu\beta}$ and $\{^{\alpha}{}_{\mu\beta}\}$ from the outset, and furthermore, from the general connection only its trace $\Gamma^{\alpha}{}_{\mu\alpha}$ appears in this derivative. Thus torsion does not sufficiently couple to spin and so its important physical role established in Einstein–Cartan theory is missing. In both theories[6,7] the fibre bundle structure of the unification is not clarified, so that T_μ in (4) cannot be gauged with U(1), as there is no U(1) bundle constructed in the theory. It is thus only a vector but not a potential.

From the discussions of general relativity, Einstein–Cartan theory, and the unified field theories we conclude that a more general linear connection would enable the space-time geometry to incorporate further physical phenomena in addition to gravitation. However, although the GL(4, $\mathbb{R}$) connection in the unified field theories[6–8] is more general than the metric connection of Einstein–Cartan theory the spin-torsion coupling is missing either because spinning matter was not considered[6,8] or because the spinor derivative was somewhat

inappropriate.[7]

In this work our aim is to obtain a new theory of gravity and electromagnetism including the spin-torsion coupling and thereby accounting for both aspects of torsion mentioned before. To achieve this we further expand the space-time geometry and allow for complex linear connections. Let us explain why this complex extension is necessary. Obviously, we must introduce a new spinor derivative containing a spin-torsion coupling besides an electromagnetic part. As a consequence, we may imagine that matter will be coupled to the connection more tightly and can therefore twist the space-time geometry so strongly that even complex degrees of freedom are excited. Another reason for the complex extension is the fact that no U(1) bundle structure could be constructed for the torsion trace T_μ of a real connection.[6,7]

Using a complex linear connection and an extended spinor derivative we arrive at a geometrical theory of gravity and electromagnetism which fully clarifies the fibre bundle structure, especially the U(1) bundle. All field equations follow from the variational principle. Due to the special spinor derivative both aspects of torsion in (3) and (4) must be revised: First, the relation $T_\mu \sim A_\mu$ turns out to be only a formal consequence of some another geometric relation, see (30). Also, the real part of torsion is not related to electromagnetism. Secondly, the spin-spin contact interaction is now found to occur only between distinct particles, thus excluding a self-interaction like in (3).

In Sec. II we establish notation and introduce the Lagrangian density. In Sections III and IV the field equations are derived using the variational method and their physical content is discussed. Here we slightly expand the theory to include differently charged particles and also consider many-particle systems to observe the new type of spin-spin contact interaction. In Section V a short summary is given. The fibre bundle geometry underlying the theory is discussed in Appendix A.

II. LAGRANGIAN DENSITY

The basic dynamical field variables of our theory are a complex tetrad $\sigma = (e_a{}^\mu \partial_\mu)$, a complex linear connection $\Gamma^a{}_{\mu b}\, dx^\mu$, and a Dirac spinor ψ on a real 4–dimensional space-time manifold M. Greek indices run from 0 to 3 and refer to local coordinates. Latin indices also run from 0 to 3 and refer to the internal Minkowski spacetime equipped with the Minkowski metric $\eta = (\eta_{ab}) = (\eta^{ab}) = \mathrm{diag}(1,-1,-1,-1)$. The (complex-valued) spacetime metric $g_{\mu\nu}$ is a dynamical variable obtained from the co-tetrads $(e^a{}_\mu dx^\mu)$ by $g_{\mu\nu} = e^a{}_\mu e^b{}_\nu \eta_{ab}$. Let $g := \sqrt{-\det(g_{\mu\nu})} = \det(e^a{}_\mu)$. Although the tetrad and the metric are complex variables, it turns out later on that they may be taken real-valued. The metric $g_{\mu\nu}$ determines the (complex) Lorentz bundle $L(M)$ consisting of orthonormal vierbeine, the structure group being given by the special Lorentz group

$$L := \left\{ \Lambda \in \mathrm{Mat}(4,\mathbb{C}) \,|\, \Lambda^T \eta \Lambda = \eta \,,\ \det \Lambda = 1 \right\} . \tag{5}$$

A tetrad $\sigma = (e_a{}^\mu \partial_\mu)$ is then a local cross section in $L(M)$.

A complex linear connection ω is a $\boldsymbol{gl}(4,\mathbb{C})$-valued connection 1-form on the complex frame bundle, denoted by $F(M)$ which is a $\mathrm{GL}(4,\mathbb{C})$ principal bundle. Since $L(M)$ is a subbundle contained in $F(M)$, the tetrad σ is naturally a cross section in $F(M)$. We pull back ω by the tetrad σ and obtain the complex linear connection in anholonomic or tetrad components

$$(\sigma^* \omega)^a{}_b =: \Gamma^a{}_{\mu b}\, dx^\mu \tag{6}$$

as our second dynamical variable. This connection $\Gamma^a{}_{\mu b}$ is neither torsion-less nor metric-compatible and is a true 1-form,transforming homogeneously under coordinate changes.

Note that ω is a general $\mathrm{GL}(4,\mathbb{C})$ connection 1-form, whereas the cross sections employed to pull back ω from $F(M)$ to M are orthonormal tetrads σ only. Since gauge transformations are generated by the changes of various cross sections and affect the pulled back connection on M but not its basic connection 1-form on the principal bundle, the gauge group considered

in our theory is only the Lorentz group L but not the whole linear group $\mathrm{GL}(4,\mathbb{C})$. (In Appendix A we shall enlarge the gauge group to $L \times \mathrm{U}(1)$.) The restriction of the gauge group to L is necessary to introduce spinors in our theory.

The coordinate components of the connection $\Gamma^a{}_{\mu b}$ are obtained by the well-known rule

$$\Gamma^\alpha{}_{\mu\beta} = e_a{}^\alpha\, e^b{}_\beta \Gamma^a{}_{\mu b} + e_c{}^\alpha \partial_\mu e^c{}_\beta \,. \tag{7}$$

The curvature tensor, Ricci tensor, curvature scalar, and the curvature trace of $\Gamma^a{}_{\mu b}$ are defined as follows

$$R^a{}_{b\mu\nu} = \partial_\mu \Gamma^a{}_{\nu b} + \Gamma^a{}_{\mu c}\Gamma^c{}_{\nu b} - \partial_\nu \Gamma^a{}_{\mu b} - \Gamma^a{}_{\nu c}\Gamma^c{}_{\mu b} \,, \tag{8a}$$

$$R_{\mu\nu} = R^a{}_{b\alpha\nu} \cdot e_a{}^\alpha\, e^b{}_\mu \,, \tag{8b}$$

$$R = R^a{}_{b\mu\nu} \cdot e_a{}^\mu\, e^{b\nu} \,, \tag{8c}$$

$$Y_{\mu\nu} = R^a{}_{a\mu\nu} = \partial_\mu \Gamma^a{}_{\nu a} - \partial_\nu \Gamma^a{}_{\mu a} \,. \tag{8d}$$

Note that $\Gamma^a{}_{\nu a}$ is a vector, contrary to $\Gamma^\alpha{}_{\nu\alpha}$. This vector vanishes only in case of metric connections because of the Lie algebra condition $\Gamma_{a\mu b} + \Gamma_{b\mu a} = 0$ which is equivalent to the metricity condition $\nabla_\alpha g_{\mu\nu} = 0$. In general however, $\Gamma^a{}_{\mu a} \neq 0$.

Every metric connection $\overset{m}{\Gamma}{}^a{}_{\mu b}$ defines a covariant differentiation of a Dirac spinor[2,9–12]

$$\overset{m}{\nabla}_\mu \psi = \partial_\mu \psi - \frac{1}{4}\, \overset{m}{\Gamma}_{a\mu b}\, \gamma^b \gamma^a \psi \,. \tag{9}$$

The γ–matrices satisfy $\gamma^a\gamma^b + \gamma^b\gamma^a = 2\eta^{ab}\mathbb{1}$. In virtue of metricity $\overset{m}{\Gamma}_{a\mu b} = -\overset{m}{\Gamma}_{b\mu a}$ the commonly used Lorentz generators $\sigma^{ba} = \frac{1}{2}(\gamma^b\gamma^a - \gamma^a\gamma^b)$ have been replaced by $\gamma^b\gamma^a$.[2,10] This replacement is not unique, and the remaining ambiguity can be usefully exploited, see Sec. IV.

Let us now extend (9) to the case of our arbitrary linear connection $\Gamma^a{}_{\mu b}$

$$\nabla_\mu \psi = \partial_\mu \psi - \frac{1}{4}\Gamma_{a\mu b}\gamma^b\gamma^a\psi \,. \tag{10}$$

Although $\mathrm{GL}(4,\mathbb{C})$ has no spinorial representation, it is nevertheless possible to construct a spin connection ω_s (A9) and define the extended covariant spinor derivative (10) from any

complex linear connection 1-form ω by means of pull-back techniques, provided the gauge group of the tetrad σ is restricted to the Lorentz group L as we have done already, see Appendix A. Thus the gauge group of (10) is still L and the spinor ψ transforms under the well-known spin-$\frac{1}{2}$ representation of L.

Due to the lack of the metricity condition, eq. (10) can be rewritten as

$$\nabla_\mu \psi = \partial_\mu \psi - \frac{1}{4}\Gamma_{a\mu b}\sigma^{ba}\psi - \frac{1}{4}\Gamma^a{}_{\mu a}\psi \,, \tag{11}$$

which contains the trace part of the connection. The pull-back techniques employed in Appendix A will produce a U(1) gauge structure for this trace part $-\frac{1}{4}\Gamma^a{}_{\mu a}$. Indeed, it is this vector which, after deriving the field equations, becomes the electromagnetic potential. As a result, eq. (10) will become covariant with respect to $L \times \mathrm{U}(1)$.

Introducing $\bar\psi := \psi^\dagger\gamma^0$, $\gamma^\mu := \gamma^a e_a{}^\mu$, the mass of the spinor particle m, $k = 8\pi G/c^4$, and a length scale l we write down the following Lagrangian density

$$\begin{aligned} \mathcal{L} &= \mathcal{L}_m + \mathcal{L}_G + \mathcal{L}_Y \\ &=: g \cdot \hbar c \left[i\bar\psi\gamma^\mu\nabla_\mu\psi - \frac{mc}{\hbar}\bar\psi\psi \right] - \frac{g}{2k}R + \frac{g}{4k}l^2 Y_{\mu\nu}Y^{\mu\nu} \,. \end{aligned} \tag{12}$$

Although it is complex valued we do not make it real, because this would restrict the contributions of the full complex connection. For an interesting example of complex Lagrangian theory see Ashtekar's formulation of general relativity,[13] which might be related to the complex structures developed in this paper.[14] Apart from being complex the three parts $\mathcal{L}_m$, $\mathcal{L}_G$, and $\mathcal{L}_Y$ resemble more or less the usual Lagrangian densities of spinorial matter, gravity, and the electromagnetic field, respectively. Whereas expressions similar to $\mathcal{L}_G$ and $\mathcal{L}_Y$ for a real connection were already used,[5–7] the matter Lagrangian $\mathcal{L}_m$ including the extended spinor derivative (10) is new and plays a key role in our theory. We stress that the gauge group of $\mathcal{L}$ (12) is not the full $\mathrm{GL}(4,\mathbb{C})$ but only L. In Appendix A we shall extend the gauge group to $L \times \mathrm{U}(1)$.

A squared length l^2 was introduced in $\mathcal{L}_Y$ from purely dimensional arguments, since the partial derivatives ∂_μ and the connection have the dimension of inverse length. To compare l with the Planck length $l_0 := \sqrt{\hbar c k}$ we rewrite (12) as follows

$$\mathcal{L} = \frac{g}{k} \cdot \left[i{l_0}^2 \bar{\psi}\gamma^\mu \nabla_\mu \psi - mc^2 k \bar{\psi}\psi - \frac{1}{2}R + \frac{1}{4} l^2 Y_{\mu\nu} Y^{\mu\nu} \right] . \tag{13}$$

In the last term we recognize l^2 as the self-coupling constant of the connection, implying that l is an intrinsic length of the space-time geometry. The first term on the right-hand side reveals ${l_0}^2$ as the coupling constant between the connection and matter. But if we regard Dirac spinors ultimately as geometrical objects, then l_0 also is a characteristic unit of the space-time. We can therefore expect l and l_0 being of the same magnitude.

III. FIELD EQUATIONS

A. Field equation for the connection

In the following we give only an outline of the computations. Details can be found elsewhere.[15] For computational convenience we shall express the field equations in terms of the holonomic components of the connection ${\Gamma^\alpha}_{\mu\beta}$ (7). Define the following complex valued third rank tensor

$${\Sigma^\alpha}_{\mu\beta} := {\Gamma^\alpha}_{\mu\beta} - \{{}^\alpha{}_{\mu\beta}\} . \tag{14}$$

A third rank tensor always admits a "4-vector decomposition"

$$\Sigma_{\alpha\beta\gamma} = Q_\alpha g_{\beta\gamma} + S_\beta g_{\alpha\gamma} + g_{\alpha\beta} U_\gamma - \frac{1}{12}\eta_{\alpha\beta\gamma\delta} V^\delta + \Upsilon_{\alpha\beta\gamma} , \tag{15}$$

where the four vectors Q_α, S_β, U_γ, V_δ and the "tensor rest" $\Upsilon_{\alpha\beta\gamma}$ with the property ${\Upsilon^\alpha}_{\alpha\gamma} = {\Upsilon^\alpha}_{\gamma\alpha} = {\Upsilon_\gamma}^\alpha{}_\alpha = \Upsilon_{[\alpha\beta\gamma]} = 0$ are defined in Appendix B. We have denoted the volume element by $\eta_{\alpha\beta\gamma\delta} := g \cdot \epsilon_{\alpha\beta\gamma\delta}$, where $\epsilon_{\alpha\beta\gamma\delta}$ is totally antisymmetric and $\epsilon_{0123} = 1$. The decomposition (14) and (15) are introduced for computational convenience and differs from those proposed by McCrea.[16] The field equation for ${\Gamma^a}_{\mu b}$ follows from (12) and (14)

$$\begin{aligned} 0 &= \frac{\delta}{\delta {\Gamma^a}_{\mu b}} (\mathcal{L}_m + \mathcal{L}_G + \mathcal{L}_Y) \cdot {\delta^\gamma}_\mu e^{a\alpha} e_b{}^\beta \cdot \frac{k}{g} \qquad \Leftrightarrow \\ 0 &= -\frac{1}{4} i{l_0}^2 \bar{\psi}\gamma^\gamma\gamma^\beta\gamma^\alpha\psi - \frac{1}{2}\left[{\Sigma^{\beta\epsilon}}_\epsilon g^{\alpha\gamma} + {\Sigma^\epsilon}_\epsilon{}^\alpha g^{\gamma\beta} - \Sigma^{\beta\alpha\gamma} - \Sigma^{\gamma\beta\alpha} \right] - l^2 g^{\alpha\beta} \nabla^*_\nu Y^{\nu\gamma} , \end{aligned} \tag{16}$$

where ∇^*_μ is the covariant differentiation with respect to $\{{}^\alpha{}_{\mu\beta}\}$. We now use (15) and the definitions of the vector current $j^\alpha := \bar{\psi}\gamma^\alpha\psi$ and the axial current $j^{5\delta} := \bar{\psi}\gamma^5\gamma^\delta\psi$, where $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$ and $1/6\eta_{\gamma\beta\alpha\delta}\gamma^\gamma\gamma^\beta\gamma^\alpha = -i\gamma^5\gamma_\delta$. Contracting (16) successively with $g_{\beta\gamma}$, $g_{\alpha\gamma}$, $g_{\alpha\beta}$ and $1/6 \cdot \eta_{\gamma\beta\alpha\delta}$ we obtain respectively

$$0 = -i{l_0}^2 \cdot j^\alpha - 3Q^\alpha - 6U^\alpha - l^2 \nabla^*_\nu Y^{\nu\alpha} , \tag{17a}$$

$$0 = \frac{1}{2} i{l_0}^2 \cdot j^\beta - 6Q^\beta - 3U^\beta - l^2 \nabla^*_\nu Y^{\nu\beta} , \tag{17b}$$

$$0 = -i{l_0}^2 \cdot j^\gamma - 4l^2 \nabla^*_\nu Y^{\nu\gamma} , \tag{17c}$$

$$0 = -\frac{1}{4}{l_0}^2 \cdot {j^5}_\delta + \frac{1}{12} V_\delta . \tag{17d}$$

It follows $-Q^\alpha = U^\alpha = -i{l_0}^2/4 \cdot j^\alpha$. Inserting this and the last two equations back into (16) we obtain $\Upsilon_{\alpha\beta\gamma} = 0$. Since the Levi–Civita connection is metric, its anholonomic components satisfy $\{_{a\mu b}\} + \{_{b\mu a}\} = 0$. Therefore, ${\Gamma^a}_{\mu a} = {\Sigma^a}_{\mu a} = Q_\mu + 4S_\mu + U_\mu = 4S_\mu$ and thus $Y_{\mu\nu} = 4S_{\mu\nu} := 4(\partial_\mu S_\nu - \partial_\nu S_\mu)$. These results amount to

$${\Gamma^\alpha}_{\mu\beta} = {\hat{\Gamma}^\alpha}_{\mu\beta} + {\delta^\alpha}_\beta \cdot S_\mu , \tag{18}$$

where

$${\hat{\Gamma}^\alpha}_{\mu\beta} := \{{}^\alpha{}_{\mu\beta}\} + \frac{1}{4}{l_0}^2 \left(i \cdot j^\alpha g_{\mu\beta} - i \cdot {\delta^\alpha}_\mu j_\beta - {\eta^\alpha}_{\mu\beta\delta} j^{5\delta} \right) \tag{19}$$

and

$$16 i l^2 / {l_0}^2 \nabla^*_\nu S^{\nu\gamma} = j^\gamma . \tag{20}$$

The last equation implies the current conservation $\nabla^*_\gamma j^\gamma = 0$.

B. The Dirac equations

The Lagrangian (12) immediately yields $0 = \delta\mathcal{L}/\delta\bar{\psi} = \partial\mathcal{L}_m/\partial\bar{\psi}$ or, equivalently, $i\gamma^\mu\nabla_\mu\psi - mc/\hbar\,\psi = 0$. With (18) and (19) this can be written as

$$i\gamma^\mu(\nabla^*_\mu - S_\mu)\psi - \frac{mc}{\hbar}\psi + \frac{3}{8}{l_0}^2(j_\mu + {j^5}_\mu\gamma^5)\gamma^\mu\psi = 0 , \tag{21}$$

where

$$\nabla^*_\mu \psi := \partial_\mu \psi - \frac{1}{4}\{_{a\mu b}\}\gamma^b\gamma^a\psi \tag{22}$$

is the covariant spinor differentiation with respect to the Levi–Civita connection. The symbol ∇^*_μ is used for the covariant derivative with respect to $\{^\alpha{}_{\mu\beta}\}$ as well as for the corresponding spinor derivative. The spinor equation for $\bar\psi$ reads[15]

$$i(\nabla^*_\mu + S_\mu)\bar\psi\cdot\gamma^\mu + \frac{mc}{\hbar}\bar\psi - \frac{3}{8}l_0{}^2\bar\psi(j_\mu + j^5{}_\mu\gamma^5)\gamma^\mu = 0 \tag{23}$$

with $\nabla^*_\mu\bar\psi = \overline{\nabla^*_\mu\psi}$. The nonlinear terms in (21) and (23) vanish due to the identity

$$(j_\mu + j^5{}_\mu\gamma^5)\gamma^\mu\psi = 0\ , \tag{24}$$

which can be derived by straightforward computations, recalling $j_\mu = \bar\psi\gamma_\mu\psi$ and $j^5{}_\mu = \bar\psi\gamma^5\gamma_\mu\psi$.

C. The field equation for the tetrad

The Lagrangian (12) contains no derivatives of $e_a{}^\mu$ and therefore we get

$$\begin{aligned} 0 &= \frac{\delta\mathcal{L}}{\delta e_c{}^\alpha}e_{c\beta} = \frac{\partial\mathcal{L}}{\partial e_c{}^\alpha}e_{c\beta} \\ &= \left[-\mathcal{L}_m\, g_{\alpha\beta} + g\, i\hbar c\bar\psi\gamma_\alpha\nabla_\beta\psi\right] - \frac{1}{2k}\left[-gR\, g_{\alpha\beta} + gR_\alpha{}^\mu{}_{\beta\mu} + gR^\mu{}_{\alpha\mu\beta}\right] \\ &\quad + \frac{1}{4k}l^2\left[-gY_{\mu\nu}Y^{\mu\nu}g_{\alpha\beta} + 4gY_{\mu\alpha}Y^\mu{}_\beta\right]\ . \end{aligned} \tag{25}$$

Using (15), (18) to (20) and (21) to (24) this can be expressed as[15]

$$T^G_{\alpha\beta} = T^m_{\alpha\beta} + T^S_{\alpha\beta}\ , \tag{26}$$

where

$$T^G_{\alpha\beta} := \frac{1}{k}\left(R^*_{\alpha\beta} - \frac{1}{2}R^* g_{\alpha\beta}\right)\ , \tag{27a}$$

$$T^m_{\alpha\beta} := \frac{i\hbar c}{4}\left[\bar\psi\gamma_\alpha(\nabla^*_\beta - S_\beta)\psi - (\nabla^*_\beta + S_\beta)\bar\psi\cdot\gamma_\alpha\psi + (\alpha\leftrightarrow\beta)\right]\ , \tag{27b}$$

$$T^S_{\alpha\beta} := \frac{16}{k}l^2\left[S_{\alpha\gamma}S_\beta{}^\gamma - \frac{1}{4}S_{\mu\nu}S^{\mu\nu}g_{\alpha\beta}\right]\ . \tag{27c}$$

In (27a) $R^*_{\alpha\beta}$ and R^* denote the Ricci-tensor and -scalar of the Levi–Civita connection.

IV. PHYSICAL INTERPRETATION

Our theory can not be immersed into a Riemann–Cartan geometry but requires the full GL(4, $\mathbb{C}$) connection 1-form, as can be seen at the resulting connection (18), which is non-metric, $\nabla_\mu g_{\alpha\beta} = -2S_\mu\cdot g_{\alpha\beta} \neq 0$. In order to obtain a physically sensible (i. e. metric compatible) parallel transport on the space-time and to associate the non-metricity vector S_μ with electromagnetism the connection (18) must be divided into its metric and non-metric parts. As shown in Appendix A the connection (18) can be viewed as a sum of the complex metric connection $\widehat\Gamma^\alpha{}_{\mu\beta}$ on the Lorentz bundle $L(M)$ and the U(1) connection S_μ on the trivial U(1) bundle $M\times$ U(1), glued together by canonical bundle mappings. Therefore, contrary to other works,[5,6] we can indeed separate the whole connection (18) into the Lorentz connection (19) and the U(1) potential S_μ.

The metric connection $\widehat\Gamma^\alpha{}_{\mu\beta}$ (19) carries out the parallel transports of (uncharged) tensors on M. But it contains a complex contorsion $1/4l_0{}^2(i\cdot j^\alpha g_{\mu\beta} - i\cdot\delta^\alpha{}_\mu j_\beta - \eta^\alpha{}_{\mu\beta\delta}j^{5\delta})$ caused by Dirac particles. Since the contorsion is a tensor, it does not vanish even in a local inertial system, which is defined to be the special coordinate system around a space-time point $p\in M$ with $g_{\mu\nu}(p) = \mathrm{diag}(1,-1,-1,-1)$ and $\partial_\sigma g_{\mu\nu}(p) = 0$. Although the physical meaning of this phenomenon is not yet clear, we remark that it is invisible to the laboratory experiments of today due to the very small magnitude of $l_0{}^2$ in the contorsion. Furthermore, $\widehat\Gamma^\alpha{}_{\mu\beta}$ is a metric connection, so that its parallel transport preserves the metric. This guarantees the invariance and conserves the real nature of physical measurements of lengths, time intervals, rest masses and various scalar products of particle momenta.

We now discuss the non-metric part S_μ. According to Appendix A, its U(1) gauge transformation is given by

$$e_a{}^\mu \mapsto e_a{}^\mu\ ,\quad S_\mu(=\frac{1}{4}\Gamma^a{}_{\mu a}) \mapsto S_\mu + \partial_\mu\lambda\ ,\quad \psi\mapsto\exp(\lambda)\psi\ . \tag{28}$$

The field equations (20), (21), (23), and (26),

$$16il^2/l_0{}^2\,\nabla^*_\nu S^{\nu\gamma} = j^\gamma\ , \tag{29a}$$

$$i\gamma^\mu(\nabla^*_\mu - S_\mu)\psi - \frac{mc}{\hbar}\psi = 0\,, \tag{29b}$$

$$i(\nabla^*_\mu + S_\mu)\bar{\psi}\cdot\gamma^\mu + \frac{mc}{\hbar}\bar{\psi} = 0\,, \tag{29c}$$

$$T^G_{\alpha\beta} = T^m_{\alpha\beta} + T^S_{\alpha\beta}\,, \tag{29d}$$

exhibit precisely the well-known structures of the Einstein–Maxwell theory, provided that S_μ is identified with the electromagnetic potential A_μ

$$S_\mu = \frac{ie}{\hbar c}A_\mu\,, \tag{30}$$

where e is the (positive) elementary charge. In this case, (28) describes the electromagnetic U(1) gauge transformation of a negatively charged spinor, which we identify with electron. Then, (29b) is the corresponding charged spinor equation in a curved space-time, (29c) being its adjoint. Note that $\nabla^*_\mu - S_\mu$ in (29b) is the U(1) gauge covariant spinor derivative. Eq. (29d) gives the energy-momentum equation involving the energy-momentum tensors of gravity (27a), charged spinor particle (27b), and the electromagnetic field (27c). We remark that the complex nature of the tetrads and the metric have never been used explicitly. Thus, we may restrict the whole theory to real tetrads. However, such a real restriction is not possible for the connection components $\Gamma^a{}_{\mu a}$, as the field equations (19) and (20) obviously render the connection complex-valued.

In order to fix the length scale l we insert (30) into the Maxwell equation (29a) and compare it with the usual expression

$$\begin{aligned} j^\gamma = 16il^2/l_0{}^2\,\frac{ie}{\hbar c}\,\nabla^*_\nu F^{\nu\gamma} &\stackrel{!}{=} \frac{1}{-e}\,\nabla^*_\nu F^{\nu\gamma} \Leftrightarrow \\ l^2 = \frac{1}{64\pi}l_0{}^2\frac{\hbar c}{e^2/4\pi} &= \frac{1}{64\pi\,\alpha}\,l_0{}^2 \Rightarrow l \approx 0.83\,l_0\,, \end{aligned} \tag{31}$$

where α is the fine structure constant. As expected in Sec. II the value of l is of the same magnitude as the Planck length, which indicates the close relation of electromagnetism to space-time geometry and to gravity. If we had taken $l := l_0$ in (12), we would have

obtained $\alpha = 1/64\pi$ and $e \approx 1.32\cdot 10^{-19}$ Coulomb. In a quantum field theoretic approach renormalization procedures could perhaps improve (31) towards $l = l_0$. When (30) and (31) are taken into account one can easily show that (29) contains exactly the equations of Einstein–Maxwell theory with an electron.

Note that the ansatz $T_\mu \sim A_\mu$ now must be changed due to (18), (19), and (30) to

$$T_\mu = \Gamma^a{}_{\mu a} - \Gamma^a{}_{a\mu} = 3S_\mu - 3U_\mu = 3\frac{ie}{\hbar c}A_\mu + \frac{3}{4}il_0{}^2\,j^\mu\,. \tag{32}$$

Moreover, we emphasize that (32) is only a formal equation, since T_μ remains invariant under U(1), while A_μ does not. In fact, A_μ is related via (30) to S_μ, which has originated from a complex linear connection through the field equations and should be understood as a U(1) connection on $M\times U(1)$ in its own right, see Appendix A: Mathematically, the linear connection 1-form on $F(M)$ has been pulled back to the bundle $(L\times\mathrm{U}(1))(M)$. With respect to this product bundle S_μ can be viewed as the pull-back of a U(1) gauge potential on the subbundle $M\times\mathrm{U}(1)$ via the trivial cross section $\hat{1}$, $S_\mu = \frac{1}{4}\Gamma^c{}_{\mu c} = \hat{1}^*\omega_c$ (A13). Therefore, S_μ can be gauged with U(1) by passing from $\hat{1}$ to another cross section $\exp(\lambda)\hat{1}$ (A18). In (32) however, the torsion trace T_μ, which is defined only on the complex frame bundle $F(M)$, has been pulled back via a coordinate reference frame $(\partial/\partial x^\mu)$ and therefore can not be gauged with U(1). In order to relate T_μ to A_μ via (32), the special cross section $\hat{1}$ must be held fixed. Since (32) is valid only in this U(1) gauge, it is rather a formal consequence of the true geometric equation (30) and has only a limited validity. We observe that in (32) the real part of torsion is not involved. Note that since the gauge transformation of S_μ takes place on $M\times$ U(1) only, the metric part (19) of the connection and also the tetrads on $L(M)$ remain unaffected. Especially, the metric $g_{\mu\nu}$ remains unchanged. Therefore, the gauge transformation (28) is by no means related the so-called conformal transformations of the metric.

Despite the before mentioned agreements of our theory with the Einstein–Maxwell theory our interpretation of electromagnetism is geometric in that we regard S_μ as the true electromagnetic vector potential rather than A_μ, and describe the electromagnetic interaction

through the field equations (29) together with the definite l (31) only, thereby completely disregarding (30). This geometrical viewpoint respects the way the U(1) potential S_μ together with the "gravitational" metric part (19) originated from a single space-time connection. Note that the occurrence of the trivial group structure $L \times \mathrm{U}(1)$ above (and, more explicitly, in Appendix A) accounting for both gravity and electromagnetism is in accord with the well-known theorem that, algebraically, it is impossible to combine space-time and internal symmetry in any but a trivial way.[17] But the aim of our theory is the understanding, how gravity and electromagnetism emerge geometrically from the space-time structure using an extended spinor derivative.

So far we have considered only an electron. In order to include other, differently charged particles we observe that in case of a metric connection with $\Gamma_{a\mu b} = -\Gamma_{b\mu a}$ the spinor derivative (10) can be written as

$$\nabla_\mu \psi = \partial_\mu \psi - \frac{1}{4}\Gamma_{a\mu b}\sigma^{ba}\psi + \frac{\varepsilon}{4}\Gamma^a{}_{\mu a}\psi \,, \tag{33}$$

where $\varepsilon \in \mathbb{R}$. Inserting back our complex connection, this spinor derivative corresponds to the charge εe.

To discover the spin-spin contact interaction missing so far we briefly discuss a many-particle system consisting of spinors ψ_z having charges $\varepsilon(z)e$, $\varepsilon(z) \in \mathbb{R}$, and masses m_z, where z is an index. In (12) only the matter Lagrangian $\mathcal{L}_m$ changes. It becomes now a sum of Lagrangians for each spinor ψ_z, its spinor derivative given by (33) with $\varepsilon = \varepsilon(z)$. The field equations for this system can be handled in the same fashion as in Sec. III.[15] Among other changes, the field equation for energy-momentum (26) gain a new spin-spin interaction term $W_{\alpha\beta}$ on its right-hand side,

$$W_{\alpha\beta} = \frac{3}{8k} l_0{}^4 \sum_{z \neq z'} \left(j_{z\,\mu}\, j_{z'}^{\mu} + j_{z\mu}^{5} j_{z'}^{5\mu} \right) g_{\alpha\beta} \,, \tag{34}$$

where $j_z^\mu = \bar{\psi}_z \gamma^\mu \psi_z$ and $j_z^{5\,\mu} = \bar{\psi}_z \gamma^5 \gamma^\mu \psi_z$. Corresponding terms appear in the Dirac equations

$$i\gamma^\mu(\nabla_\mu^* - \varepsilon(z) S_\mu)\psi_z - \frac{m_z c}{\hbar}\psi_z + \frac{3}{8} l_0{}^2 \sum_{z' \neq z} \left(j_{z'}^{\mu} + j_{z'}^{5\mu}\gamma^5 \right) \gamma_\mu \psi_z = 0 \,. \tag{35}$$

Contrary to the Einstein–Cartan theory (3) the contact interaction occurs here only between distinct particles. This and the vanishing of cubic terms in (21) and (23) are due to (24) and have their origin in our special choice of $\mathcal{L}_m$ in (12), where the adjoint covariant derivative of $\bar{\psi}$ is missing. Usually, the matter Lagrangian is required to be real, necessitating the inclusion of both derivatives of ψ and $\bar{\psi}$.[2] Since in (12) the Lagrangians $\mathcal{L}_G$ and $\mathcal{L}_Y$ were already complex, there was no need to make $\mathcal{L}_m$ alone real valued by including the adjoint spinor derivative. In our opinion, self-interactions of Dirac particles like in (3) should be avoided because of the Fermi–Dirac spin-statistics they obey. We remark that the spin-spin interaction is far too weak to be observed by laboratory experiments, and its influence on cosmological phenomena remains to be studied.[18] These experimental limitations are true also for propagating torsion theories, see Ref. 19. On the other hand, such contact interaction may be regarded as a low-energy (below the Planck scale) approximation to the fundamental dynamics of the Einstein–Cartan theory.[20]

V. CONCLUSIONS

We have employed a complex linear connection and an extended spinor derivative to embed the gravitational and electromagnetic interactions into the space-time geometry. Contrary to earlier works at unification the fibre geometric background could be clarified (Appendix A). Especially, the long standing relation $T_\mu \sim A_\mu$ were shown to be only a formal remnant of the geometric relation (30). Also, the real part of torsion is not related to electromagnetism even formally. Further, it follows from our work that electromagnetism can not be immersed into a Riemann–Cartan geometry contrary to the work by Hammond.[21]

The field equations are derived from a variational principle and exhibit precisely the structures of the Einstein–Maxwell theory with a negatively charged spinor particle. However, the many-particle system revealed a new type of spin-spin contact interaction without self-interaction, which was necessarily absent in the single-particle case. Furthermore, our theory differs from the usual Einstein–Maxwell theory in the geometrical understanding of

electromagnetism and its physical consequences. From the fibre bundle structure of our theory it follows that a metric connection and an electromagnetic vector potential emerge from the single complex linear connection by pull-back techniques. Accordingly, we interpret electromagnetism purely geometrically and use as the only physical constant a characteristic length l close to the Planck length. This geometrization scheme involves the "melting" of the charge q and A_μ into one single expression. Using a further extension of the covariant spinor derivative spinors with arbitrary charges εe could be treated in the theory. Finally, the parallel transports of uncharged tensors on space-time are carried out by the resultant metric connection mentioned before, which contains a complex contorsion. Physical measurements remain unaffected by this contribution because the metric is preserved under the parallel displacements.

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APPENDIX A: BUNDLE GEOMETRY

We establish the fibre bundle structure underlying our theory and especially introduce the spin structure[15,10,11] essential to define the extended spinor derivative (10). We then explain the pull-back techniques used to obtain a special spin connection 1-form ω_s from any given linear connection 1-form ω. Considering local cross sections we show that ω_s indeed results in (10). We stress that the gauge group of the spinor ψ is of course not the full $\mathrm{GL}(4,\mathbb{C})$ but only the product $L \times \mathrm{U}(1)$.

In the following Lie group homomorphisms and their Lie algebra homomorphisms will be denoted by the same letter.

1. Fibre Bundle Structure

Since our goal is to construct a spin connection ω_s from a linear connection ω by pull-backs, we must establish a diagram of bundle mappings between a certain spin bundle and the frame bundle $F(M)$ in order to obtain the necessary pull-back mappings. To introduce such a bundle diagram, we first concentrate only on the various structure groups involved and consider the following diagram of Lie groups to be translated later into the level of fibre bundles, see (A4):

$$\mathrm{GL}(4,\mathbb{C}) \xleftarrow{\zeta} \mathrm{Spin} \times \mathbb{C}^\times \xrightarrow{\xi_o \times id} L \times \mathbb{C}^\times \xrightarrow{\theta_o} G \xrightarrow{j_o} \mathrm{GL}(4,\mathbb{C}) \tag{A1}$$

In this diagram, $\mathbb{C}^\times \cong \mathrm{GL}(1,\mathbb{C})$ is the group of non-zero complex numbers, and the multiplication map θ_o is defined by

$$\theta_o : L \times \mathbb{C}^\times \to \mathrm{GL}(4,\mathbb{C}) \ , \quad (\Lambda, c) \mapsto \Lambda c \ . \tag{A2}$$

We then define the group G to be the image $G := \theta_o(L \times \mathbb{C}^\times)$ and denote its canonical inclusion into $\mathrm{GL}(4,\mathbb{C})$ by j_o. In (A1) Spin is the spin group of the complex Lorentz group L (thus, $\mathrm{Spin} \cong \mathrm{SL}(2,\mathbb{C}) \times \mathrm{SL}(2,\mathbb{C})$, see Ref. 23), and

$$\xi_o : \mathrm{Spin} \to L$$

is the corresponding twofold spin map. Finally, the *representation map* ζ is defined by

$$\zeta : \mathrm{Spin} \times \mathbb{C}^\times \to \mathrm{GL}(4,\mathbb{C}) \ , \quad (A, c) \mapsto \zeta_o(A) c^{-1} \ , \tag{A3}$$

where $\zeta_o : \mathrm{Spin} \to \mathrm{GL}(4,\mathbb{C})$ is the usual spin representation, see e.g. Ref. 23.

Having explained (A1) we now translate it into the level of (principal) bundles:

$$S(M) \xleftarrow{(*)} (\mathrm{Spin} \times \mathbb{C}^\times)(M) \xrightarrow{\xi \times id} (L \times \mathbb{C}^\times)(M) \xrightarrow{\theta} G(M) \xrightarrow{j} F(M) \ . \tag{A4}$$

In this diagram, generally, the $G_1 \times G_2$ product bundle $(G_1 \times G_2)(M)$ of two principal bundles $G_1(M)$ and $G_2(M)$ is given by the restriction of $G_1(M) \times G_2(M)$ to the diagonal $\Delta \subset M \times M$, Δ being identified with M itself.[22]

$\mathbb{C}^\times(M)$ is the trivial $\mathbb{C}^\times$ bundle $M \times \mathbb{C}^\times$. Because of this triviality, it is possible to define the bundle map θ analogously to θ_o (A2) by

$$\theta : (L \times \mathbb{C}^\times)(M) \to F(M)\,, \quad ((X^a), c) \mapsto (cX^a)\,, \tag{A5}$$

and define the principal bundle $G(M)$ as the image bundle $G(M) := \theta((L \times \mathbb{C}^\times)(M))$, j being its canonical inclusion into $F(M)$. In (A4), $\mathrm{Spin}(M)$ is the spin bundle of $L(M)$ with the corresponding spin structure

$$\xi : \mathrm{Spin}(M) \to L(M)\,.$$

Finally, $(*)$ in (A4) denotes the building of the associated vector bundle $S(M) := (\mathrm{Spin} \times \mathbb{C}^\times)(M) \times_\zeta \mathbb{C}^4$, which is the spinor bundle where Dirac spinors are defined as cross sections. An element ϕ of $S(M)$ is an equivalence class[22]

$$\phi = [u, \phi_o] = [u(A,c)\,,\ \zeta(A,c)^{-1}\phi_o]\,, \tag{A6}$$

where $u \in (\mathrm{Spin} \times \mathbb{C}^\times)(M)$, $\phi_o \in \mathbb{C}^4$, and $(A,c) \in \mathrm{Spin} \times \mathbb{C}^\times$.

We remark that the bundle mappings $\xi \times id$, θ, and j have as their corresponding Lie group homomorphisms exactly $\xi_o \times id$, θ_o, and j_o, respectively.

2. Spin Connection

With the help of the special bundle diagram (A4) we now construct a spin connection ω_s from a linear connection ω on $F(M)$ by means of pull-backs. Since pull-back mappings of connection 1-forms involve not only the geometrical mappings between the corresponding spaces but also mappings between the Lie algebras of the structure groups, let us consider the Lie algebras first.

Define the vector subspace $\boldsymbol{m}$ in the Lie algebra $\boldsymbol{gl}(4,\mathbb{C})$ of $\mathrm{GL}(4,\mathbb{C})$

$$\boldsymbol{m} := \left\{ A \in \boldsymbol{gl}(4,\mathbb{C}) \mid A^T\eta - \eta A = 0 \text{ and } \mathrm{tr}(A) = 0 \right\}\,. \tag{A7}$$

Denoting the Lie algebra of the Lorentz group L by $\boldsymbol{l}$, we can easily verify the following (vector space) decomposition

$$\boldsymbol{gl}(4,\mathbb{C}) = \quad \boldsymbol{l} \quad \oplus \quad \mathbb{C}\cdot\mathbb{1} \quad \oplus \quad \boldsymbol{m} \tag{A8a}$$

$$A \quad = \frac{1}{2}(A - \eta A^T\eta) \;+\; \frac{1}{4}\mathrm{tr}A\cdot\mathbb{1} \;+\; \frac{1}{2}(A + \eta A^T\eta - \frac{1}{2}\mathrm{tr}A\cdot\mathbb{1})\,. \tag{A8b}$$

Now, $\boldsymbol{l} \oplus \mathbb{C}\cdot\mathbb{1} =: \boldsymbol{g}$ is precisely the Lie algebra of G, and for every $\Lambda c \in G$ it follows $(\Lambda c)\boldsymbol{m}(\Lambda c)^{-1} \subset \boldsymbol{m}$. In this case the $\boldsymbol{g}$ component of the pull-back $j^*\omega = \omega|_{G(M)}$ in (A4) is a G connection on $G(M)$,[22] which we denote by ω_G.

We move in (A4) to the left and further pull back this connection ω_G via θ. Since obviously, θ_o (A2) induces a Lie algebra isomorphism (it is however not an isomorphism of groups), we obtain a connection $\theta_o^{-1}\theta^*\omega_G$ on the product bundle $(L \times \mathbb{C}^\times)(M)$.[22] By the same token

$$\omega_s := (\xi_o \times id)^{-1}(\xi \times id)^*(\theta_o^{-1}\theta^*\omega_G) \tag{A9}$$

is a connection on the spin bundle $(\mathrm{Spin} \times \mathbb{C}^\times)(M)$, which we call an extended spin connection.

In order to decompose the connection 1-forms on the product bundles in (A4), let $p : (L \times \mathbb{C}^\times)(M) \to L(M)$ and $q : (L \times \mathbb{C}^\times)(M) \to \mathbb{C}^\times(M)$ be the canonical projections. Then there exist a unique L connection ω_l on $L(M)$ and a unique $\mathbb{C}^\times$ connection ω_c on $\mathbb{C}^\times(M)$, which decompose the connection 1-form on $(L \times \mathbb{C}^\times)(M)$ as[22]

$$\theta_o^{-1}\theta^*\omega_G = p^*\omega_l \oplus q^*\omega_c\,. \tag{A10}$$

This leads to a corresponding decomposition of the extended spin connection (A9), where we omit the bundle projections for the sake of simplicity

$$\omega_s = \xi_o^{-1}\xi^*\omega_l \oplus \omega_c\,. \tag{A11}$$

Note the fact that no use have been made of the orthonormal tetrads σ so far.

3. Spinor Derivative

This spin connection ω_s defines a covariant differentiation on the associated vector bundle $S(M)$. We now show that (10) is exactly this derivative. Let σ be an orthonormal tetrad field which is a local cross section in L, and let $\hat{\sigma}$ be one of its liftings to the spin bundle $\mathrm{Spin}(M)$. Thus, we have $\sigma = \xi(\hat{\sigma})$. Furthermore let $\hat{1}$ be the trivial cross section on $M \times \mathbb{C}^\times$ prescribing to each $p \in \mathcal{U}$ the value $\hat{1}(p) := (p, 1)$.

Since $\Gamma^a{}_{\mu b}\, dx^\mu = (\sigma^*\omega)^a{}_b$ (6) is the pull-back of the whole linear connection, the pull-backs of the L connection ω_l and of the $\mathbb{C}^\times$ connection ω_c are obtained using (A10) and the algebraic decomposition (A8)

$$(\sigma^*\omega_l)^a{}_b = \frac{1}{2}(\Gamma^a{}_{\mu b} - \Gamma_{b\mu}{}^a) \qquad \text{and} \tag{A12}$$

$$\hat{1}^*\omega_c = \frac{1}{4}\Gamma^c{}_{\mu c}\,. \tag{A13}$$

Thus for the whole connection $\theta_o^{-1}\theta^*\omega_G$ on $(L \times \mathbb{C}^\times)(M)$ the decomposition (A10) implies

$$\left((\sigma, \hat{1})^*\theta_o^{-1}\theta^*\omega_G\right)^a{}_b = \frac{1}{2}(\Gamma^a{}_{\mu b} - \Gamma_{b\mu}{}^a) + \frac{1}{4}\Gamma^c{}_{\mu c}\cdot\delta^a{}_b\,. \tag{A14}$$

The solution (18) has exactly this structure and can thus be understood as a sum of two connections (A12) and (A13) on two different bundles according to (A10). This viewpoint enables the separation of the whole connection (18) into the $\mathbb{C}^\times$ part S_μ (which actually is a U(1) part, see below) and the L part $\widehat{\Gamma}^\alpha{}_{\mu\beta}$ (19).

We now define the covariant spinor derivative of Dirac spinors using the extended spin connection ω_s (A9). A Dirac spinor ψ is a cross section in $S(M)$, which may be written according to (A6) as an equivalence class with the help of the local cross section $(\hat{\sigma}, \hat{1})$ in $(\mathrm{Spin} \times \mathbb{C}^\times)(M)$ as

$$\psi = \left[(\hat{\sigma}, \hat{1})\,,\, \psi_o\right]\,, \tag{A15}$$

where ψ_o is a $\mathbb{C}^4$-valued function usually denoted simply by the same letter ψ and referred to as the Dirac spinor itself. This convention was already used in (10). The local trivialization of $S(M)$ in (A15) allows us to express the covariant derivative of ψ through[12]

$$\nabla_\mu\psi = \left[(\hat{\sigma}, \hat{1})\,,\, \partial_\mu\psi_o + \zeta\left((\hat{\sigma}, \hat{1})^*\omega_s\right)\psi_o\right]\,, \tag{A16}$$

where (see (A3), (A11), (A12), and (A13))

$$\begin{aligned}\zeta\left((\hat{\sigma}, \hat{1})^*\omega_s\right) &= \zeta \circ \xi_o^{-1}\xi(\hat{\sigma})^*\omega_l \oplus (-\mathbb{1})\cdot\hat{1}^*\omega_c \\ &= \frac{-1}{4}\gamma^b\gamma_a\cdot\frac{1}{2}(\Gamma^a{}_{\mu b} - \Gamma_{b\mu}{}^a) - \frac{1}{4}\Gamma^c{}_{\mu c}\cdot\mathbb{1} \\ &= -\frac{1}{4}\Gamma^a{}_{\mu b}\gamma^b\gamma_a\,.\end{aligned} \tag{A17}$$

This shows the required agreement with (10). Note that we have used here the (usual) explicit form of the Lie algebra homomorphism $\zeta\xi_o^{-1}$,[11,15] leading to the matrix expression $\frac{-1}{4}\gamma^b\gamma_a$. The $-\mathbb{1}$ in front of $\hat{1}^*\omega_c$ is due to the special representation c^{-1} in (A3), whereas the important factor $\frac{1}{4}$ in front of $\Gamma^c{}_{\mu c}$ comes from the algebraic decomposition (A8).

4. U(1) Gauge Transformation

If we change the section $\hat{1}$ to $\exp(\lambda)\cdot\hat{1}$, λ being a $\mathbb{C}$-valued function, then from standard theories on gauge transformations it follows

$$\left(\exp(\lambda)\cdot\hat{1}\right)^*\omega_c = \frac{1}{4}\Gamma^c{}_{\mu c} + \partial_\mu\lambda \tag{A18}$$

and (see (A3))

$$\psi = \left[(\hat{\sigma}, \hat{1})\,,\, \psi_o\right] = \left[(\hat{\sigma}, \exp(\lambda)\cdot\hat{1})\,,\, \exp(\lambda)\cdot\psi_o\right]\,. \tag{A19}$$

Since this $\mathbb{C}^\times$ gauge transformation takes place on $M \times \mathbb{C}^\times$ the tetrad σ and the Lorentz connection (A12) on $L(M)$ remain unchanged. Thus the metric is also invariant under $\mathbb{C}^\times$. It is easy to show that the adjoint spinor $\bar{\psi}$ transforms to $\exp(\bar{\lambda})\cdot\bar{\psi}$, and that in (12) only the matter Lagrangian $\mathcal{L}_m$ is affected by the change and transforms to $\exp(\lambda + \bar{\lambda})\cdot\mathcal{L}_m$. The invariance of $\mathcal{L}_m$ thus implies $\lambda + \bar{\lambda} = 0$ or $\exp(\lambda) \in \mathrm{U}(1)$. We must therefore replace $M \times \mathbb{C}^\times$ by its reduced bundle $M \times \mathrm{U}(1)$.

APPENDIX B: 4-VECTOR DECOMPOSITION

Given a third rank tensor $\Sigma_{\alpha\beta\gamma}$ define $\Upsilon_{\alpha\beta\gamma}$, Q_α, S_β, U_γ, and V_δ as follows:

$$\begin{aligned}\Sigma_{\alpha\beta\gamma} =: \frac{1}{18}\Big[& (5\Sigma_\alpha{}^\epsilon{}_\epsilon - \Sigma^\epsilon{}_{\alpha\epsilon} - \Sigma^\epsilon{}_{\epsilon\alpha})g_{\beta\gamma} \\ & +(-\Sigma_\beta{}^\epsilon{}_\epsilon + 5\Sigma^\epsilon{}_{\beta\epsilon} - \Sigma^\epsilon{}_{\epsilon\beta})g_{\alpha\gamma} \\ & +(-\Sigma_\gamma{}^\epsilon{}_\epsilon - \Sigma^\epsilon{}_{\gamma\epsilon} + 5\Sigma^\epsilon{}_{\epsilon\gamma})g_{\alpha\beta}\Big] + \Sigma_{[\alpha\beta\gamma]} + \Upsilon_{\alpha\beta\gamma} \\ =: \quad & Q_\alpha\, g_{\beta\gamma} + S_\beta\, g_{\alpha\gamma} + U_\gamma\, g_{\alpha\beta} - \frac{1}{12}\eta_{\alpha\beta\gamma\delta}V^\delta + \Upsilon_{\alpha\beta\gamma}\,.\end{aligned}$$

For V^δ we have then $V_\delta = 2\eta_{\alpha\beta\gamma\delta}\cdot\Sigma^{\alpha\beta\gamma}$. From the first equation we conclude

$$\begin{aligned}\Sigma_\alpha{}^\epsilon{}_\epsilon = \frac{1}{18}\Big[& 20\Sigma_\alpha{}^\epsilon{}_\epsilon - 4\Sigma^\epsilon{}_{\alpha\epsilon} - 4\Sigma^\epsilon{}_{\epsilon\alpha} \\ & -\Sigma_\alpha{}^\epsilon{}_\epsilon + 5\Sigma^\epsilon{}_{\alpha\epsilon} - \Sigma^\epsilon{}_{\epsilon\alpha} \\ & -\Sigma_\alpha{}^\epsilon{}_\epsilon - \Sigma^\epsilon{}_{\alpha\epsilon} + 5\Sigma^\epsilon{}_{\epsilon\alpha}\Big] + \Upsilon_\alpha{}^\epsilon{}_\epsilon \quad \Leftrightarrow \quad \Upsilon_\alpha{}^\epsilon{}_\epsilon = 0\,.\end{aligned}$$

Similarly, $\Upsilon^\epsilon{}_{\beta\epsilon} = \Upsilon^\epsilon{}_{\epsilon\gamma} = 0$. Furthermore, $\Sigma_{[\alpha\beta\gamma]} = \Sigma_{[\alpha\beta\gamma]} + \Upsilon_{[\alpha\beta\gamma]} \Leftrightarrow \Upsilon_{[\alpha\beta\gamma]} = 0$.

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