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Collective Issues Related To B-Factories

D.V. PESTRIKOV

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Collective Issues Related To B-Factories

D.V. Pestrikov*
National Laboratory for High Energy Physics,
Oho, Tsukuba-shi, Ibaraki, 305, Japan

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*On leave of absence from Budker Institute of Nuclear Physics, 630090 Novosibirsk, Russia.

Abstract

This report reviews the limitations on the performance of the planned B- and Φ-factories due to the collective interactions of stored bunches. Together with the traditional revision of the limitations due to the single- and multi-bunch interaction with surrounding electrodes, we briefly discuss in this report the limitations on the performance due to collective beam-beam instabilities of colliding beams and instabilities due to the collective interaction of beams with the ions.

1 Introduction

The construction of widely discussed now new generation of colliders, the so-called B- and Φ-factories [1]÷[4], which have to reach huge luminosities ($10^{33} \div 10^{34}$ 1/cm^{2}s) in the .5 - 5 GeV energy range, presents very challenging opportunities both for particle accelerator physics and technology. With such a high goal luminosity, the performance of these colliders will very likely be limited by the current-dependent phenomena. If the total (average) beam current is $I = Ne/T_0$, $E = \gamma Mc^2$ is the particle energy, and β^* is a β-function at the interaction point (IP), then, with the given threshold value of the so-called beam-beam parameter ξ the luminosity of the collider does not exceed

$$\mathcal{L} \leq \mathcal{L}_0 = \frac{(I/e)\gamma\xi}{4r_0\beta^*}, \quad r_0 = e^2/Mc^2. \tag{1.1}$$

This equation is written while assuming the lengths of colliding bunches σ_s be shorter than β^*. Otherwise, the luminosity $\mathcal{L}(\sigma_s/\beta^*)$ gets additional decay, as is shown in Fig.1. Thus, to increase the luminosity of a ring, we have to increase the threshold values of the beam current I and ξ. Except for the other reasons, these values can be limited by the collective interaction of colliding bunches as well as by their interaction with surrounding elements of the vacuum chamber. Although the limitations on the performance of B-factories due to the interaction of the beam with its environment have been reviewed at least several times (see, for instance, in [5,6]), a closer inspection of a possible limitations due to various effects related to particular designs adds more effects to worry about.

In this report, after pointing out some more or less ideas we focus on the following items:

- The use of the passive tools to cure the instabilities due to the interaction of the beam with surrounding electrodes;
- Landau damping due to the nonlinearity of betatron oscillations;
- Collective beam-beam effects; and
- The interaction of electron and positron beams with ions.

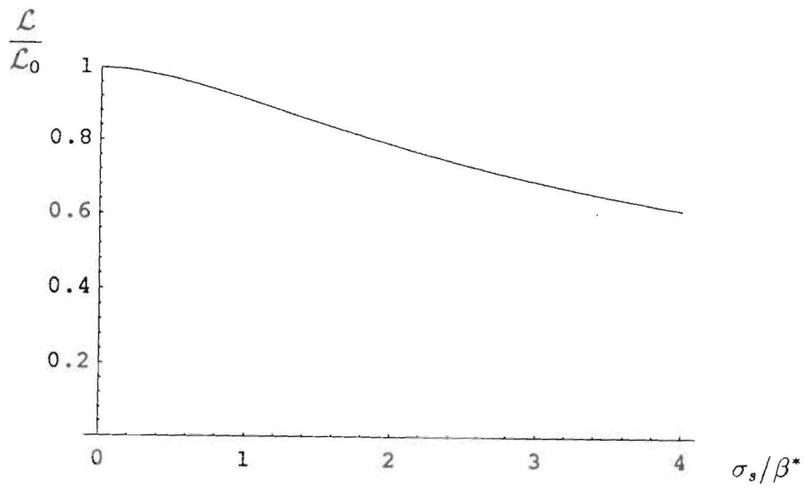

Figure 1: Dependence of the luminosity on the bunch length.

Once the main goal of this paper is to review collective effects which are specific to B-factories, we typically use relevant approximations which hold in the region of the parameters defined in the Table. For the same reason, except in some particular cases, many equations are given in their final form. In these cases the corresponding detailed calculations can be found in the cited references.

To obtain a luminosity 10^{34} $1/(\mathrm{cm}^2\mathrm{s})$ in a ring with $E = 5\mathrm{Gev}$, $\beta^* = 1\mathrm{cm}$ and $\xi = .05$ by Eq.(1.1) the total beam current should be $I \simeq 3.5\mathrm{A}$. Since such a large current cannot be obtained in one bunch with a reasonably small length and emittances, in all projects it is planed to work with beams containing many bunches, such that $I = \kappa I_b$ and when current I_b of the bunch ensures its desirable parameters. The choice of the number of bunches in the beam κ is determined by a reasonable compromise between the limitations due to single- and multi-bunch instabilities, which can occur due to the interaction of the beam and the surrounding electrodes. Since the use of various feedback systems is typically considered as the possible way to cure these instabilities, in, for instance, SLAC [3] and KEK [4] projects this number of bunches was chosen so as to simplify single-bunch problems. The Table can give the reader some impression on the discussed beam parameters.

Another scope of the limitations on bunches and ring performance can be caused by coherent beam-beam instabilities. The first careful study of this phenomenon [7] indicated it as being a severe instability, which generally limits the value of ξ. Due to the strong nonlinearity of the beam-beam kick, the design has to ensure the stability not only of the dipole, but also of some initial multipole coherent oscillations. As recently found in [8,9], for short bunches with $\sigma_s \ll \beta^*$ the Landau damping due to a nonlinearity of the beam-beam force does not help very much against this instability. Qualitatively, this was explained in [10], were it was shown that roughly only 15% of ξ contributes to Landau damping. As shown in [11], the stability of colliding bunches which have a round cross section can be improved in the region $\sigma_s \geq \beta^*$ due to an effect which was predicted by

S.Krishnagopal and R.Siemann in [12]. However, this suppression of collective beam-beam resonances does not work for flat bunches (since typically $\beta_x^* \gg \sigma_s$). The development of a feedback system which is capable to stabilizing coherent beam-beam oscillations presents a very difficult problem.

The limitations due to beam-ion interactions are typically associated with the possibility of the accumulation of ions in the electron beam, when these ions can cause very strong perturbations of the particle motion. However, in the case of a B-factory with a very close location of bunches inside the beam, the production of ions can provide undesirable coupling and, hence, an instability of the coherent oscillations of bunches, even when ions can not be trapped by the beam. This effect can limit the performance of both the electron and positron rings of a B-factory.

As a general conclusion of this section we remind the classification of collective instabilities, which was suggested in [13], and as far seems to be in agreement with practical estimations and expectations. By this classification, depending on the memory of an element interacting with the beam, all instabilities can be divided into 3 groups. The 1st contains the resonant instabilities due to the interaction with wakes, which decay during many revolution periods. This means that the beam interacts with some high-Q value cavity (or another resonant element) and oscillations become unstable, provided that the frequencies of the cavity ω_k and of the beam ($m_\alpha\omega_\alpha$, $\alpha = x, z, s$) satisfy a summation-type resonant condition [14]

$$\omega_k = m_\alpha\omega_\alpha + n\omega_0, \quad \omega_0 = 2\pi/T_0. \tag{1.2}$$

Since in this case the coupling of coherent oscillations of the bunch with oscillations of the induced fields can be very strong, the instabilities of this type are the most severe. With such a milti-turn interaction the increments of unstable modes are determined by the total current of the beam. The obvious way to cure these instabilities by detuning the cavity or the beam from the resonance is typically embarrassed by the fact that they appear accidentally and can hardly be calculated beforehand. This difficulty is well known and, once again, was pointed out by P. Morton in [5]. The interaction with a high-Q passive cavity in a differential-type resonance obviously damps the coherent resonant mode due to redistribution of the cavity-mode decrement between the field and beam oscillations. If λ_k is the decrement of a cavity-mode, the maximum decrement, which can be transferred into coherent mode, is obviously $\lambda_k/2$.

The instabilities of the 2nd group are caused by the interaction of the beam with wakes, which decay during time intervals comparable to the revolution period. The most famous example gives the resistive wall instability [15]. The increments of these instabilities contain both single- and multi-bunch (multi-turn) parts. The ratio between these two parts generally depends on the working point of the ring, on the ring chromaticity and other parameters. A careful calculation of decrements as well as a proper choice of the working point of the ring can simplify the damping of multi-bunch modes of the beam. Multi-bunch modes can be damped using a relevant feedback systems (see in [1]÷[4]). In the described B-factory projects [1]÷[4] the instabilities of the 2nd group are typically associated with the interaction of the beam with the higher order modes of various cavities.

Finally, the coherent effects of the 3d group are caused by the interaction of bunches

with wakes, which decay much faster than the revolution period in the ring (in the case of a multi-bunch beam, the wake must disappear within the bunch-to-bunch distance). These are the so-called single-turn (or single-bunch) effects. One of the specific features of these effects is the very weak dependence of decrements (increments) of coherent oscillations on the working point of the ring. In this case, when such an interaction damps coherent oscillations, corresponding devices can be used as wideband dampers for instabilities of the first two groups [16]. The practical use of such damping systems usually demands a careful high-frequency matching of the elements.

2 Single-Bunch Effects

In existing designs of B- and Φ-factories these issues usually provide the basis for estimating the longitudinal and transverse wideband impedance budgets of the ring. The instabilities due to single-bunch effects are more important for rings, where the number of bunches is not very high and, therefore, the current in a single bunch can be relatively high. The damping single-bunch effects due to interaction with the either passive, or active devices seems be useful in all cases (see, for instance, in [3,4]).

A careful analysis of single-bunch effects is usually troublesome due to the inevitable necessity to solve very complicated integral equation, or a system of integral equations which describe the interaction of particles within a bunch. Apart from evaluating the safe margins for the coupling impedance, these equations must be solved numerically. In these circumstances an important supplement to numerical calculations can provide solvable examples (see in [10]), as well as a calculation of values which do not require any direct solution of these equation. As an example, we mention the calculations of various sums of the decrements of the collective modes, which are defined as the traces of the kernels of relevant equations. Since these sums do not depend on both the coupling of bunches and on the coupling between particle degrees of freedom [17], its calculation becomes especially important for estimating the effectiveness of the damping systems of coherent oscillations.

2.1 Transverse Single-Bunch Effects

Before listing some typical effects which can limit the parameters of the rings in particular projects, we make the following general remark. Typically, the wake-fields induced by a bunch in the environment are directed against its total current. In many cases the part of the wake related to the excitation of fields by the longitudinal (synchronous) motion of the bunch dominates. Then, for the most important dipole modes of transverse oscillations the description of wakes in terms of the transverse impedance yields a suitable description of the instability. Meanwhile, there is another important class of elements of the vacuum chamber, such as matched plates, when the excitation of wakes by transverse current gives the dominant contribution to the decrements of coherent oscillations of the bunch (see, for instance in [16], [18] and [19]). This is the so-called effect of the fast damping of coherent oscillations, which was first observed in VEPP-2 in Novosibirsk (see in [20]).

Usually, the interaction with such an element damps coherent oscillations of the bunched beam with the damping decrement, which is proportional to the length of this element. The possibility of using such devices as passive damping systems against both the single- and the multi-bunch instabilities of the beam is an important issue of any design with the goal of a high beam intensity. Since bunches should not overlap electromagnetically when they interact with such a device, the usage of these dampers will be simplified in rings, working with a larger bunch spacing. [1]

Another class of damping devices use suitable frequency characteristics to enhance the mentioned contribution from differential-type resonances in the decrements of coherent oscillations. The possibility of using passive wideband systems to damp collective instabilities in B-factories was not seriously considered anywhere.

Specific features of single-bunch collective phenomena are strongly defined by the ratio of increments (decrements) of coherent oscillations δ to the angular frequency of synchrotron oscillations ω_s. In practice, however, the decrements of coherent oscillations are not know in advance, and can only be estimated by some parameter Ω_m using, say, perturbation theory . For the most important case of dipole coherent oscillations, Ω_m can be defined as follows

$$\Omega_m = -\frac{I_b e}{2p\nu_\perp} \int_{-\infty}^{\infty} dn Z_\perp(n). \tag{2.1}$$

Here, $I_b = N_b e/T_0$ is the bunch current, $p = E/c$ the particle momentum, $\nu_\perp$ the betatron tune, and $Z_\perp(\omega)$ the transverse impedance of the pipe. Physically, Ω_m yields coherent frequency shift of the bunch with the zeroth length. If the ratio

$$b = \frac{|\Omega_m|}{\omega_s} \tag{2.2}$$

is small , a coherent interaction only slightly distorts the unperturbed spectra. In this case, if $\vec{m}_\perp$ and m_c are the multipole numbers of transverse and synchrotron oscillations, the eigenfrequencies of the coherent oscillations occur close to the combinations:

$$\omega = \vec{m}_\perp \vec{\omega}_\perp + m_s \omega_s, \tag{2.3}$$

while the modes of oscillations can be classified using the synchrotron multipole number m_s into the betatron, synchrotron and synchrobetatron modes.

Since both coherent frequency shifts and increments of coherent oscillations increase with an increase in the beam current, it may occur that a coherent interaction will couple the synchrotron modes of the bunch, and the multipole number m_c will no longer classify its coherent modes. Physically, this fact corresponds to the development of coherent oscillations during time intervals comparable to or faster than the periods of synchrotron oscillations of the bunch particles. Typically, the coupling of synchrotron modes of the bunch breaks the stability of the coherent oscillations. In the region of rather moderate mode coupling ($b \simeq 1$) the study of the collective stability of the beam usually requires the

[1] For instance, to prevent the wideband bunch-to-bunch interaction, in KEK B-factory [4] the length of such devices should not exceed 30 cm, in SLAC project [3] – 60 cm.

solution of an infinite system of the coupled integral equations (see, for instance, in [21]). Its truncation yields both the eigenmodes and eigenfrequencies, as well as stability criteria of coherent oscillations. Although careful calculations of the collective spectra in cases of practical interest requires the use of numerical methods, [2] the lowest instability thresholds can be evaluated using perturbation theory. Simple calculations (see, for example, in [23])

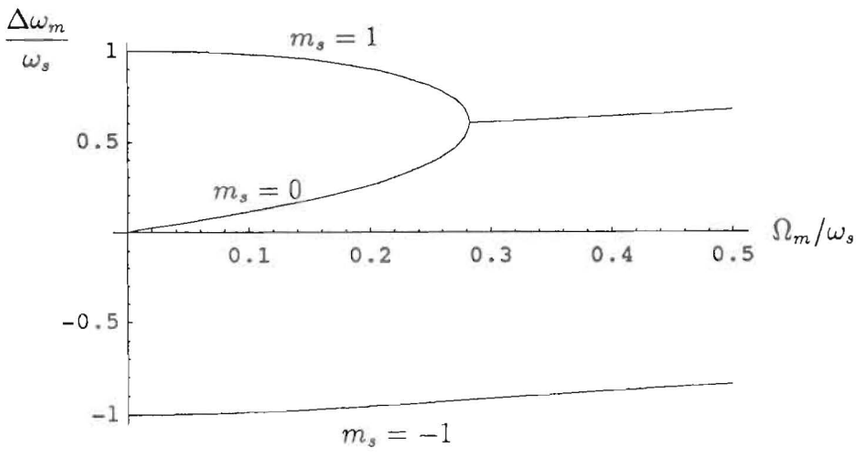

Figure 2: Dependences of coherent frequency shifts of the betatron and first two synchro-betatron modes on the bunch current.

show that the betatron and the first synchrobetatron modes merge (see in Fig.2) when

$$|\Omega_m| \simeq .3\omega_s. \tag{2.4}$$

Using the relationship between the transverse and longitudinal impedances,

$$Z_\perp(\omega) = \frac{iv}{\omega}\left(\frac{\partial^2}{\partial z \partial z'} Z_\parallel(r_\perp, r'_\perp, \omega)\Big|_{\vec{r}_\perp=0}\right), \tag{2.5}$$

and estimating the derivatives by the order of value,

$$\frac{\partial Z_\parallel}{\partial z} \simeq \frac{Z_\parallel}{l_\perp}, \tag{2.6}$$

where $l_\perp$ is the pipe radius, we can also write

$$Z_\perp(\omega) \simeq \frac{c}{l_\perp^2}\frac{iZ_\parallel(\omega)}{\omega}. \tag{2.7}$$

Substituting this relation in Eqs(2.1) and (2.4) yields

$$\left|\int_{-\infty}^{\infty} dn Z_\parallel(n)/n\right| \leq 0.6\nu_s\nu_\perp \frac{E}{eI_b}\frac{R_0^2}{l_\perp^2}, \tag{2.8}$$

[2] A simple model, which enables exact solution of the synchrobetatron mode-coupling equations, has been recently proposed by V. Danilov and E. Perevedentsev in [22]. Relevant solutions of the mode-coupling equations have been obtained in [23].

where $2\pi R_0$ is the perimeter of the orbit. For example, for the KEK B-factory in the TRISTAN MR tunnel (see in the Table), and for $l_\perp = 3$ cm, the integral in the left-hand side of Eq.(2.8) should not exceed 20 KOhm. Since below the pipe cutoff frequency $Z_{\|}/n$ is approximately constant, and then decreases roughly $\propto R_0^2/(nl_\perp)^2$, Eq.(2.8) estimates the threshold impedance of the ring as

$$(Z_{\|}/n) \leq 250 \text{ mOhm}, \tag{2.9}$$

Note that the increase in both the transverse and synchrotron tunes generally suppresses the mode-coupling instability and, therefore, increases the acceptable value of $Z_{\|}/n$.

Generally, the stability of coherent oscillations of bunches can be provided by synchrotron radiation damping (SRD) and by Landau damping of the collective modes. As can be seen from the Table, the decrements due to SRD in the present B-factory designs are expected to be rather weak. The strength of the Landau damping is defined by the frequency spread of the beam. The frequency spread due to the nonlinearity of synchrotron oscillations is

$$\Delta\omega_s = \frac{\pi^2}{8}\omega_s\left(\frac{\sigma_s}{\lambda_{RF}}\right)^2, \quad \Pi = h\lambda_{RF}.$$

This value is even smaller than the SRD-decrements for all projects. For this reason we may conclude that the stability of the synchrotron coherent oscillations of a bunch is determined by the SRD, or by a suitable feedback system.

For betatron (synchrobetatron) coherent oscillations , useful information concerning the capability of the Landau damping can be obtained by inspecting the stability diagrams in the plane of the complex variable Ω_m . If $g_m(\omega)$ is the frequency distribution function, which corresponds to the mode m, the parametric equation of this curve has the form (see in [10] and in Appendix A for details)

$$\frac{1}{\Omega_m(\Delta\omega_m)} = \int_{-\infty}^{\infty} \frac{dw g_m(w)}{\Delta\omega_m - w + i0}. \tag{2.10}$$

The frequency spread of the betatron coherent oscillations can be defined by two kinds of nonlinearities. First, in a colliding-beam mode this is a nonlinearity of the beam-beam kick. The relevant stability diagram [10] for dipole betatron oscillations is shown in Fig.3. Although, it indicates that only 0.15ξ contributes to the damping effect, the nonlinearity from the main interaction point (IP) cancels the coupled-mode instability if ξ exceeds ν_s at least twice.

Second, in the single-beam operation mode Landau damping of coherent oscillations can be provided by the cubic nonlinearity, produced by a family of octupoles. If $\sqrt{J_x\beta_x}$ and $\sqrt{J_z\beta_z}$ are the amplitudes of the horizontal and vertical betatron oscillations of a particle, the tune shift of the, for example, vertical oscillations due to that family, has the form:

$$\Delta\nu_z = \frac{\partial\nu_z}{\partial J_z}J_z + \frac{\partial\nu_z}{\partial J_x}J_x, \tag{2.11}$$

where

$$\frac{\partial\nu_z}{\partial J_{z,x}} = \mp\frac{e}{8\pi E}\int_0^{\Pi} ds\beta_z(s)\beta_x(s)\left(\frac{\partial^3 H_z}{\partial x^3}\right)_s. \tag{2.12}$$

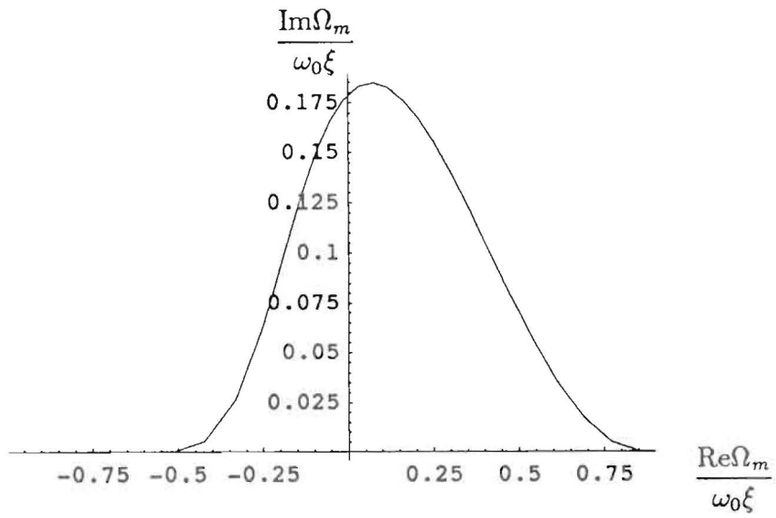

Figure 3: Stability diagram of vertical coherent oscillations for the case when the beam-beam nonlinearity defines the frequency spread.

Since for flat bunches $\varepsilon_x \gg \varepsilon_z$, except for the special location of the octupoles, the tune spreads of both vertical and horizontal oscillations will be determined by the horizontal emittance ε_x. In this case the stability diagram for vertical oscillations essentially limits the use of devices, producing the positive coherent frequency shifts (see in Fig.4). The stability diagram for horizontal coherent oscillations has the usual shape (Fig.5). Once the working point in the plane of Ω must be found inside the stability region, the use of Landau damping limits both the real and imaginary parts of the coherent frequency shift.

The tune spread, which is equivalent to that from the IP, can be produced by the family with the strength

$$\int_0^{\Pi} ds \left(\frac{\partial^3 H_z}{\partial x^3}\right)_s \simeq 120 \text{ T/m}^2.$$

If the ring impedance is below the limit, which is defined by Eq.(2.9), transverse coherent oscillations can still be unstable due, for instance, to head-tail instability. In the region $b \ll 1$ and neglecting Landau damping, the instabilities limit the real part of the impedance. For the case of short bunches, which is specific for e^+e^--factories, the increments of head-tail modes can be estimated by the following simple formula (dipole vertical betatron oscillations are assumed):

$$\delta_{m_s} = -\mathrm{Im}\omega = m_z \frac{eI_b}{p\nu_z}\frac{R_0}{2l_\perp^2}\int_{-\infty}^{\infty} dn \frac{\mathrm{Re}Z_{\|}(n - m_z\zeta)}{n - m_z\zeta} J_{m_s}^2(n\varphi_0), \quad m_z = \pm 1. \tag{2.13}$$

Here, $\zeta = d\nu_z/d\ln R_0$, $J_m(x)$ is the Bessel function and $\varphi_0 = \sigma_s/R_0$. Since $\mathrm{Re}Z_{\|}(n)$ is an even function of n, and

$$\sum_{m_s=-\infty}^{\infty} J_{m_s}^2 = 1, \tag{2.14}$$

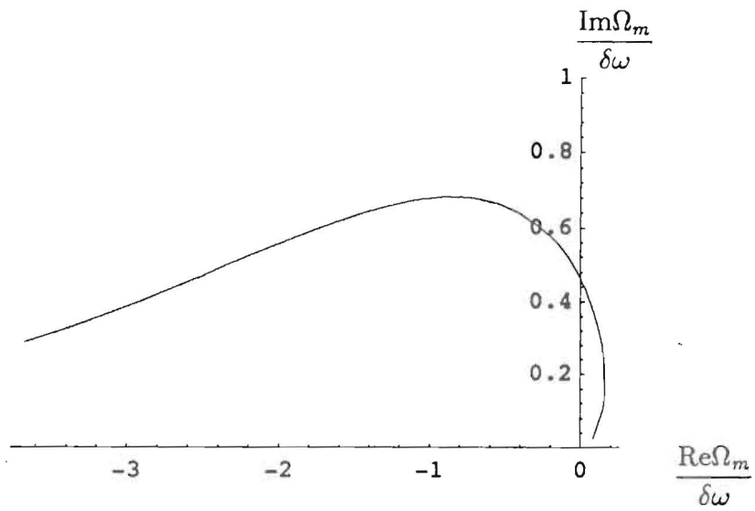

Figure 4: Stability diagram of the vertical dipole coherent oscillations for the case when the horizontal emittance defines the frequency spread of vertical oscillations $\delta\omega$. The negative sign of the nonlinearity prevents the minimum in the total tune shift and the corresponding dynamic aperture reduction.

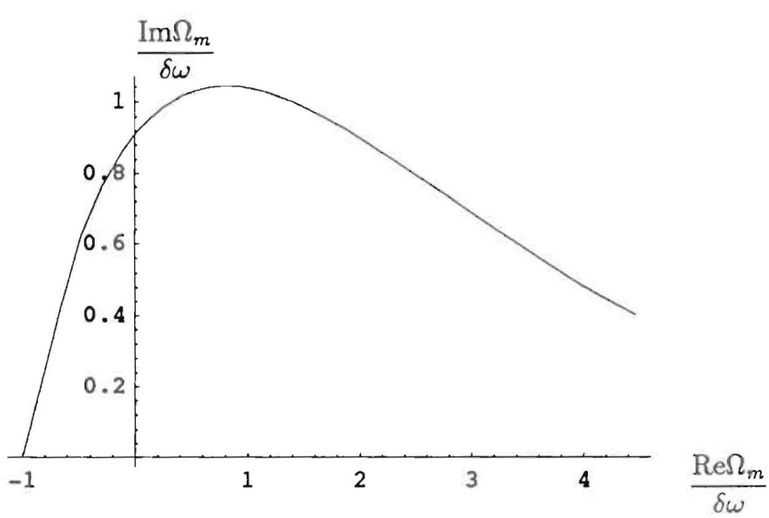

Figure 5: Stability diagram of the horizontal dipole coherent oscillations for the case when the horizontal emittance defines the frequency spread of horizontal oscillations $\delta\omega$.

the sum of the decrements in Eq.(2.13) is equal to zero:

$$\sum_{m_s=-\infty}^{\infty} \delta_{m_s} = m_z \frac{eI_b}{p\nu_z} \frac{R_0}{2l_\perp^2} \int_{-\infty}^{\infty} dn \frac{\mathrm{Re}Z_{\|}(n - m_z\zeta)}{n - m_z\zeta} \equiv 0. \tag{2.15}$$

For short bunches ($|\zeta|\varphi_0 < 1$) this sum rule separates the signs of the decrements of the betatron ($m_s = 0$) and of the synchrobetatron ($m_s \neq 0$) modes. If the betatron oscillations are stable, the synchrobetatron oscillations occur unstable and vise versa. For a pure resistive impedance ($Z_{\|}(n) = Z$) the integration in Eq.(2.13) results in the following very known formula [3]

$$\delta_{m_s} = \omega_0 \frac{eI_b Z}{E} \frac{R_0\sigma_s}{l_\perp^2} \frac{\zeta}{4m_s^2 - 1}. \tag{2.16}$$

However, the limitations due to this instability on the impedance or the ring chromaticity cannot be discussed without additional assumptions concerning the stabilizing effects. As an example, we give an estimation of the threshold impedance in the KEK LER due to Landau damping, which is caused by the nonlinearity of the beam-beam force,

$$Z \leq .15\xi \frac{E}{eI_b} \frac{l_\perp^2}{\zeta R_0 \sigma_s} = 18 \text{ KOhm}. \tag{2.17}$$

Another stabilizing effect can result from the interaction of a bunch with a system of matched plates which was mentioned above. Such a system can be installed in the ring for different purposes: for the ion cleaning, for the detection of coherent oscillations, etc. A piece of the vacuum chamber with such a plate presents the segment of a double-connected waveguige. Provided that it is terminated by the characteristic impedance Z_0, the interaction with the fundamental TEM-wave of this waveguide damps the *betatron* coherent oscillations of a short bunch. The relevant damping decrement is defined by the equation [16] [4] (l is the length of the plate)

$$\delta = \omega_0 \frac{eI_b Z_0}{2E} \frac{\Pi l}{l_\perp^2}, \quad \sigma_s < l, \quad \gamma \gg 1. \tag{2.18}$$

The fact that this decrement does not depend on the bunch length σ_s is specific for the effect of fast damping [20] and indicates it as being an effective tool to damp coherent oscillations of short bunches. The practical usage of this effect to damp coherent oscillations demands a long bunch-to bunch distance, careful matching of the plates until the

[3] Both this equation and Eq.(2.13) are written for the so-called hollow-bunch distribution function in the amplitudes of synchrotron oscillations, when all particles in the bunch have the same amplitude of the synchrotron oscillations. For more realistic, smooth distribution functions decrements can be estimated by the extrapolation of the collective spectra from the high frequency region $|n| \gg (|m_s|R_0/\sigma_s)$ (see, for instance, in [10], or [19]). This results in the equation similar to Eq.(2.16), except maybe for additional factor $1/(|m_s| + 1)$, which is not important here.

[4] Note that, since the main contribution to the decrement yield the excitation of the TEM-mode by transverse current of the bunch, this effect cannot be described using traditional concept of the transverse coupling impedance.

harmonic numbers of at least $n \sim R_0/l_\perp$, working with a definite sign of the chromaticity to stabilize synchrobetatron modes and sufficient space for the plates. As can be seen from Eq.(2.18), it is more beneficial to locate such plates in places with a small value of the β-function. For instance, the interaction of a bunch which contains $I_b = .5$ mA with a system of plates, which have $Z_0 = 50$ Ohm, $l_\perp = 3$ cm and the total length 0.1 of the ring perimeter, gives $\delta \simeq 3.5\ 10^{-3}\omega_0$. For the B-factory in the TRISTAN MR tunnel this corresponds to $\tau = 1/\delta = 0.4$ ms; for the Novosibirsk project it is 0.1 ms. We also remind that the use of damping systems instead of Landau damping does not blow-up the phase-space volume of the bunch.

2.2 Longitudinal Single-Bunch Effects

In most of the B and Φ-factory projects the limitations due to longitudinal single-bunch effects are associated with the growth of the bunch length with its intensity. The theory of this phenomenon is yet far from its completion (see, for example, in [24]), and particular designs are typically based on the half-empirical rules, or the simulations. Generally, the interaction of a bunch with the wideband environment results in two kinds of effects: in a distortion of the RF potential well, and in an instability of synchrotron coherent oscillations. It seems that the experimental results indicate that a flattening of the RF-well and associated bunch lengthening is observed at a lower beam current. This gives at least some basis to consider these two mechanisms of bunch lengthening separately.

If the bunch only slightly distorts the potential well and, therefore, the bunch lengthening is weak, it can be calculated using perturbation theory (see, for instance, in Appendix B). The result is [5]

$$\frac{\sigma_s^2}{\sigma_{s0}^2} = 1 + \sqrt{2\pi}\frac{eI_b(Z/n)_0\Pi}{\sigma_s heV \sin\varphi_s}\frac{R_0^2}{l_\perp^2}, \tag{2.19}$$

where h is the harmonic number of the RF system and $(Z/n)_0$ is the value of the impedance below the cutoff frequency.. Taking into account that at relativistic energies $V \propto \gamma^4$, we can rewrite Eq.(2.19) in the form which was used to scale the bunch lengthening experimental data in early papers (see, for example, in [25])

$$\frac{\sigma_s^2}{\sigma_{s0}^2} = 1 + C\frac{N_b}{\gamma^4\sigma_s}, \tag{2.20}$$

where C is a factor which depends on the ring parameters. Eq.(2.19) can be used to estimate the limitation on the production $I_b(Z/n)_0$, when the desirable bunch lengthening does not exceeds, say 20%. If we take again as an example the ring in the TRISTAN MR tunnel, this yields $I_b(Z/n)_0 \leq 1.3\ 10^{-4}$ V, or with $I_b = .5$ mA, $(Z/n)_0 \leq 260$ mOhm.

The calculations related to bunch lengthening due to potential-well distortion within a wider region of the parameters are based on the assumption that due to the balance between the excitation of the synchrotron oscillations due to quantum fluctuations of the

[5] To simplify equations in this section we write Z_n instead of $Z_\parallel(n)$

synchrotron radiation and synchrotron radiation damping the bunch reaches the Boltzmann equilibrium distribution with the linear density

$$\rho(\varphi) = \rho_0 \exp\left(-\frac{U(\varphi)}{T}\right). \tag{2.21}$$

Here, $\varphi = s/R_0 - \omega_0 t$ gives the longitudinal position of a particle inside the bunch, $U(\varphi)$ is the potential energy of the particle, and T is the longitudinal temperature of the bunch. It is obvious that in using Eq.(2.21) we assume the collective stability of the bunch beforehand. The substitution in Eq.(2.21) of the potential energy of the synchrotron oscillations after its modification by the bunch wake results in the so-called Haissinski equation [26]

$$\rho(\varphi) = \rho_0 \exp\left(-\frac{\varphi^2}{2\sigma_0^2} - \int_{\varphi}^{\infty} d\varphi' w(\varphi' - \varphi)\rho(\varphi')\right). \tag{2.22}$$

Here, $R_0\sigma_0 = \sigma_{s0}$ is the unperturbed bunch length and

$$w(\varphi) = \frac{N_b e^2 \omega_0}{heV \sin\varphi_s \sigma_0^2} \int_{-\infty}^{\infty} dn(-iZ_n/n)e^{-in\varphi}. \tag{2.23}$$

Eq.(2.22) is a nonlinear Volterra-type integral equation. For reasonable wakes

$$\overline{w^2} = \int_{-\infty}^{\infty} d\varphi w^2(\varphi) < \infty \tag{2.24}$$

it always has a unique solution, which can be found by at least using successive iterations [27](see also in Appendix C). In the case of strong bunch lengthening $(\overline{w^2})^{1/2} \gg 1$, the interaction with dissipative surrounding results in a strong asymmetry of the linear density along the bunch. For the case when the bunch interacts with a pure resistive impedance $Z_n = Z_0$, the Haissinski equation can be solved directly [28]. The result is

$$\rho(x) = \frac{1}{\xi'} \frac{\exp(-x^2/2)/(\sqrt{2\pi}\sigma_0)}{\coth(\xi') + \mathrm{erf}(x/\sqrt{2})}, \quad x = \varphi/\sigma_0. \tag{2.25}$$

Here, the value

$$\xi = 2\sigma_0\xi' = \pi \frac{eI_b Z_0 \alpha}{\nu_s^2 E \sigma_0}. \tag{2.26}$$

specifies the strength of the lengthening. Eq.(2.26) has simple asymptotes. In the region $\xi' \ll 1$ the distortion of the bunch length is small, while $\rho(x)$ is close to a Gaussian distribution

$$\rho(x) \simeq \exp(-x^2/2)/(\sqrt{2\pi}\sigma_0)[1 - \xi'\mathrm{erf}(x/\sqrt{2})].$$

On the contrary, in the region $\xi' \gg 1$ the linear density $\rho(x)$ gets a specific triangle shape (see, for instance, in[24,30] and in Fig.6):

$$\rho(x) = \begin{cases} 0, & x > 0\ , \\ \dfrac{|x|}{\xi} \dfrac{g(x)}{1 + g(x)(1 - 1/x^2)}, & x < 0\ , \end{cases} \tag{2.27}$$

where

$$g(x) = \frac{\exp(-x^2/2 + 2\xi')}{\sqrt{2\pi}|x|}. \tag{2.28}$$

The width of $\rho(x)$ in this case is obviously

$$-2\sqrt{\xi'} \leq x \leq 0,$$

and significantly exceeds the natural length of the bunch σ_{s0}

$$\sigma_s(N) \simeq R_0\sqrt{\frac{eI_b Z_0 \alpha}{\nu_s^2 E}} \gg \sigma_{s0}. \tag{2.29}$$

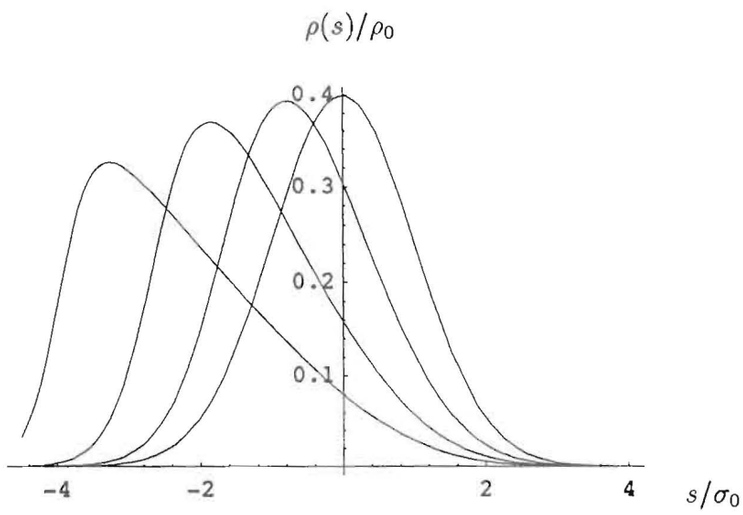

Figure 6: Deformation of the shape of the bunch linear density with the bunch intensity. From left to right ξ': 5, 2.5, 1, 0.01.

A strong asymmetry of the linear density in Eq.(2.27) generally indicates that such states of the bunch can be reached only through a strong instability of synchrotron coherent oscillations, which roughly stops in the region where the amplitudes $\varphi_{coh} \sim \sigma_\phi(N)/2$, while phases ψ_s are concentrated around π. Moreover, in this region of parameters the wakes significantly decrease the frequencies of synchrotron oscillations, which simplify the conditions for the synchrotron mode-coupling instability of the bunch.

The instability of synchrotron coherent oscillations can result in additional heating of the beam. Starting from paper [31] the bunch lengthening due to coherent instabilities of a bunch was called as the turbulent one. A closer inspection of this phenomenon indicated (see, for instance, in [10] and [32,33]) the importance of two effects for the beam heating by its unstable coherent oscillations: the turbulent stabilization of the oscillations due to a nonlinear dependence of the increments of unstable modes on the amplitudes of coherent oscillations; and absorption of the energy of coherent oscillations by particles, when the collective oscillations are stabilized by Landau damping, or some other damping effect.

Both the analytic and numerical predictions related to the turbulent bunch lengthening are still too speculative (see, for example in [34]), and are typically focused on the most convincing calculations of the stability thresholds for the linear synchrotron coherent oscillations. Below the threshold of the synchrotron mode-coupling instability the calculation of the safe margins for the longitudinal impedance uses typically Z/ncriterion, or its suitable modification. However, since for the most B-factories the SRD-decrements exceed the frequency spread of synchrotron oscillations, the threshold currents, or impedances, are defined by the balance between the increment of the instability and $1/\tau_s^{SR}$. For the data given in the Table, this typically results in roughly one order of magnitude higher thresholds than that given by the Z/n-criterion.

In the case of strong bunch lengthening the RF potential well distortion due to the bunch wakes significantly decreases the instability thresholds for the longitudinal coherent oscillations [35]. For these reasons it seems that the region $\xi \gg 1$ cannot be reached without some extraordinary efforts. It is clear, however, that the use of passive or active damping systems to increase the thresholds of instabilities is desirable anyway. Except, maybe, for the possibility to use the systems, including notch-filters, synchrotron oscillations usually do not provide strong enough modulation of the coherent energy loses. If the damping of transverse oscillations is strong enough, the damping of the synchrotron coherent oscillations can be enhanced by the elements with the radial gradient of the coupling impedance [16]. The relevant addition to the decrements of synchrotron modes can be estimated by the following expression

$$\delta_{m_s} = c\frac{eI_b}{\sigma_s E}\left(\eta\frac{\partial \mathrm{Re}(Z)}{\partial x}\right)_0, \tag{2.30}$$

where η is the value of the dispersion function at the position of the damping system, and the derivative is calculated on the closed orbit. The damping effect takes place if

$$\left(\eta\frac{\partial \mathrm{Re}(Z)}{\partial x}\right)_0 > 0. \tag{2.31}$$

This condition has a clear physical sense: the sign of the gradient of the impedance must increase the energy loses of synchrotron coherent oscillations with an increase in their amplitudes. As an example, the instability with the growth rate $\tau = 100$ μs in the LER in the TRISTAN MR tunnel could be damped by a system with $\nabla Z = 32$ KOhm/cm. Since the sum of decrements of coherent oscillations does not depend, in particular, on the radial-longitudinal coupling [17], the interaction with this system will simultaneously decrease the decrements of the dipole horizontal synchrobetatron coherent oscillations $\delta^H_{m_s}$ to maintain the sum rule

$$\sum_{m_s}\Delta(\delta^H_{m_s}) = -\sum_{m_s}\delta_{m_s}. \tag{2.32}$$

The increase in the total sum of the decrements is a general requirement to a design of all damping systems. With the use of wideband dampers this demands an increase in the contributions in decrements of the effect of fast damping. Note that the decrements, or increments, in Eq.(2.30) do not depend on the ring momentum compaction factor α. The

instabilities due to an interaction with the environment having a radial gradient of the longitudinal impedance can limit the performance of the rings with extraordinary small values of momentum compaction [36]. A non-symmetric position of the closed orbit in a vacuum chamber is typical for many B-factories. Such a gradient can also be caused by the closed-orbit distortions.

3 Multi-Bunch Effects

If the bunch wakes last longer than the bunch-to-bunch distance the interaction of bunches couples their coherent oscillations, which can cause a multi-bunch instability of the beam. It is clear in advance that the worst stability has a beam containing κ identical bunches. In this case, the symmetry of the stationary state relative to the rotation on the angle $2\pi/\kappa$ results in the propagation along the beam of uncoupled multi-bunch modes with wave-vectors of

$$k_a = \frac{2\pi a}{\kappa}, \quad a = 0, 1, \ldots, \kappa - 1. \tag{3.1}$$

This increase in the degrees of freedom of a multi-bunch beam makes its dynamical features more similar to the case of a coasting beam and, therefore, even more unstable when κ increases. The main difficulty in this case is caused by the multiplication by the bunch-to bunch interaction of the multi-turn stability diagram of a single bunch. Let $\delta(\nu)$ be the multi-turn part of the decrement of a dipole mode of a single bunch. Due to the multi-turn interaction it is a periodic function of the tune ν

$$\delta(\nu + 1) = \delta(\nu). \tag{3.2}$$

Somewhere between $\nu = 0$ and $\nu = 1$ the function $\delta(\nu)$ usually changes sign, which defines the width of the stopband of the instability. As an example, we can take that this happens when ν crosses the point $\nu = 1/2$, as in the case of a resistive wall instability, or an instability due to interaction of the beam with a low-Q cavity. If, now, the beam contains κ identical bunches, and interacts with the same system, the decrement of the a-th multi-bunch mode is defined by the same function, which, however, depends on ν and a through the following combination:

$$-\,\mathrm{Im}\omega(\nu) = \kappa\delta\left(\frac{\nu + a}{\kappa}\right). \tag{3.3}$$

If, for instance, the stability condition for a single bunch is

$$0 \leq \nu \leq 1/2,$$

for κ bunches we must simultaneously satisfy κ stability conditions of the form

$$0 \leq \nu + a \leq \kappa/2, \quad a = 0, 1, \ldots, \kappa - 1. \tag{3.4}$$

This means that, if some mode with a mode-number of $a < \kappa/2$ is stable, its reflecting partner $\kappa/2 - a$ would be unstable. Although the fact of instability seems to be

independent of ν, the sum of the decrements of all multi-bunch modes

$$\sum_{a=0}^{\kappa-1} \delta_a,$$

does not depend on the bunch-to-bunch coupling. Moreover, it is exactly equal to the sum of the multi-turn parts of the decrements, calculated for particular bunches while neglecting their coupling. In this sense the correct choice of ν can simplify the stabilization of unstable modes.

The multi-bunch instabilities can usually be suppressed by a relevant decrease of the Q-values; by the varying the parameters of the bunches (typically the tunes) along the beam [6]; and by using suitable feedback systems. The 1-st and the 3-d possibilities have been more carefully inspected to overcome the multi-bunch instability in B-factories (see, for instance, in [3], or [4]).

In some sense, multi-bunch instabilities due to the interaction of the beam with some parasitic cavities or modes can be considered to be less dangerous. Once the parasitic element is specified, definite efforts can be spent to decrease its impedance and to cure the instability. It becomes less easy, however, for the case when the beam interacts with the fundamental mode of the accelerating RF-system. Recently [4], the instability of this kind has been reported as a serious limitation on the performance of B-factories with long rings. For short bunches ($\sigma_s \ll \lambda_{RF}$, where λ_{RF} is the wavelength of the accelerating field), the maximum increment of coherent oscillations of the beam due to its interaction with the fundamental mode of the RF-system, containing N_c accelerating cells, is defined by the parameter:

$$\Omega = N_c \frac{eI(Z/Q)}{T_0 E} \frac{h^2 \alpha}{4\nu_s}, \quad h = \omega_{RF}/\omega_0. \tag{3.5}$$

Taking $(Z/Q) = 197$ Ohm, the beam current $I = 1$ A and all other necessary parameters from the Table for the LER in the TRISTAN MR tunnel, we can estimate the corresponding growth rate of the dipole synchrotron modes as

$$1/\tau_g \sim \Omega \simeq 2.92\ 10^4\ 1/\text{s}. \tag{3.6}$$

Since this value is very close to $\omega_s = 4.5\ 10^4$ 1/s, the instability is indeed fast. On the other hand, direct calculations of the dipole multi-bunch decrements ($\delta = -\text{Im}\omega$) due to nonresonant interaction of the beam with the cavity result in (see, for example, in [4], or [10])

$$\delta_m = m\Omega \exp\left(-\frac{\pi h}{\kappa Q_L}\right) F(\nu_k, m\nu_s + a), \tag{3.7}$$

$$m = \pm 1,$$

$$F(x,y) = \frac{\pi}{\kappa}\left\{\frac{\cos(2\pi\alpha_-) - p}{1 - 2p\cos(2\pi\alpha_-) + p^2} - \frac{\cos(2\pi\alpha_+) - p}{1 - 2p\cos(2\pi\alpha_+) + p^2}\right\}, \tag{3.8}$$

[6] As was already mentioned, the stability of a single bunch is determined by the proper choice of the working point, and in this sense coherent oscillations of a single bunch can be done stable easily. Hence, we may conclude that a multi-bunch system can be stabilized by a suitable distribution of bunches along the orbit.

$$\alpha_{\pm} = \frac{x \pm y}{\kappa}, \quad p = \exp\left(-\frac{\pi h}{\kappa Q_L}\right).$$

Here, $\nu_k = \omega_k/\omega_0$ (ω_k is the frequency of the cavity mode k), and Q_L is the loaded Q-value of the cavity. From Eqs (3.7) and (3.8) one can see that integer ratios h/κ are the roots of the decrements (increments) of all multi-bunch modes. Therefore, tuning of the cavity in exact resonance with the beam ($\nu_k = h$) eliminates this instability. In real operation, however, to compensate for the reactance due to beam loading and to minimize the reflected power, the frequency of the fundamental mode of the cavity must be shifted down the resonant frequency ω_{RF} to

$$\frac{\Delta\omega}{\omega_{RF}} = N_c \frac{I(Z/Q)}{2V \sin\varphi_s}, \tag{3.9}$$

where, $V\cos\varphi_s$ is the accelerating voltage. Due to this detuning, the phase advance of the wake of the fundamental mode on the bunch spacing gets a fractional part of 2π, which enhances both the increments of unstable modes and the decrements of the stable modes of the beam. One can see from Fig.7 that the minimum value of the growth time can reach the μs region. Calculations of the increments (decrements), assuming a definite

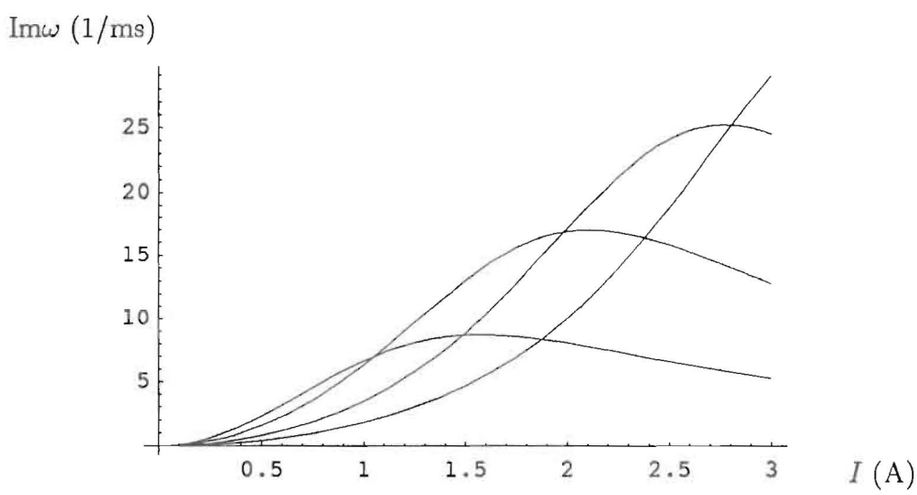

Figure 7: Dependence of increments of the dipole synchrotron coherent oscillations on the beam current. From the top to the bottom $a = 1023, 1022, 1021, 1020$; $\kappa = 5120$.

multipolarity of the synchrotron coherent oscillations m, gives reliable results only in the case when the values increments do not exceed the frequency of synchrotron oscillations ω_s. This condition limits the range of the detuning of ω_{RF} from the resonant value. For instance, the requirement that

$$\Omega < \omega_s/2$$

simultaneously demands that

$$\Delta\omega = \omega_{RF} N_c \frac{I(Z/Q)}{2V \sin\varphi_s} < \omega_0.$$

The fact that the beam has collective degrees of freedom makes its properties similar to that of a dielectric media [10]. In the case of the interaction with a cavity, this results in an increase of ω_k on the frequency shift, given by Eq.(3.9) (see in Appendix D). This effect must be taken into account in the injection and in the cavity tuning scenario.

As a general conclusion of this section we may state that except for the case when modification of the bunch-to-bunch coupling is used to cure multi-bunch instability, their damping demands an enhancement of the collective single-bunch stability. In spite of the necessity to damp only some unstable modes this can require the use a damping system with bandwidths corresponding to distances shorter than the bunch spacing in the beam.

4 Collective Beam-Beam Effects

Usually, the limitations due to a collective beam-beam instability are not so widely discussed like, say, the limitations predicted by a weak-strong analysis of beam-beam instability. This is probably due to the fact that an analysis of the strong-strong interaction and its first step – the collective stability of colliding bunches – is not easy for either analytic and numerical methods. Moreover, except for general expectations, there are not many direct experimental indications of the limitations due to a collective beam-beam instability. However, the understanding that the collective interaction of colliding bunches can dramatically shrink the stability diagram, if the rings have different perimeters [37,38], has completely canceled discussions of the B-factory schemes involving a short LER.

Both analytic and numerical studies of the collective beam-beam interaction are embarrassed by two complementary effects. The 1st is the strong tune dispersion across the bunch due to the nonlinearity of the beam-beam kick. This results in the necessity to take into account in calculations the Landau damping, which strongly complicates the equations. The 2d difficulty comes from the incoherent beam-beam interaction of the bunches. The calculation of the interference of coherent and incoherent beam-beam instabilities is very complicated problem even for numerical calculations, and is ignored in all analytic approaches. This means that predictions of such calculations should not invade in region of the incoherent beam-beam instability.

In the models, ignoring this interference and the Landau damping of modes, the stability of colliding bunches can be analyzed more or less easily. First, the rates of the time dependencies of the amplitudes of coherent oscillations are defined by $\xi\omega_0$, where ξ is the beam-beam parameter. This means that for the initial modes, say, the dipole and quadrupole can be rather fast. If the tunes of oscillations are far enough from the resonant values, for instance

$$m_x\nu_x + m_z\nu_z = n, \tag{4.1}$$

where m_x, m_z and n are integer numbers, the oscillations of bunches are stable, while their interaction results in a coherent frequency shift. The value of this shift $\omega_0\Delta\nu$ generally depends on the unperturbed distribution function and on the geometry of the mode. For example, for dipole oscillations it can be estimated by $\Delta\nu = 2\xi\Lambda$ with $\Lambda \simeq 1$. This fact can be used to measure ξ. However, the interpretation of such measurements requires

exact knowledge of the unperturbed distribution function (to calculate Λ) as well as the taking into account the nonlinearity of coherent beam-beam oscillations [39].

In the vicinity of resonances coherent oscillations of colliding bunches can be unstable. The strength of this instability strongly depends on the tunes of oscillations as well as on the symmetry of bunches. The interaction of identical bunches separates the modes, describing relative oscillations of bunches (π-modes), from modes, when bunches oscillate with the same phase (o-modes). The increments and widths of stopbands of the instability can be estimated using the dispersion equation (the sign + corresponds to π-mode)

$$1 = \pm\frac{2\Delta_m \Xi_m}{\nu^2 - \Delta_m^2}, \tag{4.2}$$

$$\Xi_m = \frac{m_x\xi_x(1+r) + m_z\xi_z(1+1/r)}{|m_x m_z|(m_x^2 r + m_z^2/r)}, \tag{4.3}$$

Here, $\Delta_m = m_x\nu_x + m_z\nu_z - n$; ξ_x and ξ_z are the partial beam-beam parameters:

$$\xi_z = \frac{Ne^2\beta_z^*}{2\pi E\sigma_z(\sigma_x + \sigma_z)}, \tag{4.4}$$

$$\xi_x = \frac{Ne^2\beta_z^*}{2\pi E\sigma_x(\sigma_x + \sigma_z)}, \tag{4.5}$$

σ_x and σ_z are the horizontal and vertical rms beam sizes and $r = \sigma_z/\sigma_x$ is the beam aspect ratio. According to Eq(4.2) the positions and the widths of stopbands in the plane (ν_x, ν_z) are defined by the following inequalities

$$|\Delta_m| < 2|\Xi_m|, \tag{4.6}$$

$$\Delta_m\Xi_m < 0, \quad \pi - \text{modes}, \tag{4.7}$$

and

$$\Delta_m\Xi_m > 0, \quad o - \text{modes} \tag{4.8}$$

Since the tunes in Δ_m include the incoherent beam-beam tune shift, for one-dimensional oscillations the stopbands of π-modes occur below and for o-modes – above of $\nu_0 = (n/m) - \xi$, where ν_0 is an unperturbed betatron tune. This means that the stopbands for o-modes can invade inside the stopband of the incoherent oscillations

$$(n/m) - \xi \leq \nu_0 \leq n/m.$$

In this case, more reliable predictions concerning the behaviour of colliding bunches can give the numerical analysis.

According to Eqs(4.3) and (4.6) the relative importance of various two-dimensional resonances depends on both the symmetry of the mode m_x, m_z and on the aspect ratio of the beam r. The instability of a mode with a multipole number of $\vec{m}$ is caused by a sum-type resonance of this mode and the $-\vec{m}$ mode . For flat bunches ($r \ll 1$) the widths of stopbands are mainly related to the excitation of vertical oscillations:

$$\Delta_0 \simeq \frac{2\xi_z}{m_z^2|m_x|}. \tag{4.9}$$

The broadest are sum-type resonance $\nu_x + \nu_x = 1$, and the main coupling resonance $\nu_x = \nu_z$

For round beams ($\sigma_x = \sigma_z$) there is no stopband along the main coupling resonance, while the most powerful two-dimensional is the linear sum-type resonance. In this case with an increase in the beam current the working point moves along the line $\nu_x = \nu_z$; its vicinity may become more preferable for the position of the working point of the ring.

The difference in transverse beam sizes, say $\sigma^{(1)} \gg \sigma^{(2)}$, generally suppresses the coherent beam beam instability [7]. For instance, for a water-bag distribution and one-dimensional oscillations with a multipole number of m, the value ξ in Eq.(4.2) must be replaced by

$$\xi_{eff} = \xi(\sigma^{(2)}/\sigma^{(1)})^m \ll \xi. \tag{4.10}$$

This suppression of the collective beam-beam instability for bunches with unequal transverse sizes can be one of the reasons for their flip-flop instability.

The calculation of the Landau damping for the beam-beam instability can be done more or less easily for horizontal oscillations of identical, very flat bunches which have a Gaussian distribution in the amplitudes of horizontal oscillations $\sqrt{J\beta}$. If $x = \sqrt{J/\varepsilon}$, where ε is the horizontal emittance of the bunch, and $X(x)$ is the Fourier-amplitude of the π-mode with a multipole number of m [7], then one can find that the function $w = \sqrt{x}X(x)$ satisfies the following differential equation [8]:

$$w'' + \left(2V_m(x) - \frac{m^2 - 1/4}{x^2}\right) w(x) = 0. \tag{4.11}$$

Here,

$$V_m(x) = \frac{4\Delta(x)\xi}{(\nu/m)^2 - \Delta^2(x)} e^{-x^2/2}, \tag{4.12}$$

$\omega_0\nu$ is an unknown frequency of the coherent mode, $\Delta(x) = \nu_x(x) - n/m = \Delta_0 + \Delta\nu_x(x)$,

$$\Delta\nu_x(J_x) = \frac{2\xi}{x^2}\left[1 - \exp(-x^2/2)\right], \quad \xi = \frac{Ne^2}{2\pi E\varepsilon}. \tag{4.13}$$

Although Eq(4.11) is still difficult to solve directly, many important properties of the eigenfunctions $w(x)$ and of the spectrum of eigenfrequencies ν can be obtained using the fact that Eq(4.11) has the form of the Schrödinger equation in quantum mechanics, which is written for a particle with zeroth energy, moving in an effective potential well

$$U_{eff}(x) = \frac{m^2 - 1/4}{x^2} - 2V_m(x). \tag{4.14}$$

[7] If we use, for instance, Vlasov's approach, when the collective oscillations of bunches are described by the harmonics of the distribution function in phases of incoherent oscillations ψ

$$f^{(1,2)} = f_0 + \sum_{m\neq 0} f_m^{(1,2)}(J)\exp(im\psi - i\omega t),$$

then

$$X(J) = f_m^{(1)}(J) - f_m^{(2)}(J).$$

Since $w(x)$ is concentrated between the stop-points of the mechanical problem when $U_{eff}(x_{1,2}) = 0$; and outside decreases like

$$w(x) = \begin{cases} x^{1/2+m}, & x \to 0 \ , \\ x^{1/2-m}, & x \to \infty \ , \end{cases}$$

the analysis of the stability problem can be reduced to an inspection of the variations of the potential curves with a variation of ν. In particular, this indicates (see in Fig.8) that at least for initial modes of $m = 1, 2, 3$ Landau damping does not completely suppress the instability. This result was recently confirmed by numerical calculations in [9].

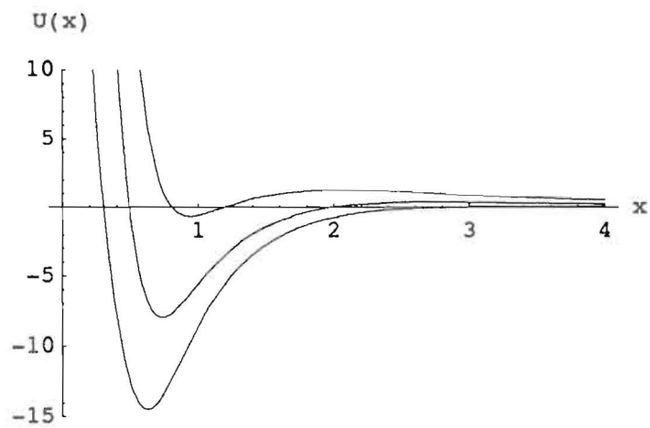

Figure 8: Effective potential wells for (from bottom to top) the dipole, quadrupole and sextupole modes; $\Delta_0/\xi = -1$; $\nu/\xi = 0.1i$.

As far as Eq.(4.11) presents a particular case of the more general Sturm-Liouville equation, its spectrum can be estimated using the general relationships. For instance, exact solutions $w(x) = w_j(x)$ minimize the integral

$$I[w] = \int_0^\infty dx \left[(w')^2 + \frac{m^2 - 1/4}{x^2} w^2 - 2V_m w^2 \right] \tag{4.15}$$

This yields the normalization condition for eigensolutions

$$\int_0^\infty dx \left((w_j')^2 + \frac{m^2 - 1/4}{x^2} w_j^2 \right) = 2, \tag{4.16}$$

and the dispersion equation of the problem

$$\int_0^\infty dx |w_j|^2 \frac{4\Delta(x)\xi}{(\nu/m)^2 - \Delta^2(x)} e^{-x^2/2} = 1, \quad \mathrm{Im}\nu > 0. \tag{4.17}$$

If eigenfunctions w_j are not known, both they and spectra can be found using minimization routines starting from more or less suitable set of probe functions $w(x)$.

Another issue, which can be especially important for B- and Φ-factories, is the influence of the bunch lengths on the beam-beam instability. As was previously mentioned, in the region $\beta^* \sim \sigma_s$, the strong modulation of the betatron phase on the IP suppresses the beam-beam resonances of synchronous particles [12]. Direct calculations [11] show that a similar suppression is valid for the coherent beam-beam instability when it is described by a renormalization of the beam-beam parameter:

$$\xi_{eff} = \xi Y_m(\sigma_s/\beta^*). \tag{4.18}$$

where for identical bunches with a round cross section

$$Y_m(\zeta) = \int_{-\infty}^{\infty} \frac{du}{\sqrt{\pi}} \exp[-u^2 + 2iq \arctan(\zeta u)], \tag{4.19}$$

$$q = (m_x + m_z), \quad \zeta = \sigma_s/\beta^*,$$

and for vertical oscillations of identical bunches with a flat cross section

$$Y_m^f(\zeta) = \int_{-\infty}^{\infty} \frac{du}{\sqrt{\pi}} \exp[-u^2 + 2im_z \arctan(\zeta u)]\sqrt{1+\zeta^2 u^2}, \tag{4.20}$$

In some cases the calculation of the functions Y_m and Y_m^f can be simplified by their representation in terms of the confluent hypergeometric function $\Psi(a,b,z)$ [40]

$$\Psi(a,b,z) = \frac{1}{\Gamma(a)} \int_0^{\infty} dt t^{a-1}(1+t)^{b-a-1} e^{-zt}.$$

Direct calculations (see in Appendix E) yield

$$Y_m(\zeta) = \frac{(2m)!}{4^m} \sum_{k=0}^{m} (-1)^{m-k} \frac{(4/\zeta^2)^k}{(2k)!(m-k)!} \Psi(m, k+1/2, 1/\zeta^2) \tag{4.21}$$

and

$$Y_m^f(\zeta) = \frac{(2m)!\zeta}{4^m} \sum_{k=0}^{m} (-1)^{m-k} \frac{(4/\zeta^2)^k}{(2k)!(m-k)!} \Psi(m-1/2, k, 1/\zeta^2). \tag{4.22}$$

One can see from Eqs(4.19) and (4.20) that for round beams due to $|Y_m| \leq 1$, the modulation of the betatron phase along the IR always suppresses coherent resonances. For flat bunches due to a mismatching between the modulation of the beam-beam kick and the beam cross section we can write

$$|Y_m^f| \leq \sqrt{1 + (\sigma_s/\beta^*)^2}.$$

Hence, the modulation can amplify the coherent beam-beam kick for very long bunches. However, Fig.9 shows that the region $\sigma_s \simeq \beta^*$ is quite safe for the vertical coherent oscillations of flat bunches. Schematically, this can be demonstrated by Fig.10, where we plotted $\mathcal{L}/\mathcal{L}_0$ assuming that ξ in Eq.(1.1) is limited by the resonance $m = 4$. One can see that $\mathcal{L}/\mathcal{L}_0$ significantly exceeds 1 in the region $0.5 \leq \sigma_s/\beta^* \leq 2$.

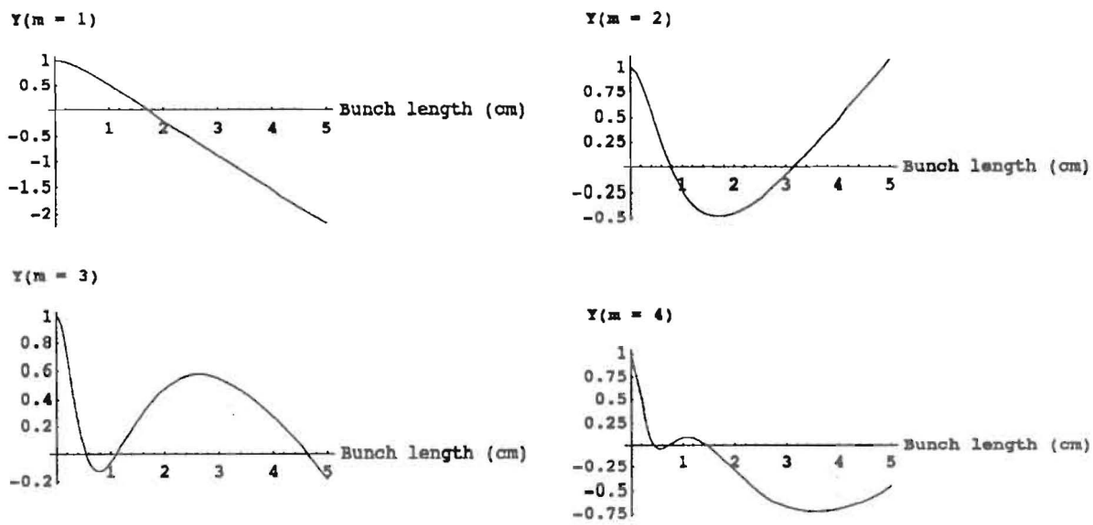

Figure 9: Dependence of Y_m for flat bunches on the bunch length; $\beta^* = 1$ cm.

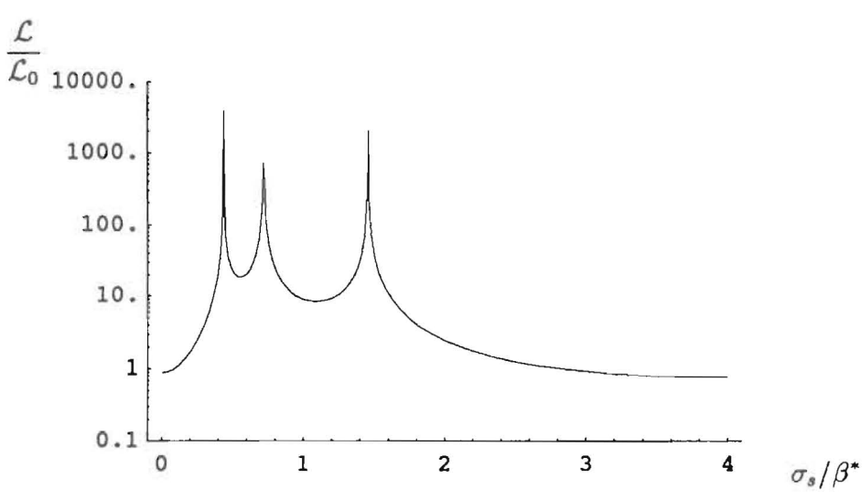

Figure 10: Dependence of the luminosity on the bunch length; limited by the 4-th order resonance.

Since an incoherent tune shift does not depend on the phase advance of oscillations on the IR, the described suppression of coherent beam-beam resonances in the region $\sigma_s \simeq \beta^*$ will enhance the Landau damping of the instability in round beams for both vertical and horizontal oscillations. For flat beams, due to the fact that $\beta_z^* \simeq \sigma_s \ll \beta_x^*$ this will be valid for only vertical oscillations; horizontal coherent beam-beam oscillations, however, will still be unstable. This fact gives one more dynamical advantage to round colliding bunches.

5 Instabilities Due To Ions

When the beam moves along a closed orbit the collisions of particles with atoms of the residual gas produce positively charged ions. Among others possibilities, the interaction with these ions can cause various instabilities of coherent oscillations of the beam. The specific features of these instabilities strongly depend on the lifetime of the ion inside the beam and, in particular, on the possibility for ions to be trapped by the beam. By this reason, we discuss below the issues related to the interaction with ions in electron and positron rings separately.

5.1 Ions in an Electron Ring

A train of electron bunches presents for ions a sequence of focusing lenses, separated by time intervals, in which ions are defocused due to their space charge. Provided that the betatron oscillations of the ions are stable, they can be trapped in the beam and can perturb the motion of electrons. Generally, two effects are associated with the ion-trapping. The first is a tune shift and spread due to ions, which can be estimated by (see, for instance, [41])

$$\Delta\nu = \frac{N_i r_0}{2\pi\gamma\varepsilon}, \quad \varepsilon = \sqrt{\varepsilon_x \varepsilon_z}, \tag{5.1}$$

where N_i is the number of the stored ions. For realistic parameters [41] the value of $\Delta\nu$ can reach .05. Due to the nonlinearity of the Coulomb force this effect increases the Landau damping of coherent oscillations of an electron beam and, in this sense, can be considered as being a positive one. On the other hand, since Landau damping dilutes the phase-space volume of the beam (see, for example, in [10]) it generally can limit the performance of the ring.

However, stronger limitations can be caused by the interaction of stored ions with coherent oscillations of the beam. Although the description of instabilities due to this interaction is very speculative, some of its general properties can be predicted using the analogy of this instability and the instability of colliding bunches. In particular, we may expect the unstable coherent oscillations below the resonances

$$m_z\nu_z + m_i\nu_i = n, \quad m_z m_i > 0, \tag{5.2}$$

where integers $m_{z,i}$ define the multipole numbers of collective modes of the electron and

ion beams. The increments of this instability would be proportional to [7]

$$\delta_0 \propto \omega_0 \sqrt{\Delta\nu\xi_i} \propto \omega_0 \sqrt{\frac{N N_i r_0 r_p}{A_i \gamma \varepsilon^2}}, \tag{5.3}$$

and will slowly depend on the multipole numbers (m_z, m_i). Here, A_i is the atomic weight of the ion and $r_p = 1.5\ 10^{-16}$ cm is the classical radius of the proton. For the reasons described in the previous section, we can expect that such an instability will be hardly suppressed by Landau damping. Since the configuration of electron and ion beams, which have different transverse sizes, is more stable [7], we may also expect the flip-flop instability of the beams.

Ions will definitely not be accumulated inside the beam if the bunch spacing is chosen to make the betatron oscillations of the ions unstable. This criterion defines the critical ratio $\mathcal{R} = N_b/\kappa$ (see, for instance in [41,42]) when the ions are swept out of the beam due to their overfocusing:

$$\mathcal{R} > \mathcal{R}_0 = \frac{2A_i \sigma_z \sigma_x (1+r)}{r_p \Pi}. \tag{5.4}$$

Here, $r = \sigma_z/\sigma_x$ is the bunch aspect ratio. If we take, for example, $A_i \simeq 30$ (CO) and $\varepsilon \simeq 2\ 10^{-7}$ cm, Eq.(5.4) yields $\mathcal{R}_0 = 1.3\ 10^9$. This value significantly exceeds the corresponding ratios for all of the presently discussed B-factories and, without special efforts, ions definitely will be trapped in electron rings. Additional cleaning can be achieved by using either clearing field electrodes, or missing the necessary amount of bunches from the train. As was mentioned, the matched clearing field electrodes can be used as a single bunch dampers of coherent oscillations. In most design reports simpler method with the missing of roughly 10% of bunches from the beam is used to prevent ion trapping.

5.2 Ions in a Positron Ring

Positively charged ions cannot be trapped in a positron beam. However, the ions, once produced by a bunch, cannot be removed from the orbit immediately. As long as the ion trace of the bunch carries information about its coherent oscillations, it can cause at least a multi-bunch instability of the beam. Since the instability of this kind is not associated with the trapping of ions, it can be important for electron beams in cases when the motion of ions is unstable. In the estimations given below we use the results of a paper [43] where this problem was studied in many detail for proton bunches.

Once ions are not stored in the beam, it is obvious that the strength of the instability significantly depends on the pressure of the residual gas in the ring and on the loss factor of the ions from the beam. The first factor defines the ion production rate, while the second defines the bunch-to-bunch memory. In the case of positron or proton beams, the most important aspect is the deflecting of ions from the orbit by the beam space-charge. If we assume that at low pressure the residual gas mainly consists of hydrogen atoms, the rate of the increase in the radius of the ion spot can be estimated by the dimensionless increment

$$\nu_i \simeq \sqrt{\frac{2N r_p \nu}{\varepsilon}}, \quad \varepsilon \sim \sqrt{\varepsilon_x \varepsilon_z}. \tag{5.5}$$

Then, the standard calculations of the multi-turn part of the increments of coherent oscillations of a single bunch results in [43]

$$\delta_m = (n_A c\sigma_i)\Lambda_m \frac{N_b e^2}{E\varepsilon} 8\pi \exp[-2\pi\nu_i] m \sin(2\pi m\nu_z). \tag{5.6}$$

Here, $n_A \simeq 4\ 10^{16} P$ [Torr] is the density of the residual gas, σ_i is the ion production cross section, m is the multipole number of the coherent mode, and

$$\Lambda_m \simeq \frac{2}{\pi}\frac{\psi(m-1/2)}{m^2}, \quad \psi(x) = \frac{d\ln\Gamma(x)}{dx}. \tag{5.7}$$

For the sake of simplicity we write the decrement of one-dimensional (vertical) coherent oscillations. Eq.(5.6) indicates at least two important features of the discussed instability. The first is that in the working plane ν_x, ν_z the position of the stopbands of this instability is inverse to that of the resistive wall instability. For instance, one dimensional oscillations are stable, when

$$k - 1/2 \leq |m|\nu_z \leq k, \quad k = 1, 2, \ldots \tag{5.8}$$

The second is that due to a strong nonlinearity of the Coulomb force the increments slowly depend on the multipole mode number m, and, apart from the dipole, some initial higher multipole modes can contribute in the limitations.

Now, combining Eqs (5.6) and (3.3) yields the decrements of multi-bunch modes:

$$\delta_{m,a} = (n_A c\sigma_i)\Lambda_m \frac{8\pi N e^2}{E\varepsilon} \exp\left(-\frac{2\pi\nu_i}{\kappa}\right) m \sin\left(\frac{2\pi[m\nu_z + a]}{\kappa}\right), \tag{5.9}$$

$$a = 0, \ldots, \kappa - 1.$$

For the LER with a vacuum of $P = 10^{-9}$ Torr ($n_A \simeq 4\ 10^7$ cm^{-3}), and the data, given in Table, Eq.(5.9) estimates the increment of the dipole mode by $\delta_1 \simeq 12$ 1/s for a bunch spacing of 3 m, and by $\delta_1 \simeq 10^4$ 1/s for a bunch spacing of .5 m. The first number is smaller than the decrements due to synchrotron radiation damping and hence, is quite acceptable. In the second case the instability must be suppressed by a suitable combination of feedback damping and a decrease of the pressure in the ring. We point out that due to slow dependence of increments in Eq.(5.9) on the mode number m, the feedback system must ensure the damping except for the dipole also quadrupole and maybe sextupole modes. The relevant gap in the bunch train definitely must help against this instability. However, this possibility must be studied more carefully, because the breaking of the feedback between the head-on and tail-on bunches still does not prevent a beam breakup instability of the train.

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A Stability Diagrams of Betatron Coherent Oscillations

The stability diagram of coherent oscillations is defined by the dispersion equation, when the frequency of oscillations tends to the real axes ($\mathrm{Im}\omega \to +0$). For a bunch interacting with surrounding electrodes, the dispersion equation for, say, vertical dipole betatron coherent oscillations reads

$$1 = -\Omega_m \int dJ_x dJ_z \frac{J_z \partial f_0/\partial J_z}{\omega - \omega_z(J_x, J_z)}, \quad \mathrm{Im}\omega > 0. \tag{A.1}$$

Here, $f_0(J_x, J_z)$ is the stationary distribution function of the beam. If we define the frequency distribution function

$$g(\omega) = -\int dJ_x dJ_z J_z \frac{\partial f_0}{\partial J_z} \delta(\omega - \omega_z(J_x, J_z)), \tag{A.2}$$

we can rewrite Eq.(A.1) in a form similar to that of Eq.(2.10)

$$1 = \Omega_m \int_{-\infty}^{\infty} dw \frac{g(w)}{\omega - w}, \quad \mathrm{Im}\omega > 0. \tag{A.3}$$

The solution of Eq.(A.3) yields the unknown frequencies of the collective modes $\omega = \omega_m$.

Another way to use Eq.(A.3) is to plot stability diagrams of coherent oscillations – i.e. curves in the plane of the complex variable Ω_m, which separate the region containing stable solutions ($\mathrm{Im}\omega_m < 0$) from the region containing unstable solutions. The parametric plot of this curve is defined by Eq.(A.3), when ω tends to the real axes from above ($\omega \to \omega + i0$) and when Eq.(A.3) yields Eq.(2.10)

$$\frac{1}{\Omega_m(\omega)} = \int_{-\infty}^{\infty} dw \frac{g(w)}{\omega - w + i0}. \tag{A.4}$$

The calculation of the particular stability diagrams demands knowledge of the stationary distribution function f_0 and the dependence of the frequency of incoherent oscillations on amplitudes J_x and J_z. If, for instance,

$$f_0 = \frac{1}{4\varepsilon_x\varepsilon_z}\exp\left(-\left[\frac{J_x}{2\varepsilon_x}+\frac{J_z}{2\varepsilon_z}\right]\right),$$

and the frequency spread of the vertical oscillations is defined by the cubic nonlinearity and, moreover,

$$\left|\frac{\partial\nu_z}{\partial J_x}\varepsilon_x\right| \gg \left|\frac{\partial\nu_z}{\partial J_z}\varepsilon_z\right|, \tag{A.5}$$

Eqs(A.2) and (A.3) yield

$$\frac{\delta\omega}{\Omega_m(\omega)} = F_0(y) = e^{-y}\left[\mathrm{Ei}(y) - i\pi\theta(y)\right], \tag{A.6}$$

where

$$y = \omega/\delta\omega \quad \delta\omega = \frac{\partial\omega_z}{\partial J_x}2\varepsilon_x,$$

$\mathrm{Ei}(x)$ is the exponential integral function

$$\mathrm{Ei}(x) = -\int_{-x}^{\infty} dt\frac{e^{-t}}{t},$$

and

$$\theta(x) = \begin{cases} 1, & x > 0\ , \\ 0, & x < 0\ . \end{cases}$$

Similar calculations result in the stability diagram for the horizontal coherent oscillations:

$$\frac{\delta\omega}{\Omega_m(\omega)} = F_1(y) = yF_0(y) - 1. \tag{A.7}$$

The case when the nonlinearity of the beam-beam force defines the frequency spread of the beam is more difficult due to a more complicated dependence of the tunes on the amplitudes of incoherent oscillations. Apart from the possibility of numerical integration in Eq.(A.1), the main features of the stability diagram can be found using the simplified model, where

$$\omega_z = \omega_{z0} + \omega_0\frac{\xi}{1+J/\varepsilon}. \tag{A.8}$$

Substitution of this expression in Eq.(A.1) yields

$$\frac{\omega_0\xi}{\Omega_m} = F_{bb}(y), \quad y = \frac{\omega-\omega_{z0}}{\omega_0\xi},$$

where

$$F_{bb} = \frac{1}{y}+\frac{1}{y^2}-\frac{1-y}{y^3}\exp\left(-\frac{1-y}{y}\right)\left[\mathrm{Ei}\left(\frac{1-y}{y}\right)+i\pi\theta(y)\theta(1-y)\right]. \tag{A.9}$$

B Weak Bunch Lengthening

We start with the equations of the synchrotron oscillations of the particle, which are distorted by the bunch wake:

$$\frac{d\Delta\omega}{dt} = -\omega_{s0}\varphi - 2\lambda\Delta\omega - \frac{N_b e^2\omega_0^2\alpha}{\Pi p}\int_{-\infty}^{\infty} dn Z_n \rho_n (e^{in\varphi} - 1) + X(t), \tag{B.1}$$

$$\frac{d\varphi}{dt} = \Delta\omega, \quad \Delta\omega = -\frac{\omega_0\alpha}{E}\Delta E, \quad \lambda = \frac{1}{2}\frac{\partial W}{\partial E_s}. \tag{B.2}$$

Here, φ is the deviation of the particle's phase from the synchronous value φ_s,

$$\omega_{s0}^2 = \omega_0^2 \frac{heV|\sin\varphi_s|\alpha}{2\pi E} \tag{B.3}$$

is the frequency of unperturbed synchrotron oscillations, $W(E)$ is the average power of the synchrotron radiation of the particle and $X(t)$ is the random force describing the excitations of synchrotron oscillations due to quantum fluctuations of the synchrotron radiation. Below we use the following properties of X:

$$< X(t) >= 0, \quad < X(t)X(t') >=< X^2 > \delta(t - t'). \tag{B.4}$$

Assuming that the wake is a slow function of φ, and can, therefore, be expanded in a Taylor series, we can write in the linear approximation

$$\ddot{\varphi} + 2\lambda\dot{\varphi} + \omega_s^2\varphi = X(t), \tag{B.5}$$

$$\omega_s^2 = \omega_{s0}^2 \left\{ 1 - i\frac{N_b e^2\omega_0}{heV|\sin\varphi_s|}\int_{-\infty}^{\infty} dn Z(n) n\rho(n) \right\}. \tag{B.6}$$

The driven solution of this equation has the form

$$\varphi(t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{X(\omega)e^{-i\omega t}}{\omega_s^2 - 2i\lambda\omega - \omega^2}, \tag{B.7}$$

or

$$\varphi(t) = \int_0^t dt_1 X(t_1) e^{-\lambda(t-t_1)} \frac{\sin\omega_s(t - t_1)}{\omega_s}. \tag{B.8}$$

Its mean square has the asymptote

$$< \varphi^2(t \to \infty) >= \frac{1}{\omega_s^2}\frac{< X^2 >}{2\lambda}. \tag{B.9}$$

Since for Gaussian distribution functions we have $< \varphi^2 >= \sigma_\phi^2$, and defining

$$\sigma_0^2 = \frac{1}{(\omega_s^2)_0}\frac{< X^2 >}{2\lambda} \tag{B.10}$$

as the unperturbed bunch length, we can write

$$\frac{\sigma_\phi^2}{\sigma_0^2} = \frac{(\omega_s^2)_0}{\omega_s^2} = \frac{1}{\left[1 - i\frac{N_b e^2 \omega_0}{heV|\sin\varphi_s|}\int dn Z(n) n \rho(n)\right]}. \tag{B.11}$$

For more detailed calculations we need some particular expressions for Z/n and $\rho(n)$. Taking, for instance, [8]

$$\frac{-iZ}{n} = (Z/n)_0 \frac{n_0^2}{n^2 + n_0^2}, \quad \rho(n) = \exp\left(-\frac{n^2\sigma_\phi^2}{2}\right), \tag{B.12}$$

we can estimate the integral in Eq.(B.11) for short bunches ($n_0\sigma_\phi \ll 1$) as follows:

$$\int_{-\infty}^{\infty} dn \left(\frac{-iZ(n)}{n}\right) n^2 \rho(n) \simeq \frac{\sqrt{2\pi}}{\sigma_\phi} n_0^2 (Z/n)_0.$$

Substituting this expression into Eq.(B.11), we can write

$$\frac{\sigma_\phi^2}{\sigma_0^2} = \frac{1}{\left[1 + \sqrt{2\pi}\frac{eI_b(Z/n)_0}{heV|\sin\varphi_s|}\frac{\Pi}{\sigma_s}\frac{R_0^2}{l_\perp^2}\right]}. \tag{B.13}$$

From Eq.(B.13) one may expect the bunch lengthening for the case that $(Z/n)_0 < 0$, i.e. when the bunch interacts with an inductive-type impedance; vice verse, bunch shortening may occur when the bunch interacts with a capacitive-type impedance. This result is not very surprising, since the inductive-type coupling decreases and capacitive, thus increases the rigidity of the potential well due to a driving RF. If the bunch lengthening is small, the expansion of Eq.(B.13) in a linear approximation yields Eq.(2.19).

C On the Iteration of the Haissinski Equation

Let us prove that the nonlinear Haissinski equation has a unique solution, if the bunch wake satisfies the condition in Eq.(2.24). We also can rewrite this condition as follows

$$\overline{w^2} = 2\pi \left(\frac{N_b e^2 \omega_0}{heV \sin\varphi_s \sigma_0^2}\right)^2 \int_{-\infty}^{\infty} \frac{dn}{n^2} |Z_n|^2 < \infty. \tag{C.1}$$

[8] Note that, if $\rho(\varphi)$ is an even function of φ, only reactive part of Z is important

First, we write Eq.(2.22) in the form

$$\begin{aligned}
\rho(x)/\rho_0 = &\ \exp(-x^2/2 - u(x)), \quad x = \varphi/\sigma_0, \\
u_0 = &\ 0, \\
u_1(x) = &\ \int_x^\infty dx_1 w(x_1 - x) e^{-x_1^2/2}, \\
&\ \vdots \\
u_{n+1}(x) = &\ \int_x^\infty dx_1 w(x_1 - x) \exp(-x_1^2/2 - u_n(x_1)).
\end{aligned} \tag{C.2}$$

Due to condition (2.24), the function $u(x)$ has the following properties. It is limited

$$\begin{aligned}
u^2(x) &\le \int_x^\infty dx_1 w^2(x_1 - x) \int_x^\infty dx_1 \frac{\rho^2(x_1)}{\rho_0^2} \\
&\le \mathcal{N}\overline{w^2} = u_{\max}^2,
\end{aligned} \tag{C.3}$$

$$\mathcal{N} = \int_{-\infty}^\infty dx_1 \frac{\rho^2(x_1)}{\rho_0^2}, \tag{C.4}$$

and can be normalized:

$$\begin{aligned}
\int_{-\infty}^\infty dx u^2(x) \le &\ \exp(2u_{\max}) \int_{-\infty}^\infty dx u_1^2(x) \\
\le &\ \exp(2u_{\max}) \int_{-\infty}^\infty \frac{dk}{2\pi} |w(k)|^2 e^{-k^2}, \\
w(x) = &\ \int_{-\infty}^\infty \frac{dk}{2\pi} e^{ikx} w(k).
\end{aligned} \tag{C.5}$$

This, in particular, means that the potential well distortion $u(x)$ is distributed within the finite interval of x with a width of

$$\Delta x \sim \left(\overline{w^2}\right)^{1/4}. \tag{C.6}$$

If $|\Delta x| \gg 1$, this relation estimates the lengthening of the bunch as follows:

$$\sigma_s \sim R_0 \sqrt{\frac{eI_b(Z/n)_0}{heV \sin\varphi_s} \frac{\Pi}{l_\perp}}. \tag{C.7}$$

Since the function $|u(x)| \le u_{\max}$ is limited, we can write

$$|e^{-u_1} - e^{-u_2}| \le e^{2u_{\max}} |u_1 - u_2|. \tag{C.8}$$

Then, the convergency of iterations in Eq.(C.2) can be proved using the standard calculations [27], which result in

$$|u_{n+1} - u_n| \propto \frac{\left(\sqrt{\overline{w^2}}\right)^n}{n!}, \quad n \gg 1. \tag{C.9}$$

This expression shows that the iterations in Eq.(C.2) converge and, thus, Eq.(2.22) has a unique solution [27], provided that the condition in Eq.(2.24) is valid. On the other hand, this estimation indicates that, if the potential well distortion is high ($\overline{w^2} \gg 1$), higher order corrections in the iteration row

$$u(x) = u^{(1)} + (u^{(2)} - u^{(1)}) + \ldots + (u^{(n+1)} - u^{(n)}) + \ldots \tag{C.10}$$

will decay after

$$n \gg n_0 \sim \sqrt{\overline{w^2}}. \tag{C.11}$$

We also note that the discussed estimation concerning the convergency of the iterations in Eq.(C.2) is based on a rather rough evaluation the integrals involved in these equations. Therefore, Eq.(2.24) specifies the condition when Eq.(2.22) has a solution beforehand. Its violation does not necessarily means that Eq.(2.22) has no solutions.

D Dynamic Beam Loading

In this section we calculate the modification of the impedance of the fundamental mode of the cavity from the side of the fundamental coupler due to a collective reaction of the beam. For the sake of simplicity, we do the calculations for a single bunch.

First, we write the longitudinal component of the vector-potential of the fundamental mode in the form

$$A(\mathbf{r}, t) = cA_k(\mathbf{r})q_k(t), \tag{D.1}$$

and assume that the eigenfunctions A_k obey the equation

$$\Delta A_k(\mathbf{r}) + \frac{\omega_k^2}{c^2} A_k(\mathbf{r}) = 0, \quad \int d^3\mathbf{r}|A_k(\mathbf{r})|^2 = 4\pi c. \tag{D.2}$$

Here, ω_k is the frequency of mode k. For the mode with the Q-value equal to $Q = \omega_k/(2\lambda_k)$ from Maxwell equations we find

$$\ddot{q}_k + 2\lambda_k \dot{q}_k + \omega_k^2 q_k = Ne \int d\Gamma f A_k^*(s) + j_k^{ext}(t), \quad d\Gamma = R_0 d\Delta p d\varphi, \tag{D.3}$$

where the current $j_k^{ext}(t) = j_k^{ext} \exp(-i\omega_{RF} t)$ describes an external excitation of the cavity. Below we use the expansions:

$$A_k(s) = \sum_n A_{kn} \exp(in\theta), \quad s = R_0\theta = ct + \varphi,$$

and

$$f(I,\psi,t) = \sum_m f_m(I,t)\exp(im\psi),$$

where I and ψ are the action-phase variables of the synchrotron oscillations. For synchrotron oscillations the variable I and the amplitude of oscillations φ_c are related through

$$I = \frac{E\nu_s}{2\omega_0\alpha}\varphi_c^2. \tag{D.4}$$

From the Vlasov equation, written in action-phase variables ($\gamma\alpha > 1$)

$$\frac{\partial f_m}{\partial t} + im\omega_s f_m = im\frac{\partial f_0}{\partial I}ecq_k(t)\sum_n A_{kn}J_m(n\varphi_c)\exp(in\omega_0 t), \tag{D.5}$$

we express the amplitudes of the expansion

$$f_m = \sum_n f_{mn}\exp(-i\omega_{RF}t + in\omega_0 t), \tag{D.6}$$

through the the Fourier-harmonic of $q_k(\omega_{RF})$:

$$f_{mn} = -\frac{m(\partial f_0/\partial I)}{\omega_{RF} - n\omega_0 - m\omega_s}ecA_{kn}J_m(n\varphi_c)q_k(\omega_{RF}). \tag{D.7}$$

Now, we rewrite Eq.(D.3) in the form

$$q_k(\omega_{RF})[\omega_k^2 - 2i\lambda_k\omega_{RF} - \omega_{RF}^2] = 2\pi Ne^2\sum_m\sum_n A_{kn}^*\int_0^\infty dI f_{mn}J_m(n\varphi_c) + j_k^{ext}. \tag{D.8}$$

Substituting in this equation f_{mn} from Eq.(D.7), we obtain

$$\begin{aligned} &q_k(\omega_{RF})[\omega_k^2 - 2i\lambda_k\omega_{RF} - \omega_{RF}^2] = j_k^{ext} + q_k(\omega_{RF})\times \\ &2\pi Ne^2c\sum_{m=1}^\infty\sum_n\frac{2m^2\omega_s|A_{kn}|^2}{(\omega_{RF} - n\omega_0)^2 - m^2\omega_s^2}\int_0^\infty d\varphi_c\varphi_c\frac{\partial f_0}{\partial I}J_m^2(n\varphi_c). \end{aligned} \tag{D.9}$$

Since $\omega_{RF} = h\omega_0$, the contribution of the resonant term $n = h$ in the sum over n dominates. Neglecting the contributions of the nonresonant terms and using the relationships

$$|A_{kn}|^2 = \frac{\omega_0\omega_k}{\Pi}\left(\frac{Z}{Q}\right), \tag{D.10}$$

$$\frac{\partial f_0}{\partial I} = \frac{\omega_0\alpha}{E\nu_s}\frac{\partial f_0}{\partial(\varphi_c^2/2)} = -\frac{\omega_0\alpha}{E\nu_s}\frac{\exp(-\varphi_c^2/2\sigma_\phi^2)}{2\pi\sigma_\phi^4}, \tag{D.11}$$

we obtain for short bunches $h\sigma_\phi \ll 1$:

$$q_k(\omega_{RF})\left[\omega_k^2 + \omega_k\frac{eI(Z/Q)}{T_0E}\frac{\alpha h^2}{\nu_s^2} - 2i\lambda_k\omega_{RF} - \omega_{RF}^2\right] = j_k^{ext}. \tag{D.12}$$

This equation shows that due to a collective reaction of the beam the frequencies of all modes of the cavity (including the fundamental one) become higher than ω_k on the amount, defined by Eq.(3.9) . Without special efforts this frequency shift decreases both the accelerating voltage, induced by the fundamental coupler, and the voltage, which the beam induces during its stationary rotation along the orbit (the stationary beam loading).

E Calculation of Collective Beam-Beam Suppression Factors

The factors $Y_m(\zeta)$ (with $\zeta = \sigma_s/\beta^*$), describing the suppression of the collective beam-beam resonances, can be expressed in terms of the confluent hypergeometric function $\Psi(\alpha, \gamma, z)$ [40]. This can sometimes simplify the estimation and plotting of the function $Y_m(\zeta)$. Let us start with calculations for the flat beam. Using

$$\sin(\arctan(x)) = \frac{x}{\sqrt{1+x^2}}, \qquad \cos(\arctan(x)) = \frac{1}{\sqrt{1+x^2}},$$

we can rewrite Eq.(4.20) in the form

$$Y_m^f(\zeta) = \int_{-\infty}^{\infty} \frac{du}{\sqrt{\pi}} \frac{e^{-u^2}}{(1+\zeta^2 u^2)^{m-1/2}} \mathrm{Re}[(1+i\zeta u)^{2m}]. \tag{E.1}$$

Then, using

$$\mathrm{Re}[(1+i\zeta u)^{2m}] = \sum_{k=0}^{m} \frac{(2m)!}{(2k)!(2m-2k)!}(-1)^{m-k}(\zeta u)^{2(m-k)}, \tag{E.2}$$

and the substitution

$$\frac{1}{(1+\zeta^2 u^2)^{m-1/2}} = \frac{1}{\Gamma(m-1/2)} \int_0^{\infty} ds s^{m-3/2} e^{-s(1+\zeta^2 u^2)}, \tag{E.3}$$

we can rewrite Eq.(E.1) in the form

$$Y_m^f(\zeta) = \frac{2}{\sqrt{\pi}} \sum_{k=0}^{m} \frac{(2m)!}{(2k)!(2m-2k)!}(-1)^{m-k}\zeta^{2(m-k)} W_{m,k}, \tag{E.4}$$

where

$$W_{m,k} = \frac{1}{\Gamma(m-1/2)} \int_0^{\infty} du u^{2(m-k)} e^{-u^2} \int_0^{\infty} ds s^{m-3/2} e^{-s(1+\zeta^2 u^2)}. \tag{E.5}$$

Now, upon changing in this equation the order of integration and by substituting $u^2 = t$, we find that the integral over u yields

$$\int_0^{\infty} du u^{2(m-k)} e^{-u^2(1+\zeta^2 s)} = \frac{\Gamma(m-k+1/2)}{2(1+\zeta^2 s)^{m-k+1/2}}.$$

Substituting this expression into Eq.(E.5) gives

$$W_{m,k} = \frac{\Gamma(m-k+1/2)}{2\zeta^{2m-1}\Gamma(m-1/2)} \int_0^{\infty} dt t^{m-3/2}(1+t)^{-m+k-1/2} e^{-t/\zeta^2} \tag{E.6}$$

or [40]

$$W_{m,k} = \frac{\Gamma(m-k+1/2)}{2\zeta^{2m-1}}\Psi(m-1/2,k,1/\zeta^2). \tag{E.7}$$

By also using

$$\frac{\Gamma(m-k+1/2)}{2(2m-2k)!} = \frac{\sqrt{\pi}/2}{(m-k)!4^{(m-k)}},$$

we obtain

$$Y_m^f(\zeta) = \frac{(2m)!\zeta}{4^m}\sum_{k=0}^{m}(-1)^{m-k}\frac{(4/\zeta^2)^k}{(2k)!(m-k)!}\Psi(m-1/2,k,1/\zeta^2). \tag{E.8}$$

Similar calculations for the round beam result in

$$Y_m(\zeta) = \frac{(2m)!}{4^m}\sum_{k=0}^{m}(-1)^{m-k}\frac{(4/\zeta^2)^k}{(2k)!(m-k)!}\Psi(m,k+1/2,1/\zeta^2). \tag{E.9}$$

Table 1: Some Parameters of the B-Factories Projects.

	BINP(Nov-sk)	Cornell	SLAC	KEK	
Energy (GeV)	7/4	8/3.5	9/3.1	8/3.5	8/3.5
Perimeter Π (m)	765.0	765.0	2199	3018	1273
Vertical emittance ε_z (nm)	.91	1.95	1.9/3.9	.19	.19
Horizontal emittance ε_x (nm)	5.8/4.0	130	48/96	19	19
$\Delta p/p$ ($\times 10^4$)	10	8.4/6.5	6.1/9.5	7.2/7.7	7.2/7.9
β_V^* (cm)	.9	1.5	3/1.5	1	1
β_H^* (cm)	40	100	75/37	100	100
σ_s (cm)	.7	1	1	.5	.5
ξ_V	.05	.03	.03	.05	.05
ξ_H	.017	.03	.03	.05	.05
$N_b(\times 10^{-10})$	11/19	6/13.7	4/6	1.4/3.3	1.4/3.3
Bunch spacing (m)	4.2	3	1.26	.6	.6
Beam current (A)	1.2/2.1	.9/2	1.5/2.1	1.1/2.6	1.1/2.6
Number of bunches	177	230	1658	4950	2090
Vertical damping time τ_z^{SR} (ms)	11/33	7.8/24	37/36	16/32	16/32
Horizontal damping time τ_x^{SR} (ms)	5.9/33	7.8/24	37/36	16/32	16/32
Long. damping time τ_s^{SR} (ms)	10/17	3.9/12	19/18	8.3/16	8.3/16
Vertical tune ν_z	14.27/15.27		24.18/35.18	39.19	25.19/26.19
Horizontal tune ν_x	27.58/28.58		25.28/32.28	39.15	25.15/27.15
Synchrotron tune ν_s	.025	.085	.05	.07	.047
Momentum compaction ($\times 10^3$)	1.44	8.4/11	2.4/1.5	1	1.8/1.5
RF frequency (MHz)	500	500	476	508	508
RF voltage (MV)	15.4/8.61	35/12	18/19.5	48/22	35/15
Harmonic number	1260		3492	5120	2160
Number of cavities	12/4		20/10	56 (8)	56 (8)
Luminosity ($\times 10^{34}$ cm^{-2}s^{-1})	1	.3	.3	1	1