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ON QUANTIZATION OF LIOUVILLE THEORY AND RELATED CONFORMAL FIELD THEORIES

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Zusammenfassung

Themen dieser Arbeit sind die Quantisierung der Liouville-Theorie im Operatorformalismus und die Klärung der Zusammenhänge zwischen quantisierter Liouville-Theorie und und verwandten konformen Feldtheorien. Analyse der klassischen Theorie und bestimmte Probleme bisheriger, auf der Verwendung von Fock-Räumen beruhender Quantisierungsverfahren motivieren eine Definition des Zustandsraumes mittels irreduzibler Darstellungen der Virasoro-Algebra. Es wird ein geeigneter Operatorformalismus basierend auf chiralen Vertex-Operatoren als elementaren Objekten vorgestellt. Eine Reformulierung dieses Formalismus in Termen von Fusionsprodukten erlaubt es, sowohl lokale wie globale Aspekte von Korrelationsfunktionen transparent zu machen. Fusions- und Braidmatrizen werden mithilfe einer Realisierung durch freie Felder berechnet und auf Einhaltung der Fusionsregeln untersucht. Sie erweisen sich als durch die Racah-Wigner-Koeffizienten der Quantengruppe $\mathcal{U}_q(sl(2))$ darstellbar. Die Bedeutung des Zusammenhangs zwischen Quantengruppen und konformen Feldtheorien wird dadurch unterstrichen, daß sich die Korrelationsfunktionen letzterer als Lösung eines mithilfe der Quantengruppen-Strukturdaten definierten Riemann-Hilbert-Problems gewinnen lassen, wie für den vorliegenden Fall gezeigt wird.

Abstract

Subjects of this thesis are the quantization of Liouville theory and clarification of the relations between quantum Liouville theory and related conformal field theories. Analysis of the classical theory and certain problems of previous quantization schemes based on Fock spaces motivate a definition of the space of states in terms of irreducible representations of the Virasoro algebra. A suitable operator formalism bases on chiral vertex operators is presented. The reformulation of this formalism in terms of fusion products of representations allows to describe both local and global properties of correlation function in a transparent manner. Fusion and braid matrices are calculated via a free field representation, and compatibility with the fusion rules is checked. They turn out to be related to Racah-Wigner coefficients of the quantum group $\mathcal{U}_q(sl(2))$. The importance of the connection between quantum groups and conformal field theories is emphasized by showing that the correlation functions of the latter can be obtained as solutions of a Riemann-Hilbert problem defined from the structural data of the quantum group. This is explicitely carried out in the present case related to $\mathcal{U}_q(sl(2))$.

Meinen Eltern
in Dankbarkeit
gewidmet

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SECTION I: INTRODUCTION

There are at least two important motivations to study Liouville theory: One of them originates from the fact that Liouville theory plays an important rôle for the quantization of 2D gravity or string theory in a target space of dimension $d \neq 26$.

2D gravity may be an interesting toy model, in which those complications in the quantization of gravitational theories which are mainly due to diffeomorphism invariance, such as the problem of time, may be studied in a technically simpler context.
String theories with target spaces of dimension $d \neq 26$ have been studied in the search for string models that are phenomenologically interesting. Since first-quantized string theory may be considered as 2D gravity on the world-sheet, one has an intimate relation between these two problems.

Liouville theory arises in these models in the following way: A quantum anomaly results in non-decoupling of the conformal factor of the metric, which leads to non-trivial quantum gravitational effects [Pol]. In the so-called conformal gauge [DK] one has almost decoupled dynamics of matter sector and conformal factor: There are no interaction terms in the lagrangian, but a residual gauge-invariance forces one to build physical states as non-trivial combinations of matter and conformal factor degrees of freedom [DK]. The dynamics of the conformal factor is described by the Liouville Lagrangian

$$\mathcal{L} = \int d^2x(\partial_\mu\phi\partial^\mu\phi + e^\phi). \tag{1.1}$$

Construction of correlation functions in this approach therefore requires construction of certain correlation functions of the Liouville field theory.

Despite a lot of effort in the last years, it is still hard to draw physically relevant conclusions from these models. At least one of the main problems is that quantization of Liouville theory raises a number of hard technical problems, which in most of the physically relevant cases are not yet resolved.

The second motivation for studying Liouville theory is that it is an interesting example of an integrable conformal field theory in two dimensions. It will be seen that the minimal models may be considered as closed subsectors of Liouville theory. However, it is to be expected that the complete quantum Liouville theory allows to consider much more operators than those required for the minimal models. It may well be that the complete construction of quantum Liouville theory will shed some light on the non-rational conformal field theories, on which not much is known yet.

The starting point of the present work was an attempt to construct correlation functions for 2D gravity in the continuum approach. I soon realized that presently available techniques are not powerful enough for a rigorous understanding of the relevant Liouville correlators. This led me to reconsider those cases where a rigourous construction of correlation functions is possible. The hope is of course that some structure that explains solvability in the known cases may be generalized to provide information on the unknown cases. It turned out that, despite much effort, in the explanation of solvability of the known cases there still are open questions, the investigation of which is worthwhile in itself.

In order to state my aims more precisely, I will now shortly describe the general strategy for quantization of Liouville theory: Crucial for integrability of the classical theory is the feature that reparametrizations of the light-cone coordinates x^+, x^- are realized as a symmetry of the theory (conformal invariance). This is reflected in the fact that there exist chiral (anti-chiral) fields $f_i^+(x^+)$, $f_i^-(x^-)$, $(i = 1, 2)$ from which the Liouville field may be reconstructed, and which transform covariantly under reparametrizations of x^+ (resp. x^-). These are good building blocks for a solution of the theory.
Upon quantization, it is a promising ansatz to try to preserve integrability by preserving conformal invariance. The Hilbert space factorizes into spaces representing the chiral (antichiral) excitations. The quantum analogs of the fundamental fields $f_i^+(x^+)$, $f_i^-(x^-)$ (and generalizations thereof) will then be operators that seperately act on the two chiral halfs. Their construction turns out to be equivalent to the construction of the so-called chiral vertex operators (CVO's), which describe transitions between different sectors of the chiral (resp. anti-chiral) parts of the space of states. These are the basic building blocks for operators that transform covariantly under the conformal symmetry, such as the quantum analogs of exponential functions of the Liouville field.

Construction of chiral vertex operators and investigation of their algebra and correlation functions will be one primary theme of this work. This provides the necessary requisites for discussion of issues such as locality, hermiticity of field operators or unitarity. Information on CVO's and their correlation functions has previously been obtained by several methods. In essence they all primarily exploit conformal invariance[1]. In their development new structures related to quantum groups appeared.

One aim is therefore to reconsider the existing methods for construction of CVO's, to clarify the relations between them and to explore the range of applicability. Furthermore, I will attempt to clarify the relation between Gervais-Neveu approach to Liouville theory and the conformal field theory techniques that were initiated by Belavin, Polyakov and Zamolodchikov [BPZ] and reformulated in terms of chiral vertex operators by Moore and Seiberg [MS] as well as Felder, Fröhlich and Keller [FFK].
A second aim is to understand the role of the quantum group structures. To this end, I will compare the different disguises in which these structures have appeared and try to clarify the relations between them.
A primary aim is to investigate the possibility to construct correlation functions from the quantum group structure. The latter provides the monodromies of the correlation functions of chiral vertex operators, so that one has a kind of Riemann-Hilbert problem of reconstruction of multivalued analytic functions from their monodromies.

The first two aims concern an area that has already been under extensive investigation. I should therefore state what I consider to be the new aspects of the present work:

1. I will make a proposal on the relation of the conformal field theory techniques to the Gervais-Neveu approach to Liouville theory. The latter approach uses Fock spaces that are labelled by a variable called momentum for the space of states. There are problems to achieve hermiticity of the energy momentum tensor on a discrete subseries of these Fock spaces. This subseries is distinguished by the property that the representation of the Virasoro algebra is no longer irreducible on the respective Fock spaces. I propose to avoid the problems with hermiticity of energy momentum by considering irreducible representations from the very beginning. In the course of this work I will find several pieces of evidence in favor of this proposal.

2. Construction of an operator formalism for minimal models and related conformal field theories: The formalism of [BPZ] or [MS] is not an operator formalism in the usual sense. Rather one defines three point functions (or equivalently CVO's) by the conformal Ward identities and builds higher point functions by "sewing" three point functions together.
   Instead, I will define CVO's purely algebraically by their covariance properties and examine in some detail the question of their existence: If there are null vectors in the representations between which the chiral vertex operator maps, it turns out that conformal covariance is impossible unless the conformal dimensions involved satisfy certain restrictions (fusion rules). A complete answer is possible thanks to certain partial information on general null vectors due to Feigin and Fuchs.
   Then the notion of a CVO is reformulated as a product operation mapping two Virasoro representations into a third. This idea goes back to Moore and Seiberg, but a constructive definition has, to the best of my knowledge not yet been given. It is shown that the notion of a fusion product allows to formulate the conformal covariance properties of arbitrary descendants of chiral vertex operators in a very concise way. In this language, the relation between monodromies of conformal blocks and quantum groups is just the statement that commutativity and associativity operations of fusion product and quantum group tensor product are equivalent.
   A careful analysis of fusion and braiding matrices shows that the chiral vertex operators satisfying the fusion rules form an algebra that is closed under fusion and braiding, i.e. that no unphysical representations appear.

3. The next part is an investigation of free field realizations. CVO's are constructed as integrals of products of vertex operators over certain multiple contours on the universal cover of the punctured Riemann sphere. The space of such contours was shown by Felder and Wiescerkowski [FW] to carry a natural quantum group structure. By combining this quantum group structure with a suitable operator formalism one gets very efficient techniques for the calculation of fusion and braiding matrices: they turn out to be given in terms of q-6j symbols. Previously used techniques have not been capable for calculation of fusion matrices from free field realizations.

4. Drinfeld's theorem on the uniqueness of quasi-triangular quasi Hopf algebras is used to find an explicit relation between minimal model correlators and solutions of the Knizhnik-Zamolodchikov (KZ) equations, i.e. WZNW-model conformal blocks. Techniques of this kind may provide a starting point for the reconstruction of conformal field theories from their quantum group structure by characterizing candidates for the conformal blocks as solutions to a Riemann problem defined from the quantum group structure. Moreover, one may hope that these methods still work in cases where no free field representations are available to provide integral representations for the correlation functions.

In the conclusions I will make some speculative remarks on possible extensions of the present work. After all, the rigorous construction of 2D gravity correlation functions is still elusive.

[1] This is not so obvious in the Gervais-Neveu approach. More on this below.

SECTION II:

CLASSICAL THEORY

Let classical Liouville theory be defined by the Lagrangian

$$\mathcal{L} = k\partial_\mu\phi\partial^\mu\phi - Me^{\beta\phi} \tag{2.1}$$

Two of the three parameters may be absorbed by simple shifts and rescalings of ϕ. I will thus take:

$$\mathcal{L} = \frac{1}{\gamma^2}(\tfrac{1}{2}\partial_\mu\phi\partial^\mu\phi - 8e^\phi) \tag{2.2}$$

The Liouville equation of motion then reads

$$\partial_+\partial_-\phi + 2e^\phi = 0, \tag{2.3}$$

where light-cone coordinates have been introduced according to

$$x^\pm \equiv \tau \pm \sigma \qquad \partial_\pm \equiv \tfrac{1}{2}(\partial_\tau \pm \partial_\sigma). \tag{2.4}$$

Throughout this article I shall only consider the case that ϕ is periodic in σ, i.e. $\phi(\sigma + 2\pi, \tau) = \phi(\sigma, \tau)$.
A first important observation is the invariance of the action under following transformations:

$$\delta\phi := v^\mu\partial_\mu\phi + \partial_\mu v^\mu \qquad \text{where} \qquad \partial_\mu v_\nu + \partial_\nu v_\mu = g_{\mu\nu}\partial_\lambda v^\lambda. \tag{2.5}$$

In light-cone coordinates the condition on v^μ simply reads $\partial_- v^+ = 0$ and $\partial_+ v^- = 0$. One therefore finds reparametrizations of the light-cone coordinates as a symmetry of the theory. This infinite-dimensional symmetry implies the existence of infinitely many conserved charges. These may be constructed in the following way: Starting from the canonical energy-momentum tensor

$$\Theta_{\mu\nu} := \frac{1}{\gamma^2}\partial_\mu\phi\partial_\nu\phi - g_{\mu\nu}\mathcal{L}, \tag{2.6}$$

one may construct a *traceless* conserved tensor by adding a total derivative term:

$$\mathcal{T}_{\mu\nu} := \Theta_{\mu\nu} - C(\partial_\mu\partial_\nu - g_{\mu\nu}\partial_\lambda\partial^\lambda)\phi \tag{2.7}$$

This tensor is on-shell conserved and traceless if C is chosen as $C = \frac{2}{\gamma^2}$. These are all the properties needed to prove that the charges

$$Q[v^\mu] := \int_0^{2\pi} d\sigma v^\mu \mathcal{T}_{\mu 0} \qquad \text{with} \qquad \partial_\mu v_\nu + \partial_\nu v_\mu = g_{\mu\nu}\partial_\lambda v^\lambda \tag{2.8}$$

are indeed conserved.
Let me anticipate a parametrization of the general solution:

$$\phi(\tau, \sigma) = \ln\left(\frac{\partial_+\mathcal{A}(x_+)\partial_-\mathcal{B}(x_-)}{(\mathcal{A}(x_+) + \mathcal{B}(x_-))^2}\right). \tag{2.9}$$

Since the variables $\mathcal{A}$, $\mathcal{B}$ transform under (2.5) simply by reparametrizations of their arguments ($\delta\mathcal{A} = v^+\partial_+\mathcal{A}$, $\delta\mathcal{B} = v^-\partial_-\mathcal{B}$) one sees that starting from simple solutions a whole class of others can be generated by applying diffeomorphisms of the light-cone coordinates.

1. Solutions of the Liouville equation

I will now start a more detailed analysis of the solutions to the Liouville equation, which is a important preliminary for both Hamilton formalism and quantization. The aim is to derive different parametrizations for the space of solutions, each of which has certain virtues and drawbacks.

1.1 ENERGY-MOMENTUM PARAMETRIZATION

The first step towards the solution of the Liouville equation is to express solutions of the Liouville equation in terms of solutions of a linear equation. The result is the following:

If ϕ is a solution of $\partial_+\partial_-\phi + 2e^\phi = 0$ then there always exist functions $g_i^\pm$, $i = 1, 2$ such that

$$e^{-\frac{1}{2}\phi(\sigma,\tau)} = \sum_{i=1,2} g_i^+(x_+)g_i^-(x_-). \tag{2.1.1}$$

$g_i^\pm$ are functionals of ϕ that are determined as solutions of

$$\partial_+^2 g_i^+ = T_{++} g_i^+ \qquad \partial_-^2 g_i^- = T_{--} g_i^-, \tag{2.1.2}$$

normalized by

$$\partial_\pm g_1^\pm g_2^\pm - \partial_\pm g_2^\pm g_1^\pm = 1 \tag{2.1.3}$$

and $T_{\pm\pm}$ is given in terms of ϕ as

$$T_{\pm\pm} \equiv e^{\phi/2} \partial_\pm^2 e^{-\phi/2} = \tfrac{1}{4}\partial_\pm\phi\partial_\pm\phi - \tfrac{1}{2}\partial_\pm^2\phi. \tag{2.1.4}$$

Conversely, assume given periodic data $T_{\pm\pm}(x_\pm)$. Then any function ϕ constructed as

$$\phi(x_+, x_-) = \ln \frac{1}{\left(\sum_{i=1,2} f_i^+(x_+) f_i^-(x_-)\right)^2} \tag{2.1.5}$$

from solutions $f_i^\pm$ of (2.1.2) and (2.1.3) gives a solution of $\partial_+\partial_-\phi + 2e^\phi = 0$.

The proof of above statement is based on the following fundamental observation: If ϕ is a solution of the Liouville equation then

$$\partial_\mp T_{\pm\pm} = \tfrac{1}{2}\partial_\pm\phi\partial_\mp\partial_\pm\phi - \tfrac{1}{2}\partial_\pm^2\partial_\mp\phi = 0. \tag{2.1.6}$$

Writing the definition of T_{++} as $(\partial_+^2 - T_{++})e^{-\frac{1}{2}\phi} = 0$ then shows that $e^{-\frac{1}{2}\phi}$ can be written as a linear combination of two linearly independent solutions g_i^+, $i = 1,2$ of $(\partial_+^2 - T_{++}(x_+))g_i^+(x_+) = 0$:

$$e^{-\frac{1}{2}\phi} = \sum_{i=1,2} c_i g_i^+(x_+). \tag{2.1.7}$$

The defining equation of T_{--} then implies that $c_i =: g_i^-(x_-)$ are solutions of $(\partial_-^2 - T_{--}(x_-))g_i^- = 0$. The converse statement can be checked by direct calculation.

Now it is easy to derive the parametrization (2.9) from these results: Given $f_i^\pm$, introduce A, B as

$$A(x_+) := \frac{f_1^+}{f_2^+} \qquad B(x_-) := \frac{f_1^-}{f_2^-} \tag{2.1.8}$$

Due to the normalization conditions (2.1.3) $f_i^\pm$ may be recovered as

$$\begin{aligned} f_2^+ &= (\partial_+ A)^{-\frac{1}{2}} & f_2^- &= (\partial_- B)^{-\frac{1}{2}} \\ f_1^+ &= (\partial_+ A)^{-\frac{1}{2}} A & f_1^- &= (\partial_- B)^{-\frac{1}{2}} B. \end{aligned} \tag{2.1.9}$$

Inserting these expressions into (2.1.5) yields

$$\phi(\tau,\sigma) = \ln\left(\frac{\partial_+ A(x_+)\partial_- B(x_-)}{(1 + A(x_+)B(x_-))^2}\right). \tag{2.1.10}$$

Note that by taking into account (2.1.3) one gets

$$\partial_+ A = (f_2^+)^{-2} > 0 \qquad \partial_- B = (f_2^-)^{-2} > 0, \tag{2.1.11}$$

so that reality of ϕ is guaranteed.

The equivalent parametrization (2.9) is obtained by defining $\mathcal{A} := A$ and $\mathcal{B} := 1/B$. If one now writes $\mathcal{B} = g_1/g_2$ with $\partial_- g_2 g_1 - \partial_- g_1 g_2 = 1$ then instead of (2.1.5) one has the representation[1]

$$\phi(x_+, x_-) = \ln \frac{1}{(f_1(x_+)g_2(x_-) + f_2(x_+)g_1(x_-))^2} \tag{2.1.12}$$

A number of remarks are to be made to clarify the utility of above statements for parametrizing the space of solutions:

(1) *Initial-value problem:* This construction of solutions to the Liouville equation also gives a way to solve the Cauchy-problem[PP]: Assume regular Cauchy-data $\phi(\sigma) := \phi(\sigma, 0)$ and $\dot\phi(\sigma) := \dot\phi(\sigma, 0)$ to be given. By using the equation of motion it is easy to rewrite $T_{\pm\pm}$ as

$$T_{\pm\pm} = \tfrac{1}{16}(\dot\phi \pm \phi')^2 \mp \tfrac{1}{4}(\dot\phi \pm \phi')' + e^\phi. \tag{2.1.13}$$

$T_{\pm\pm}$ can therefore be expressed in terms of the Cauchy-data. Take two arbitrary solutions $g_i(x_-)$ of $(\partial_-^2 - T_{--})g_i = 0$. The nice trick in [PP] is to define functions f_i at $\tau = 0$ by

$$f_i(\sigma) := (-1)^i e^{-\frac{1}{2}\phi(\sigma)}\left(-g_i'(\sigma) + \frac{1}{4}(\phi'(\sigma) - \dot\phi(\sigma))g_i(\sigma)\right) \tag{2.1.14}$$

One may then check that $\partial_- f_i = 0$ and $(\partial_+^2 - T_{++})f_i = 0$. Moreover, the function $\phi(x_+, x_-)$ constructed from these f_i, g_i as in (2.1.12) reproduces the initial data. In [PP] this is used to prove existence and uniqueness of a solution to the initial value problem for a class of initial values with certain singularities.

(2) *Singular solutions:* (2.1.12) becomes singular when $f_1 g_2 + f_2 g_1$ vanishes. There is nevertheless a class of solutions where these singularities can be controlled in a well defined way (see [PP]): These are obtained from initial values which have singularities only at discretely distributed points σ_i around which neighborhoods $\mathcal{U}_i$ have to exist such that in $\mathcal{U}_i$

[1] The transition from f^- to g is nothing but exchanging the rôles of f_1^- and f_2^-.

$$\begin{aligned} \phi(\sigma) &= -\ln\left(\frac{\sigma-\sigma_i}{1-v_i^2}\right) + f_i(\sigma) \quad f_i \in \mathcal{C}^2(\mathcal{U}_i), f_i(\sigma_i)=0,\ v_i^2<1 \\ \dot{\phi}(\sigma) &= v_i\left(\frac{2}{\sigma-\sigma_i} + f_i'(\sigma)\right) + g_i(\sigma) \quad g_i \in \mathcal{C}^1(\mathcal{U}_i), g_i(\sigma_i)=0 \end{aligned} \tag{2.1.15}$$

The point is that precisely this type of singular behaviour allows to construct continuous $T_{\pm\pm}$ and therefore twice differentiable f_i, g_i similiar as indicated in the preceeding remark (for details see [PP]).
To give a qualitative picture of the time evolution of the singularities, I will use the representation (2.9) resp. (2.1.12). The singularities are the zeros of $\mathcal{A}(x_+)+\mathcal{B}(x_-)$. They may be seen to form time-like non-intersecting trajectories along which singular behaviour of the form (2.1.15) is found on every slice of constant time: $\mathcal{A}$ and $\mathcal{B}$ have singularities whenever $f_2=0$ or $g_2=0$ respectively. However, looking at (2.1.12) one sees that they generically do not lead to singularities of ϕ, but only if the light-like trajectories of $\{x_+|f_2=0\}$ and $\{x_-|g_2=0\}$ meet. These trajectories form a mesh on the cylinder and by taking into account the monotony of $\mathcal{A}$, $\mathcal{B}$ between their singularities one may convince oneself (or see [PP]) that one and only one singularity line of ϕ passes through each parallelogram forming the mesh. From such considerations one also finds that the number of singularities N is just the sum of the numbers $N_{\mathcal{A}}$ and $N_{\mathcal{B}}$ of singularities in $\mathcal{A}$ and $\mathcal{B}$ respectively.

(3) *Boundary conditions:* Above construction of solutions from the data $T_{\pm\pm}$ suggests to take them as elementary variables for a parametrization of the space of solutions. Which data besides $T_{\pm\pm}$ are needed? The obvious candidates for additional data are the initial conditions for the integration of (2.1.2). Now observe that as a consequence of the periodicity of $T_{\pm\pm}$, $T_{\pm\pm}(x_\pm \pm 2\pi) = T_{\pm\pm}(x_\pm)$ the functions $f_i^\pm$ have to be quasiperiodic, i.e. $f_i^+(x_+ + 2\pi) = f_j^+(x_+)M_{ji}^+$ and $f_i^-(x_- - 2\pi) = M_{ij}^- f_j^-(x_-)$. The matrices M are called monodromy matrices. Solutions of (2.1.2) are uniquely specified by $T_{\pm\pm}$ and initial conditions at some reference point x_0. Solutions $\tilde{f}_i^+$ corresponding to any other choice of initial conditions are linearly related to the f_i^+: $\tilde{f}_i^+ = f_j^+ A_{ji}$, where $det A = 1$ to preserve (2.1.3). If f_i^+ have monodromy M then $\tilde{f}_i^+$ have monodromy $\tilde{M} = A^{-1}MA$. The conjugacy class of M is therefore independent of the initial conditions and a functional of T_{++} only. An alternative parametrization of the boundary conditions is given by the matrix A which relates via $M = A^{-1}M_0A$ the monodromy matrix M of given solutions of (2.1.2) to a standard representative M_0 of its conjugacy class. For making convenient choices for M_0 I will distinguish three cases:

- Elliptic case ($|TrM|<2$): $M_0 = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$
- Parabolic case ($|TrM|=2$): $M_0 = \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}$
- Hyperbolic case ($|TrM|>2$): $M_0 = \begin{pmatrix} e^p & 0 \\ 0 & e^{-p} \end{pmatrix}$

Two observations clarify the dependence of $e^{-\frac{1}{2}\phi}$ on these boundary data: First, in order to have periodicity of ϕ it is necessary that M^- is inverse to M^+. Second, $e^{-\frac{1}{2}\phi}$ is invariant under inverse $sl(2,\mathbb{R})$-transformations of f_i^+ and f_i^-. It is therefore no loss of generality to demand f_i^+ to have monodromy M_0 and f_i^- to have M_0^{-1}. This fixes $f_i^\pm$ up to multiplication of $f_1^\pm$ with a factor $e^{q_\pm}$ and $f_2^\pm$ with $e^{-q_\pm}$, so that the *only* additional parameter besides $T_{\pm\pm}$ needed for a unique specification of solutions is $q = q_+ + q_-$.
Note that a $SL(2,\mathbb{R})$-transformation $f_i^+ \to f_j^+ R_{ji}$ with $R = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$ induces a Moebius transformation $A \to \frac{aA+b}{cA+d}$ on A, and similarly for f_i^-, B.

We will see below that the three different classes of solutions (elliptic, parabolic and hyperbolic) form closed sectors which require individual treatments of Hamilton formalism and quantization. One important distinguishing feature is that any regular solution is in the hyperbolic sector, whereas the elliptic solutions always have singularities (see [PP] or [ABBP]).

1.2 Free field parametrization

For Hamilton formalism and quantization it is desirable to have a parametrization of phase space (=space of classical solutions) in terms of action-angle variables (i.e. harmonic oscillators). The functions A, B can not be chiral components of a free field: The latter have additive monodromy and do not satisfy $\partial_+ A > 0$, $\partial_- B > 0$ in general. I will first consider solutions with hyperbolic monodromy, put into the form $M^+ = \begin{pmatrix} e^{\frac{1}{2}p} & 0 \\ 0 & e^{-\frac{1}{2}p} \end{pmatrix}$. Because of $\partial_+ A > 0$, $\partial_- B > 0$ it is possible to define

$$\varphi_+(x_+) := \ln(\partial_+ A(x_+)) \qquad \varphi_-(x_-) := \ln(\partial_- B(x_-)) \tag{2.1.16}$$

Because of the diagonal monodromy of A and B, $\varphi_\pm$ can be expanded as

$$\varphi_\pm(x_\pm) = q_\pm + \frac{p_\pm}{2\pi}x_\pm + \sum_{n\neq 0} \frac{e^{inx_\pm}}{in} a_n. \tag{2.1.17}$$

Single-valuedness of ϕ requires $p_+ = p_- = p$, reality $a_n^* = a_{-n}$. Now $f_2^\pm = e^{-\frac{1}{2}\varphi_\pm}$ is a solution of $(\partial_\pm^2 - T_{\pm\pm})f_2^\pm = 0$ iff

$$T_{\pm\pm} = \tfrac{1}{4}(\partial_\pm\varphi)^2 - \tfrac{1}{2}\partial_\pm^2\varphi. \tag{2.1.18}$$

In order to find $f_1^\pm$ in terms of $\varphi_\pm$ use (2.1.9) together with

$$A(x_+) = \int_0^{x_+} dx e^{\varphi_+(x)} + c_A \qquad B(x_-) = \int_0^{x_-} dx e^{\varphi_-(x)} + c_B. \tag{2.1.19}$$

Requiring these expressions to have the right monodromy fixes c_A, c_B. Then A, B can be written as follows:

$$A(x_+) = \frac{1}{2\sinh(p/2)} \int_0^{2\pi} dx e^{\frac{p}{2}\epsilon(x_+ - x)} e^{\varphi_+(x)}$$
$$B(x_-) = -\frac{1}{2\sinh(p/2)} \int_0^{2\pi} dx e^{-\frac{p}{2}\epsilon(x_- - x)} e^{\varphi_-(x)}, \tag{2.1.20}$$

where $\epsilon(x) = 1$ if $x > 0$ and $= -1$ if $x < 0$. Finally ϕ can be rewritten as

$$\phi(x_+, x_-) = \varphi_+(x_+) + \varphi_-(x_-) - \ln\left((1 - S(x_+, x_-))^2\right) \tag{2.1.21}$$
$$S(x_+, x_-) := \tfrac{1}{4\sinh^2(p/2)} \int_0^{2\pi} dy_+ dy_- e^{\frac{p}{2}(\epsilon(x_+ - y_+) - \epsilon(x_- - y_-))} e^{\varphi_+(x_+) + \varphi_-(x_-)}$$

There is an ambiguity in the map to free fields: Instead of (2.1.16) one could have chosen to define

$$\tilde{\varphi}_+(x_+) := \ln\left(-\partial_+ \frac{1}{A(x_+)}\right) \qquad \tilde{\varphi}_-(x_-) := \ln\left(-\partial_- \frac{1}{B(x_-)}\right) \tag{2.1.22}$$

One gets a second equivalent representation of the same solution. The momentum $\tilde{p}$ of $\tilde{\varphi}_\pm$ is just $-p$. This also implies that the functions $\varphi_\pm$ with negative p and those with positive p parametrize the same space of solutions. One may impose i.e. $p > 0$ to cover the Liouville space of solutions only once.

In order to introduce free field variables in the case of elliptic monodromy one also has to diagonalize the monodromy matrix, which requires a *complex* similarity transformation. Write M_0^+ as $\begin{pmatrix} e^{ir/2} & 0 \\ 0 & e^{-ir/2} \end{pmatrix}$. Then formally everything seems to work as before just by replacing p with ir, but actually there are some complications:

(1) $\varphi_\pm$ now have to be complex variables. One therefore has to find the conditions on $\varphi_\pm$ to guarantee reality of ϕ. From (2.1.5) one may see that the necessary and sufficient condition for ϕ to be real is $(f_i^+)^* = f_j^+ C_{ji}$, $(f_i^-)^* = C_{ij}^{-1} f_j^-$ with $C^* C = \mathbf{1}$. These reality conditions are compatible with the quasiperiodicity of $f_i^\pm$ iff $(M^+)^* = C^{-1} M^+ C$. For above choice of monodromy this fixes $C = \begin{pmatrix} 0 & e^{i\lambda} \\ e^{i\lambda} & 0 \end{pmatrix}$, where compatibility with (2.1.3) requires $e^{i\lambda} = \pm i$. Let me without loss of generality choose $(f_1^+)^* = i f_2^+$, $(f_1^-)^* = -i f_2^-$. Expressing this condition in terms of the free fields $\varphi_\pm$ unfortunately leads to rather complicated conditions on the oscillators: It may be rewritten as $e^{\frac{1}{2}(\varphi_+ - \varphi_+^*)} = iA$ and $e^{\frac{1}{2}(\varphi_- - \varphi_-^*)} = -iB$ or by taking derivatives as
$$\frac{1}{2i} \partial_\pm (\varphi_\pm - \varphi_\pm^*) = \pm e^{\frac{1}{2}(\varphi_\pm + \varphi_\pm^*)}. \tag{2.1.23}$$
In this form one recognizes how real and imaginary parts of $\varphi_\pm$ may be expressed in terms of each other. This condition also ensures $T_{\pm\pm}$ to be real.

(2) For single-valuedness of ϕ one does not need $r_+ = r_-$ but only $r_+ = r_+ + 2\pi k$; $k \in \mathbb{Z}$. This additional discrete degree of freedom can be interpreted as a winding number (see [PP]).

(3) In case $r_+ = 2\pi k$; $k \in \mathbb{Z}$ one observes that because of $M^+ = \mathbf{1}$ any c_A will do in (2.1.19). One does not have to use the singular expression (2.1.20). Moreover, in these cases arbitrary $SL(2,\mathbb{R})$-transformations applied on A and B will have the same monodromy, which leads to an ambiguity in the definition of the free fields: To see this, observe that an infinitesimal $sl(2,\mathbb{R})$-transformation $\delta A = \frac{1}{2}A^2$ corresponds to a transformation $\delta\varphi_+ = A$. There will be a one-parametric set $\varphi_\pm^t$, $t \in \mathbb{R}$ of free field configurations that all have $M^+ = \mathbf{1}$ and describe the same Liouville solution! A reduced set of variables will suffice to parametrize all solutions in these cases. I propose that the symmetry that expresses this redundancy is related to the classical counterpart of the Felder-BRST [Fel1]. Anyway, these facts indicate that the case $r_+ = 2\pi k$ requires special treatment.

2. Hamilton Formalism

The Hamilton formalism provides one with hints on the definition of quantum commutation relations by giving their classical limits. Especially the rôle of symmetries for the classical integrability of the theory can be made transparent. One may hope that the symmetries responsible for classical integrability can also be realized in the quantum theory and allow for its solution.

The aim of this subsection is therefore threefold:

- To exhibit the conformal symmetry in the Hamilton formalism,
- to discuss the transition to action-angle variables, and
- to derive the quadratic Poisson algebra of the variables $f_i^\pm$, which may be the starting point for a discussion of the classical origin of quantum group symmetries.

Let's start from the Lagrangian

$$\mathcal{L} = \tfrac{1}{2\gamma^2}(\partial_\mu \phi \partial^\mu \phi - 2e^\phi) = \tfrac{1}{\gamma^2}(\tfrac{1}{2}\dot{\phi}^2 - \tfrac{1}{2}\phi'^2 - e^\phi), \tag{2.2.1}$$

introduce the momentum density $\pi = \frac{\partial \mathcal{L}}{\partial \phi_\tau} = \frac{1}{\gamma^2}\phi_\tau$, and define fundamental Poisson brackets

$$\{\pi(\sigma, 0), \phi(\sigma', 0)\} = \delta(\sigma - \sigma'). \tag{2.2.2}$$

The canonical Hamiltonian is

$$H = \tfrac{1}{\gamma^2} \int_0^{2\pi} d\sigma \left(\tfrac{1}{2}(\gamma^2\pi)^2 + \tfrac{1}{2}\phi'^2 + e^\phi\right). \tag{2.2.3}$$

2.1 Conformal symmetry

The improved energy momentum tensor $\mathcal{T}_{\pm\pm}$[2] has previously been identified as a generating function for the conserved Noether charges corresponding to the conformal symmetry. It was moreover seen to be a fundamental variable for a chirally split presentation of the solutions.

Expressing $\mathcal{T}_{\pm\pm}$ at $\tau = 0$ in terms of ϕ, π,

$$\mathcal{T}_{\pm\pm} = \tfrac{1}{\gamma^2}\left(\tfrac{1}{4}(\gamma^2\pi \pm \phi')^2 \mp (\gamma^2\pi \pm \phi')' + e^{\phi}\right), \tag{2.2.4}$$

one finds from $\{\pi, \phi\} = \delta$ that

$$\begin{aligned}\{\mathcal{T}_{\pm\pm}(\sigma), \mathcal{T}_{\pm\pm}(\sigma')\} &= -(\partial_\sigma - \partial_{\sigma'})(\mathcal{T}_{\pm\pm}(\sigma)\delta(\sigma - \sigma')) + \frac{2}{\gamma^2}\delta'''(\sigma - \sigma') \\ \{\mathcal{T}_{\pm\pm}(\sigma), \mathcal{T}_{\mp\mp}(\sigma')\} &= 0\end{aligned} \tag{2.2.5}$$

or in terms of modes

$$L_n^{\pm} := \int_0^{2\pi} d\sigma e^{\pm in\sigma}\mathcal{T}_{\pm\pm} \tag{2.2.6}$$

one has

$$i\{L_m^{\pm}, L_n^{\pm}\} = -(m-n)L_{n+m} - \frac{4\pi}{\gamma^2}n^3\delta_{m,-n} \tag{2.2.7}$$

Note that the Virasoro algebra defined by the Poisson brackets (2.2.7) is centrally extended already at classical level. This expresses the fact that $\mathcal{T}_{\pm\pm}$ does not transform covariantly under conformal transformations, but rather inhomogeneously as a connection.

$$\mathcal{T} \longrightarrow \mathcal{T}^g = (g')^2(\mathcal{T} \circ g) + \frac{4\pi}{\gamma^2}S(g), \qquad S(g) = \frac{g'''}{g'} - \frac{3}{2}\left(\frac{g''}{g'}\right)^2 \tag{2.2.8}$$

This kind of transformation law is characteristic for a projective connection.

I will now consider what transformations $\mathcal{T}_{\pm\pm}$ generate on ϕ:

$$\{L_m^+, \phi(\sigma)\} = (\partial_+\phi + im)e^{im\sigma} = \left(\frac{A''}{A'} - 2\frac{A'}{A+B} + im\right)e^{im\sigma} \tag{2.2.9}$$

Phrased differently ($\phi = \phi[A, B]$)

$$\phi + \epsilon\{L_m, \phi\} = \phi[A(\sigma + \epsilon e^{im\sigma}), B(\sigma)] + \mathcal{O}(\epsilon^2). \tag{2.2.10}$$

[2] $\mathcal{T}_{+-}$ is proportional to the trace of $\mathcal{T}$ and therefore vanishes on-shell

In this form it is obvious that the L_m are the canonical generators of reparametrizations of the lightcone variables.

One may now verify conservation of the Noether charges: First note that $H = L_0^+ + L_0^-$. Then by using $v^{\pm} = v^{\pm}(x_{\pm})$ one calculates

$$\frac{d}{dt}Q[v^{\pm}] = \int_0^{2\pi} d\sigma(\partial_\tau v^{\pm}\mathcal{T}_{\pm 0} + v^{\pm}\{H, \mathcal{T}_{\pm 0}\}) = \pm\int_0^{2\pi}(\partial_\sigma v^{\pm}\mathcal{T}_{\pm 0} + v^{\pm}\partial_\sigma\mathcal{T}_{\pm 0}) = 0.$$

2.2 Free field Poisson brackets

It is in fact possible to derive the Poisson brackets for the variables $\varphi_{\pm}$ introduced in section 1 from the canonical Poisson brackets of the Liouville field [PP]. Since this requires quite some calculation, I will just scetch how this is done. The basic strategy is that of the inverse scattering method, but the peculiarities of Liouville theory require several modifications.

One basic observation is that the Liouville equation is equivalent to the zero-curvature condition for the connection $\mathcal{U}$ defined as

$$\mathcal{U}_\sigma := \begin{pmatrix} \frac{\gamma^2}{4}\pi & e^{\frac{1}{2}\phi} \\ e^{\frac{1}{2}\phi} & -\frac{\gamma^2}{4}\pi \end{pmatrix} \qquad \mathcal{U}_\tau := \begin{pmatrix} \frac{1}{4}\phi' & e^{\frac{1}{2}\phi} \\ -e^{\frac{1}{2}\phi} & -\frac{1}{4}\phi' \end{pmatrix} \tag{2.2.11}$$

The zero-curvature condition reads

$$\partial_\tau\mathcal{U}_\sigma - \partial_\sigma\mathcal{U}_\tau - [\mathcal{U}_\sigma, \mathcal{U}_\tau] = 0. \tag{2.2.12}$$

Then introduce an object called transition matrix $T(\sigma)$, defined at time $\tau = 0$ as solution of $\partial_\sigma T = \mathcal{U}_\sigma T$ with initial condition $T(0) = 1$[3]. The utility of introducing T lies in the following two facts, to be elaborated on in the sequel:

- Variables $A(x_+)$ and $B(x_-)$ from which ϕ may be reconstructed as in (2.9) can be found from ratios of matrix elements of T. After diagonalizing the monodromy free field variables are obtained as in section 1.2.
- The Poisson brackets of the matrix elements of T can be calculated from the fundamental Poisson brackets (2.2.2) at time $\tau = 0$, those of A and B follow.

To find the Poisson brackets of T note that at $\tau = 0$ the Poisson brackets of $\mathcal{U}_\sigma$ may be directly calculated and written as

$$\{\mathcal{U}_\sigma(\sigma) \overset{\otimes}{,} \mathcal{U}_\sigma(\sigma')\} = [\mathcal{R}, \mathcal{U}_\sigma(\sigma) \otimes \mathbf{1} + \mathbf{1} \otimes \mathcal{U}_\sigma(\sigma)]\delta(\sigma - \sigma'), \tag{2.2.13}$$

[3] This object is analogous to the functions $f_i^{\pm}$ which can be defined from the initial data by solving the second order differential equations (2.1.2)

with

$$\mathcal{R} = -\frac{\gamma^2}{8}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{2.2.14}$$

Now it is a standard calculation, to be found i.e. in [FT] to show that Poisson brackets

$$\begin{aligned}\{T(\sigma) \overset{\otimes}{,} T(\sigma')\} = &\Theta(\sigma' - \sigma)(\mathbf{1} \otimes T(\sigma')T^{-1}(\sigma))[\mathcal{R}, T(\sigma) \otimes T(\sigma)] \\ &+ \Theta(\sigma - \sigma')(T(\sigma)T^{-1}(\sigma') \otimes \mathbf{1})[\mathcal{R}, T(\sigma') \otimes T(\sigma')]\end{aligned} \tag{2.2.15}$$

follow from (2.2.13).

In order to find A and B from T one first introduces $\tilde{A}$ and $\tilde{B}$ by $\tilde{A} := -T_{11}/T_{12}$, $\tilde{B} := T_{21}/T_{22}$. By writing $\mathcal{U}_\sigma = \partial_\sigma T \cdot T^{-1}$ one finds that ϕ is recovered from $\tilde{A}$ and $\tilde{B}$ as in (2.9). Moreover, because of (2.2.12) there exists a continuation $T(\sigma, \tau)$ to arbitrary times which also solves $\partial_\tau T = \mathcal{U}_\tau T$ iff ϕ solves Liouville's equation. Because of

$$\begin{aligned} e^{\frac{1}{2}\phi} &= (\mathcal{U}_\tau)_{12} = (\partial_\tau T \cdot T^{-1})_{12} = T_{12}^2 \partial_\tau \tilde{A} \\ &= (\mathcal{U}_\sigma)_{12} = (\partial_\sigma T \cdot T^{-1})_{12} = T_{12}^2 \partial_\sigma \tilde{A} \end{aligned}$$

one finds that $\partial_- \tilde{A} = 0$, and similarly that $\partial_+ \tilde{B} = 0$.

However, $\tilde{A}$ and $\tilde{B}$ will in general not have diagonal monodromy. To find functions A and B from which one may define free fields one has to apply $sl(2,\mathbb{R})$-transformations to diagonalize the monodromy. What complicates the computations of the Poisson brackets for A, B with diagonal monodromy is the fact that the parameters of the diagonalizing $sl(2,\mathbb{R})$-transformation are functionals of ϕ, π and therefore have nontrivial Poisson brackets. In the end one indeed finds [PP] free field Poisson brackets for $\varphi := \ln(\partial_+ A) + \ln(\partial_- B)$, namely

$$\{\varphi(\sigma, 0), \varphi(\sigma', 0)\} = 0 \qquad \{\dot{\varphi}(\sigma, 0), \varphi(\sigma', 0)\} = \gamma^2 \delta(\sigma - \sigma'). \tag{2.2.16}$$

It is convenient to use a slight redundancy in the parametrization of phase space by introducing independent zero-modes $q_\pm$, $p_\pm$ for the chiral halves of the theory. The condition $p_+ = p_-$ is then imposed as a constraint. By letting the nonvanishing zero-mode Poisson brackets be $\{p_\pm, q_\pm\} = \gamma^2$ one gets the advantage that the two chiral sectors Poisson-commute. The fundamental Poisson brackets in terms of the variables $\varphi_\pm$ then are (see (2.1.17)):

$$\{\varphi_\pm(\sigma), \varphi_\pm(\sigma')\} = \pm\frac{\gamma^2}{2}\epsilon(\sigma - \sigma') \qquad \{\varphi_\pm(\sigma), \varphi_\mp(\sigma')\} = 0. \tag{2.2.17}$$

All of this works fine in the hyperbolic sector, where the variables $\varphi_\pm$ can be taken to be real. However, in the elliptic sector one unfortunately finds nontrivial Poisson brackets (see [GN2]) $\{\partial_+\varphi_+(\sigma), (\partial_+\varphi_+(\sigma'))^*\}$ in addition to (2.2.16), such that neither real nor imaginary part of φ_+ can be expanded in terms of ordinary real oscillators. One possible way to deal with that problem [GN2] is to introduce a larger phase space with independent real and imaginary parts of φ. The condition of reality of the energy momentum tensor is imposed as a constraint. For non-integer values of the momentum it is shown in [GN2][GN3] that the constraint may be solved such that the Liouville phase space is recovered. The solution given in [GN2][GN3] breaks down for integer values of the momenta[4]. I strongly suspect that this is related to the ambiguity in the definition of the free fields for integer momenta. I will discuss a quantum counterpart of this phenomenon later. However, even classically it seems that the correct hamiltonian treatment of the subleties at integer momenta is still to be worked out. Maybe a classical version of Felder-BRST is at work.

2.3 Poisson structure for the chiral components

The possibility of writing solutions of the Liouville equation in terms of the chiral variables $f_i^\pm$ is crucial for it's integrability, because the action of the symmetries becomes simple in these variables. To get a preparation for exploiting these symmetries in quantization it is useful to study their realization in the Hamilton formalism.

In the following a slightly more general class of variables will be considered, namely

$$\begin{aligned}\psi_m^j(x_+) &:= N_m^j (f_1^+)^{j-m}(f_2^+)^{j+m} \\ &= N_m^j e^{-j\varphi_+(x_+)}(A(x_+))^{j-m}.\end{aligned} \tag{2.2.18}$$

These variables are useful for a chirally split representation of Liouville exponentials such as

$$e^{-j\phi(x_+,x_-)} = \sum_m C_m^j \psi_m^j(x_+)\bar{\psi}_m^j(x_-) \tag{2.2.19}$$

First of all these variables transform simple under conformal transformations: By expressing $\mathcal{T}_{++}$ in terms of free fields and using the Poisson brackets (2.2.16) one may check

$$\{L_n, e^{-j\varphi_+(x_+)}\} = e^{inx_+}(\partial_+ - inj)e^{-j\varphi_+(x_+)}. \tag{2.2.20}$$

Especially one has

$$\{L_n, e^{\varphi_+(x_+)}\} = \partial_+(e^{inx_+}e^{\varphi_+(x_+)}), \tag{2.2.21}$$

so that one finds:

$$\{L_n, A(x_+)\} = e^{inx_+}\partial_+ A(x_+). \tag{2.2.22}$$

[4] I thank J. Schnittger for drawing my attention to this fact.

From these formulae it immediately follows that the ψ^j_m do indeed transform covariantly ($T[v] := \int_0^{2\pi} d\sigma v(\sigma) T_{++}(\sigma)$):

$$\{T[v], \psi^j_m(x_+)\} = +v(x_+)\partial_+\psi^j_m(x_+) - j(\partial_+ v(x_+))\psi^j_m(x_+). \quad (2.2.23)$$

The next task will be to compute the Poisson brackets among themselves: Starting from

$$\{\varphi_+(\sigma), \varphi_+(\sigma')\} = \tfrac{\gamma^2}{2}\epsilon(\sigma-\sigma') \quad (2.2.24)$$

one immediately has

$$\{e^{\lambda\varphi_+(\sigma)}, e^{\mu\varphi_+(\sigma')}\} = \lambda\mu\frac{\gamma^2}{2}\epsilon(\sigma-\sigma')e^{\lambda\varphi_+(\sigma)}e^{\mu\varphi_+(\sigma')}. \quad (2.2.25)$$

For the computation of $\{A(\sigma), A(\sigma')\}$ one needs (2.2.25) and

$$\{f(p), e^{\lambda\varphi_+(\sigma)}\} = \lambda\gamma^2 f'(p)e^{\lambda\varphi(\sigma)}. \quad (2.2.26)$$

After some calculation one finds

$$\{A(\sigma), A(\sigma')\} = -\frac{\gamma^2}{4}\Big(\epsilon(\sigma-\sigma')(A(\sigma)-A(\sigma'))^2 - \coth(p/2)(A^2(\sigma)-A^2(\sigma'))\Big). \quad (2.2.27)$$

Finally one needs

$$\{A(\sigma), e^{\lambda\varphi_+(\sigma')}\} = \lambda\frac{\gamma^2}{2}\frac{e^{\frac{p}{2}\epsilon(\sigma-\sigma')}}{\sinh(p/2)}(A(\sigma) - A(\sigma')e^{\lambda\varphi_+(\sigma')} \quad (2.2.28)$$

to find

$$\begin{aligned}
&\{\psi^{j_1}_{m_1}(\sigma), \psi^{j_2}_{m_2}(\sigma')\} = \\
&\frac{\gamma^2}{2}\Big((j_1-m_1)(j_2-m_2)\epsilon(\sigma-\sigma') + \\
&\frac{1}{\sinh(p/2)}(m_1 j_2 e^{\frac{p}{2}\epsilon(\sigma-\sigma')} - m_2 j_1 e^{-\frac{p}{2}\epsilon(\sigma-\sigma')})\Big)\psi^{j_1}_{m_1}(\sigma)\psi^{j_2}_{m_2}(\sigma') \\
&+\frac{\gamma^2}{2}\frac{e^{\frac{p}{2}\epsilon(\sigma-\sigma')}}{2\sinh(p/2)}(j_1-m_1)(j_2+m_2)\psi^{j_1}_{m_1+1}(\sigma)\psi^{j_2}_{m_2-1}(\sigma') \\
&-\frac{\gamma^2}{2}\frac{e^{\frac{p}{2}\epsilon(\sigma-\sigma')}}{2\sinh(p/2)}(j_1+m_1)(j_2-m_2)\psi^{j_1}_{m_1-1}(\sigma)\psi^{j_2}_{m_2+1}(\sigma')
\end{aligned}$$

This can be rewritten in a more compact way in terms of the variables

$$\chi^j_m(\sigma) := (\sinh(p/2))^{\frac{-2j-m}{2}}\psi^j_m(\sigma) \quad (2.2.29)$$

if one introduces an sl_2-action on χ^j_m by

$$\begin{aligned}
F\chi^j_m &= (j+m)\chi^j_{m-1} \\
E\chi^j_m &= (j-m)\chi^j_{m+1} \\
H\chi^j_m &= 2m\chi^j_m,
\end{aligned} \quad (2.2.30)$$

uses a vector notation defined by $\chi^{j_1}_{m_1}\chi^{j_2}_{m_2} = (\chi^{j_1}\otimes\chi^{j_2})_{m_1m_2}$ and defines

$$4r^+ := H\otimes H + 4E\otimes F \quad , \quad 4r^- := -H\otimes H + 4F\otimes E. \quad (2.2.31)$$

Then one has

$$\{\chi^{j_1}(\sigma) \overset{\otimes}{,} \chi^{j_2}(\sigma')\} = \tfrac{\gamma^2}{8}(\epsilon(\sigma-\sigma')(r^+-r^-) + \coth(p/2)(r^++r^-))\chi^{j_1}(\sigma)\otimes\chi^{j_2}(\sigma').$$

These are the classical counterparts of exchange relations for the quantum operators that will become important later. Again, the occurrence of $\frac{1}{\sinh(p/2)}$ indicates trouble with the Hamilton formalism at $p = ik$, $k\in\mathbb{Z}$.

Section III:

Quantization of Liouville theory

In the present chapter I will give a general discussion of the principles the present approach to quantizing Liouville theory will be based on. The discussion has two main purposes:

First, to discuss the choice of the space of states. This question turns out to be nontrivial in the case corresponding to the elliptic sector of the classical theory. I will present evidence for my proposal that the problems of the Gervais-Neveu approach which occur at a discrete subseries of momenta may be cured by considering irreducible Virasoro representations instead of Fock spaces.

Second, I will outline the general strategy to be followed in the construction of Liouville field operators in order to identify the technical problems to be considered later.

Throughout, I will work in euclidean two-dimensional space-time, parametrized by variables $z, \bar{z}$. These are related to the coordinates τ, σ on the cylinder by $z = e^{\tau + i\sigma}$, $\bar{z} = e^{\tau - i\sigma}$.

1. Space of states

The analysis of the classical theory exhibits the crucial rôle of the conformal symmetry for the integrability of the theory. To preserve conformal symmetry upon quantization is therefore one of the most important requirements.
As a consequence, the space of states of the theory has to carry representations of the Virasoro algebra. More precisely, two commuting copies of the Virasoro algebra are required, the generators of which are collected in the chiral (resp. anti-chiral) parts of the energy momentum tensor.

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}n(n^2-1)\delta_{n,-m}$$

$$[\bar{L}_n, \bar{L}_m] = (n-m)\bar{L}_{n+m} + \frac{c}{12}n(n^2-1)\delta_{n,-m}$$

$$T(z) = \sum_n z^{-n-2} L_n \qquad \bar{T}(\bar{z}) = \sum_n \bar{z}^{-n-2}\bar{L}_n$$

As in the classical theory, the operator $H = L_0 + \bar{L}_0$ will be identified with the Hamiltonian. The space of states will be defined to be of the general form

$$\mathcal{H} = \bigoplus_I \mathcal{H}_I \otimes \bar{\mathcal{H}}_I, \tag{3.1.1}$$

where the L_n ($\bar{L}_n$) act on $\mathcal{H}_I$ (resp. $\bar{\mathcal{H}}_I$) only. In order to have energy bounded from below one only considers highest weight representations: These are representations which contain a vector v_I, called the highest weight vector, such that $L_n v_I = 0$ and $L_0 v_I = h_I v_I$. There are three different types of highest weight representations relevant for Liouville theory, called Verma modules, irreducible highest weight modules and Fock modules. The following subsections contain a review of the relevant properties of these representations.

1.1 Verma modules vs. irreducible highest weight modules

A Verma module $\mathcal{V}_h$ is defined as the highest weight representation for which the states

$$L_{-n_1} L_{-n_2} \ldots L_{-n_k} v_h \quad \text{with} \quad k \in \mathbb{N} \quad \text{and} \quad n_i \geq n_j \quad \text{for all} \quad i < j \quad i,j = 1\ldots k \tag{3.1.2}$$

form a basis. $\mathcal{V}_h$ may be decomposed into L_0 (energy) eigenstates.

$$\mathcal{V}_h = \bigoplus_{n\in\mathbb{N}} \mathcal{V}_h^{(n)} \qquad L_0 \mathcal{V}_h^{(n)} = (h+n)\mathcal{V}_h^{(n)}. \tag{3.1.3}$$

The subspace $\mathcal{V}_h^{(n)}$ with energy $h+n$ is spanned by all vectors of the form (3.1.2) such that $n = \sum_{i=1}^k n_i$. The number n will be called level in the following.

1.1.1 There is a unique bilinear form $(.,.)$ on $\mathcal{V}_h$ such that $(v_h, v_h) = 1$ and $(L_n\xi, \zeta) = (\xi, L_{-n}\zeta)$. The latter property implies that the energy momentum tensor on the cylinder $T(x) = \sum L_n e^{-in\sigma}$ is hermitian with respect to this inner product and that the inner product is invariant under conformal transformations. This inner product is in general degenerate. One may prove [KaRa] that its kernel is the maximal proper subrepresentation contained in $\mathcal{V}_h$. $(.,.)$ becomes nondegenerate on the *irreducible* representation

$\mathcal{H}_h := \mathcal{V}_h / Ker(.,.)$.

1.1.2 Reducibility is equivalent to the existence of vectors $\mathfrak{n}_h$ besides v_h that obey $L_n \mathfrak{n}_h = 0$, $n > 0$, called *null vectors*. An example is the vector $(L_{-2} - \frac{3}{2(2h+1)} L_{-1}^2) v_h$ which is a null vector if for given c the parameter h is a solution of $(2h+1)(4h+\frac{c}{2}) = 9h$. One may easily see that indeed the null vector and any vector generated from it are in $Ker(.,.)$.

1.1.3 $\mathcal{V}_h$ and $\mathcal{H}_h$ are up to isomorphism unique for given values of c and h. It is therefore important to know how the reduciblity of $\mathcal{V}_h$ depends on these parameters. The answer has been given by Feigin and Fuchs in [FF2]. The parts relevant for my purposes may be summarized as follows:

(1) *Case 1: c irrational*
In this case there is precisely one nullvector in the Verma modules to the discrete set

$$h_{n'n} = \frac{1}{4}((n'^2 - 1)\alpha_-^2 + (n^2 - 1)\alpha_+^2 + 2(1 - n'n)) \quad n', n \geq 1, \tag{3.1.4}$$

where

$$\alpha_\pm = \sqrt{\frac{1-c}{24}} \pm \sqrt{\frac{25-c}{24}}. \tag{3.1.5}$$

Note that $\alpha_\pm^2$ and consequently $h_{n'n}$ are real only if $c \leq 1$ or $c \geq 25$.
The irreducible module is obtained by dividing $\mathcal{V}_{h_{n'n}}$ by the submodule generated by this nullvector. For all other values of h the Verma module is irreducible.

(2) *Case 2:* $c = c_{p'p} = 1 - 6\frac{(p'-p)^2}{p'p}$ *with* p', p *relatively coprime*
Now the structure of the Verma-modules $\mathcal{V}_{n'n} := \mathcal{V}_{h_{n'n}}$ is more complicated. Consider first the case $1 \leq n' \leq p' - 1$, $1 \leq n \leq p - 1$. Because of the degeneracy $h_{n'n} = h_{p'-n',p-n}$ there now are two immediate[1] nullvectors in $\mathcal{V}_{n'n}$: One at level $n'n$, the other at level $(p' - n')(p - n)$. But now the weights of the null vectors are also among the $\{h_{n'n}; n', n \geq 1\}$, so that the Verma modules generated by the null vectors themselves contain nullvectors. One ends up with an infinite set of nullvectors which are contained in an infinite nested inclusion of Verma modules, one being generated by the nullvectors of the other. The structure of embeddings of Verma modules may be summarized by the following diagram:

[1] I will call nullvectors immediate if they are not contained in submodules generated by null vectors with lower level of the respective Verma module

$$\begin{tikzcd} & s_1 \arrow[r] \arrow[ddr] & s_3 \arrow[r] \arrow[ddr] & s_3 \arrow[r] \arrow[ddr] & s_3 \cdots \\ s_0 \arrow[ur] \arrow[dr] & & & & \\ & s_2 \arrow[r] \arrow[uur] & s_4 \arrow[r] \arrow[uur] & s_4 \arrow[r] \arrow[uur] & s_4 \cdots \end{tikzcd}$$

There is an arrow from a vector s_i to another vector s_j whenever s_j is an immediate nullvector in the Verma module generated by s_i. The cases $j'p' + 1 \leq n' \leq (j'+1)p' - 1$, $jp + 1 \leq n \leq (j+1)p + 1$ for $j, j' \in \mathbb{N}$ have a similiar structure, whereas the remaining cases $n' = j'p'$, $n = jp$, $j, j' \in \mathbb{N}$ also have a structure of nested inclusions of Verma modules, but with each Verma module having only one immediate null vector (instead of two in the other cases).

(3) *Case 3:* $c = \hat{c}_{p'p} = 25 + 6\frac{(p'-p)^2}{p'p}$ *with* p', p *relatively coprime*
The diagram that describes how the Verma modules are imbedded into each other is now obtained by the following remarkable symmetry: If one reverses all arrows in a diagramm for central charge $c_{p'p}$ in which the singular vectors s_i have conformal weights h_i then one gets the diagram for $\hat{c}_{p'p}$, where now the singular vectors at the corresponding vertices have conformal dimension $1 - h_i$.[2]

1.2 Fock modules

Consider the Fock space $\mathcal{F}_\alpha$ defined by acting with oscillators a_n, $n \in \mathbb{Z}$ with $[a_n, a_m] = 2n\delta_{n,-m}$ on a vector v_α which satisfies $a_n v_\alpha = 0$, $n > 0$ and $a_0 v_\alpha = \alpha v_\alpha$. One has

$$\mathcal{F}_\alpha = \bigoplus_{k=0}^{\infty} \bigoplus_{1 \leq n_1 \leq \ldots \leq n_k} \mathbb{C} a_{-n_1} \ldots a_{-n_k} v_\alpha. \tag{3.1.6}$$

There is a representation of the Virasoro algebra on this space given by

$$L_n = \frac{1}{4} \sum_{k=-\infty}^{\infty} a_{n-k} a_k - \alpha_0 (n+1) a_n, \quad n \neq 0,$$
$$L_0 = \frac{1}{2} \sum_{k=1}^{\infty} a_{-k} a_k + \frac{1}{4} a_0^2 - \alpha_0 a_0.$$

A direct calculation shows that the so defined L_n obey a Virasoro algebra with central

[2] This symmetry should have implications for 2D gravity, since it relates the structure of a module $\mathcal{H}_{n'n}$ to that which corresponds to its gravitational dressing.

charge $C = 1 - 24\alpha_0^2$. In the case $c \geq 25$ one may use $Q = -i\alpha_0$. The Fock space forms a highest weight module of the Virasoro algebra with highest weight

$$h_\alpha v_\alpha := L_0 v_\alpha = (\alpha^2 - 2\alpha_0\alpha)v_\alpha. \tag{3.1.8}$$

$\mathcal{F}_\alpha$ may also be decomposed into L_0-eigenstates:

$$\mathcal{F}_\alpha = \bigoplus_{n=0}^{\infty} \mathcal{F}_\alpha^{(n)}, \tag{3.1.9}$$

where $L_0\mathcal{F}_\alpha^{(n)} = (h_\alpha + n)$.

1.2.1 Expectation values of operators will be defined by means of the dual pairing: Denote the dual vector space to $\mathcal{F}_\alpha$ by $\mathcal{F}_\alpha^T$. It is the direct sum of the duals to the $\mathcal{F}_\alpha^{(n)}$:

$$\mathcal{F}_\alpha^T = \bigoplus_{n=0}^{\infty} (\mathcal{F}_\alpha^{(n)})^T, \tag{3.1.10}$$

Denote the dual pairing of $\xi \in \mathcal{F}_\alpha$ and $\zeta \in \mathcal{F}_\alpha^T$ by $< \zeta, \xi >$. The dual is also a Fock space:

$$\mathcal{F}_\alpha^T = \bigoplus_{k=0}^{\infty} \bigoplus_{1 \leq n_1 \leq \dots \leq n_k} \mathbb{C} a_{n_1}^T \dots a_{n_k}^T v_\alpha^T, \tag{3.1.11}$$

where a^T is the usual transpose of an operator, defined by $< a^T\zeta, \xi > = < \zeta, a\xi >$. v_α^T will be normalized by $< v_\alpha^T, v_\alpha > = 1$. There is a natural Virasoro action on $\mathcal{F}_\alpha^T$. It is defined by

$$< L_n\zeta, \xi > = < \zeta, L_{-n}\xi > . \tag{3.1.12}$$

It is important to note that $\mathcal{F}_\alpha^T$ is isomorphic as a Virasoro module to $\mathcal{F}_{-\alpha+2\alpha_0}$. The isomorphism may be defined by the mapping $a_n^T \to -a_{-n} + 4\alpha_0\delta_{n,0}$ which sends L_n^T to L_{-n}.

1.2.2 The structure of the Fock modules ([FF1]) depends on the value of α as well as the central charge c. In the generic case $h_\alpha \neq h_{n'n}$ the Fock space $\mathcal{F}_\alpha$ is isomorphic to the irreducible module $\mathcal{H}_{h_\alpha}$ (which coincides with the Verma module $\mathcal{V}_{h_\alpha}$). The situation is more complicated for those α where $h_\alpha = h_{n'n}$, $n'n > 0$. The corresponding values of α may be parametrized as

$$\alpha_{n'n} = \frac{1-n'}{2}\alpha_- + \frac{1-n}{2}\alpha_+. \tag{3.1.13}$$

I will first describe the case $c \neq c_{p'p}$. First, $\mathcal{F}_{-n',-n}$ $(n > 0, n' > 0)$ is isomorphic to the corresponding Verma module. The null vectors (also called singular vectors) are nonvanishing when expressed in terms of oscillators. Because of $-\alpha_{n'n} + 2\alpha_0 = \alpha_{-n',-n}$ and the isomorphism of $\mathcal{F}_\alpha^T$ and $\mathcal{F}_{-\alpha+2\alpha_0}$ one finds that $\mathcal{F}_{n'n}$ must be isomorphic to the dual of the Verma module $\mathcal{V}_{n'n}$. If the null state in $\mathcal{V}_{n'n}$ is expressed as some polynomial $\sigma_{n'n}$ in the L_{-n} acting on the highest weight state then the state $\sigma_{n'n}v_{n'n}$ vanishes *identically* if the L_{-n} are expressed in terms of oscillators. Instead there must be a Fock space state at the same level that is not created by the Virasoro action. This state is the dual to the singular vector in $\mathcal{F}_{n'n}$ and will therefore be called cosingular.

1.2.3 Again, everything gets much more complicated if $c = c_{p'p}$. What one needs to know is which parts of the Fock space have to be divided out in order to get the irreducible module. There is an elegant description of the irreducible modules inside the Fock space as a kind of BRST-cohomology due to Felder [Fel1]. It will be described in the next subsection. But to prepare for an explanation of how it works I will need to scetch the description given by Feigin und Fuchs [FF1] as presented in [Fel1]:
First of all $\mathcal{F}_{n'n}$ contains infinitely many singular vectors $u_1, u_2 \dots$ that generate a submodule $\mathcal{SF}_{n'n}^{(0)}$. This submodule is a direct sum of irreducible highest weight modules. A further set of infinitely many vectors $\dots v_{-1}, v_0, v_1 \dots$ become singular vectors if one divides $\mathcal{SF}_{n'n}^{(0)}$ out to get $\mathcal{F}_{n'n}^{(1)} := \mathcal{F}_{n'n}^{(0)}/\mathcal{SF}_{n'n}^{(0)}$. Here one identifies v_0 with $v_{n'n}$. Again the subspace $\mathcal{SF}_{n'n}^{(1)}$ generated by these vectors v_i is a direct sum of irreducible highest weight modules. Now there still are vectors that can not be generated by acting with Virasoro generators on $v_{n'n}$. These are contained in the quotient $\mathcal{F}_{n'n}^{(2)} := \mathcal{F}_{n'n}^{(1)}/\mathcal{SF}_{n'n}^{(1)}$, which finally also is a direct sum of irreducible highest weight modules.

1.3 Felder-BRST

Felder's characterization of the irreducilbe Virasoro module $\mathcal{H}_{n'n}$ contained in $\mathcal{F}_{n'n}$ essentially states that it is isomorphic to the cohomology of a suitably constructed BRST operator Q. It is constructed as an integral over products of vertex operators. These are operators that map one Fock space to another. Their definition therefore requires introducing an operator q conjugate to a_0 such that $e^{\alpha q}a_0 = (a_0 - 2\alpha)e^{\alpha q}$. It is defined as

$$V_\alpha := e^{\alpha q} z^{\alpha a_0} \exp\left(\alpha \sum_{n=1}^{\infty} \frac{a_{-n}}{n} z^n\right) \exp\left(-\alpha \sum_{n=1}^{\infty} \frac{a_n}{n} z^{-n}\right). \tag{3.1.14}$$

Then define

$$Q_m := \frac{1}{m} \int_C du_1 \dots du_m V_{\alpha_+}(u_1) \dots V_{\alpha_+}(u_m), \tag{3.1.15}$$

where C is a multiple contour encircling the origin. The most important property of Q in this context is that it commutes with the Virasoro algebra and therefore maps Virasoro modules to others (intertwining property).

1.3.1 Again I will start by discussing the case $c \neq c_{p'p}$. One may prove ([Fel1]) that $Q_m v_{-m',m}$ is the singular vector in $\mathcal{F}_{-m',-m}$. By the intertwining property of Q_m one therefore finds that the whole submodule generated by $Q_m v_{-m',m}$ is BRST-trivial. $\mathcal{F}_{-m',-m}/Q_m\mathcal{F}_{-m',m}$ is isomorphic to the irreducible module $\mathcal{H}_{m'm}$.
In addition one finds that

$$\mathcal{H}_{m'm} \simeq \mathrm{Ker}(Q_m : \mathcal{F}_{m'm} \to \mathcal{F}_{m',-m}) \tag{3.1.16}$$

by duality. The irredcible submodule in $\mathcal{F}_{m'm}$ is therefore the submodule of BRST-invariant states.

In the case $c = c_{p'p}$ consider the class of Fock modules $\mathcal{F}_{m'm}$ relevant for the minimal models, i.e. $1 \leq m' \leq p'-1$ and $1 \leq m \leq p$. Now the result of [Fel1] may be stated as follows:
There exist operators $Q_{p-m} : \mathcal{F}_{m',2p-m} \to \mathcal{F}_{m'm}$ and $Q_m : \mathcal{F}_{m'm} \to \mathcal{F}_{m',-m}$ such that
(i) $Q_m Q_{p-m} = 0$
(ii) $\mathrm{Ker}Q_m/\mathrm{Im}Q_{p-m}$ is isomorphic to the irreducible highest weight module $\mathcal{H}_{m'm}$.

1.4 On the Liouville space of states

All existing approaches to quantum Liouville theory use Fock-spaces to define the space of states. This is absolutely justified for quantization of the hyperbolic sector, since in that case the classical map from a real free field to the Liouville field is one-to-one, i.e. any Liouville solution has an unique representation in terms of oscillator variables[3]. However, in the present work I shall mainly be concerned with the quantization of the elliptic sector corresponding to imaginary values of the momentum. This sector is particularly interesting due to the fact that in 2D gravity models one needs states from this sector in order to construct physical states (the so-called gravitational dressing).

In trying to use Fock-spaces for the construction of the elliptic sector of quantum Liouville theory, one has the problem of finding appropriate hermiticity conditions on the oscillators that ensure hermiticity of the energy-momentum tensor. In terms of the Virasoro generators one needs $L_n^\dagger = L_{-n}$.

For the following considerations it will be useful to use the form of the Virasoro generators that is more natural when working on the cylinder instead of the complex plane:

$$L_n = \frac{1}{4}\sum_{k\in\mathbb{Z}} : p_{n-k}p_k : + iQnp_n, \tag{3.1.17}$$

[3] Unique of course only if a branch for the momentum variable has been choosen, two-to-one otherwise

where $: \ldots :$ denotes the usual normal ordering and the p_n coincide with the a_n except for a shift of the zero-mode: $p_0 = a_0 + 2iQ$. This shift may be seen to arise from the transformation from the cylinder to the complex plane, so that p_0 is the "true" momentum.

In order to describe the elliptic sector one demands p_0 to have purely imaginary eigenvalues p. p_0 will therefore be required to be antihermitian.[4] But then it is nontrivial to find hermiticity conditions on the oscillators a_n that ensure $L_n^\dagger = L_{-n}$. One has to solve the equations

$$L_n^\dagger = \frac{1}{4}\sum_{k\in\mathbb{Z}} : p_k^\dagger p_{n-k}^\dagger : + iQnp_n^\dagger = \frac{1}{4}\sum_{k\in\mathbb{Z}} : p_{-n-k}a_k : + iQnp_{-n} = L_{-n} \tag{3.1.18}$$

for $p_n^\dagger$ as function of the p_k, $k \in \mathbb{Z}$ under the condition $p_0^\dagger = -p_0$. A related problem was observed in the classical theory.

In the approach of Gervais and Neveu this problem is treated as follows: They propose to start by considering two free fields with oscillators p_n and $\tilde{p}_n$ with corresponding Virasoro generators L_n and $\tilde{L}_n$ that are related to each other via the constraints

$$L_n = \tilde{L}_n \qquad \text{and} \qquad p_0 = -\tilde{p}_0. \tag{3.1.19}$$

In [GN3] it is shown that for generic values of p_0 (i.e. $p \neq p_{n',n} \equiv \alpha_{n'n} + iQ$, $n'n > 0$) there indeed exists a solution $\tilde{p}_n = \tilde{p}_n(\{p_k; k \in \mathbb{Z}\})$ of (3.1.19) as a formal power series $\tilde{p}_n = \sum_{r=0}^{\infty} f^{(r)}(\{p_k; k \in \mathbb{Z}\})$. Hermiticity of the energy momentum tensor is now achieved by imposing $p_n^\dagger = \tilde{p}_{-n}$.

However, it has already been noted by J. Schnittger that the constraint (3.1.19) can not be solved on states with the momenta $p_{n',n}$, $n'n > 0$. The simplest example is the case $p = p_{1,1} = iQ$ where

$$L_{-1}v_{iQ} = 0 \qquad \text{but} \qquad \tilde{L}_{-1}v_{iQ} \neq 0. \tag{3.1.20}$$

More generally, hermiticity of the energy momentum tensor can not be achieved on the Fock spaces build upon $v_{\alpha_{n',n}}$ for $n'n > 0$: If the null state in $\mathcal{F}_{-n',-n}$ is written as the polynomial $\sigma_{n'n}(\{L_{-n}\})$ acting on $v_{-n',-n}$ then (see section 1.2.2):

$$\sigma_{n'n}(\{L_{-n}\})v_{-n',-n} \neq 0 \qquad \text{but} \qquad \sigma_{n'n}(\{\tilde{L}_{-n}\})v_{-n',-n} = 0. \tag{3.1.21}$$

[4] If one chooses p_0 to be hermitian, then the inner product of momentum eigenstates v_p must be off-diagonal: $(v_p, v_{p'}) = \delta_{p+p',0}$. Such an inner product can not be a scalar product. This may perhaps have a sensible interpretation in the context of 2D gravity: Seiberg has proposed in [Sei] that wave functions of states with imaginary momentum have to be non-normalizable in order to be associated with *local* states.

My proposal to avoid this problem will be to take irreducible Virasoro modules instead of Fock modules for the definition of the space of states. The problem with hermiticity of energy momentum is avoided from the very beginning: Since one does not have to realize the Virasoro generators in terms of oscillators, it is possible to achieve $L_n^\dagger = L_{-n}$ simply by definition of the inner product.
The physical heuristics behind my proposal is the following: The Fock-space $\mathcal{F}_{n',n}$ provides a redundant parametrization of the space of states: It contains unphysical states, on which energy-momentum is not hermitian. However, hermiticity of energy momentum may be maintained on a subspace of physical states, which is defined as Felder-BRS-cohomology and is isomorphic to the irredicible Virasoro module. This interpretation is supported by the observation made in the classical theory that free field variables provide a redundant parametrization of the space of solutions at integer values of the momenta (which are the classical limits of the $p_{n',n}$.).

Abovementioned problems with the free field realization do not mean that they cannot be used for explicit constructions of operators or for obtaining integral representations for correlation functions: One may define matrix elements, expectation values etc. by means of the dual pairing. Since $< L_n\zeta, \xi > = < \zeta, L_{-n}\xi >$ the dual pairing reduces to the Shapovalov form on the subspace of physical states. The problem is to take care that all relevant operators commute with the Felder-BRST operator in order to ensure decoupling of the unphysical states!

From the point of view of starting from two free fields and imposing (3.1.19) as a constraint one may argue as follows:
One starts from a space $\mathfrak{F}_{n',n}$ that is generated by applying both p_{-n} and $\tilde{p}_{-n}$ to $v_{n',n}$. Acting on $v_{n',n}$ with p_n (resp. $\tilde{p}_n$) oscillators only defines subspaces $\mathfrak{S}_{n',n}$ and $\tilde{\mathfrak{S}}_{n',n}$ which are vir-isomorphic to $\mathcal{F}_{n',n}$ and $\mathcal{F}_{-n',-n}$ respectively. Constraint (3.1.19) can only be imposed on subspaces (or rather quotient spaces) of $\mathfrak{S}_{n',n}$ and $\tilde{\mathfrak{S}}_{n',n}$ that are vir-isomorphic to each other. These are obtained by dividing out all singular and cosingular vectors of $\mathcal{F}_{n',n}$ and $\mathcal{F}_{-n',-n}$ and everything that is generated from them. In $\mathfrak{F}_{n',n}$ one will have to divide out every vector that can be obtained by acting both with p_{-n} and $\tilde{p}_{-n}$ on the singular and the cosingular vectors in the submodules $\mathfrak{S}_{n',n}$ and $\tilde{\mathfrak{S}}_{n',n}$. I suspect that the subspace obtained by dividing out the singular subspaces and imposing (3.1.19) is isomorphic to the irreducible Virarsoro module.

Anyway, taking irreducible Virasoro modules in the definition of the space of states will be seen to lead to a consistent extension of the Gervais-Neveu approach to modules corresponding to the conformal weights $h_{n',n}$: Most virtues and important achievments of the Gervais-Neveu approach will be preserved, but the problems with the hermiticity of the energy momentum tensor avoided. In the case $c < 1$ this proposal will of course naturally incorporate the minimal models into Liouville theory.

2. Construction of operators: General strategy

Before going into detailed technical discussions, I now want to give a general overview of the approach to the construction of field operators to be followed in this work. Certain general assumptions that are motivated by the analysis of the classical theory suffice to put the theory into a framework that is similar to that of Moore and Seiberg [MS] or Felder, Fröhlich and Keller [FFK] for rational conformal field theories.

The primary goal will be the construction of operators that transform covariantly with respect to the Virasoro algebras:

$$[L_n, \Phi_h(z,\bar{z})] = z^n(z\partial_z + h(n+1))\Phi_h(z,\bar{z}) \tag{3.2.1}$$
$$[\bar{L}_n, \Phi_h(z,\bar{z})] = \bar{z}^n(\bar{z}\partial_{\bar{z}} + h(n+1))\Phi_h(z,\bar{z}). \tag{3.2.2}$$

I already mentioned the main motivations for this: First, these are operators relevant for applications to noncritical strings and 2d gravity. Second, Liouville exponentials were seen to be covariant objects classically.

It can be proved [FFK][5] that the covariant operator Φ is of the form (chiral factorization)

$$\Phi_h(z,\bar{z}) = \sum_m D^h_m \phi^h_m(z)\bar{\phi}^h_m(\bar{z}). \tag{3.2.3}$$

The basic requirement on the quantum operators ϕ^h_m is that of conformal covariance

$$[L_m, \phi^h_m(z)] = z^n(z\partial_z + h(n+1))\phi^h_m(z). \tag{3.2.4}$$

Consider a field $\phi^h_m(z)$ acting on a highest-weight state $\xi \in \mathcal{H}_{h_1}$. The resulting state can be decomposed into states from different modules:

$$\phi^h_m(z)\xi = \sum_{h_2} \varphi^h_{h_2h_1}(z) \quad \text{with} \quad \varphi^h_{h_2h_1} \in \mathcal{H}_{h_2}. \tag{3.2.5}$$

It is therefore natural to consider as elementary objects the operators

$$\psi\begin{pmatrix} h \\ h_2 h_1 \end{pmatrix}(z) : \mathcal{H}_{h_1} \to \mathcal{H}_{h_2} \quad \text{and} \quad \bar{\psi}\begin{pmatrix} h \\ h_2 h_1 \end{pmatrix}(\bar{z}) : \bar{\mathcal{H}}_{h_1} \to \bar{\mathcal{H}}_{h_2} \tag{3.2.6}$$

which transform according to (3.2.4). Such operators are called chiral vertex operators. Mathematically, chiral vertex operators are intertwiners between different Virasoro representations. They will be seen to be uniquely defined by their intertwining property.

[5] The argument given in [FFK] does not use rationality of the CFT.

The physical field operators $\Phi_h(z,\bar{z})$ may be reconstructed from the chiral vertex operators in the form

$$\Phi_h(z,\bar{z}) = \sum_{h_1,h_2} D\binom{h}{h_2 h_1}\psi\binom{h}{h_2 h_1}(z)\bar{\psi}\binom{h}{h_2 h_1}(\bar{z}), \tag{3.2.7}$$

where $D\binom{h}{h_2 h_1}$ are numbers that depend on the normalizations of ψ, $\bar{\psi}$. The construction of the chiral vertex operators is therefore a decisive step in the construction of the theory, and will be one of the main themes to be treated in the present work.

It will be seen later that there is at least a large subclass of chiral vertex operators that satisfy exchange (or braid-) relations[6] of the following form:

$$\psi\binom{h_1}{h_\infty\, h}(z_1)\psi\binom{h_2}{h\, h_0}(z_2) = \sum_{h'} B_{hh'}\begin{bmatrix} h_1 & h_2 \\ h_\infty & h_0 \end{bmatrix}\psi\binom{h_2}{h_\infty\, h'}(z_2)\psi\binom{h_1}{h'\, h_0}. \tag{3.2.8}$$

The matrix B is called braid matrix. For the anti-chiral vertex operators $\bar{\psi}$ one then has a similiar relation, the corresponding braid matrix denoted $\bar{B}$. In the present case it turns out that the anti-chiral operators $\bar{\psi}$ can always be constructed such that B and $\bar{B}$ are related by complex conjugation.
Two physical field operators Φ_{h_1} and Φ_{h_2} will then be local to each other if a relation of the form

$$\sum_h D\binom{h_1}{h_\infty\, h}D\binom{h_2}{h\, h_0}B_{hh'}\begin{bmatrix} h_1 & h_2 \\ h_\infty & h_0 \end{bmatrix}\bar{B}_{hh''}\begin{bmatrix} h_1 & h_2 \\ h_\infty & h_0 \end{bmatrix} = \delta_{h'h''}D\binom{h_2}{h_\infty\, h'}D\binom{h_1}{h'\, h_0} \tag{3.2.9}$$

holds. It turns out that the braiding matrices of all chiral vertex operators needed for the construction of Liouville exponentials may be expressed in terms of the Racah-Wigner coefficients of the quantum group $\mathcal{U}_q(sl(2))$, in which case (3.2.9) reduces to the well-known orthogonality relations for the q-Racah-Wigner coefficients, see [GS].

These considerations put quantum Liouville into a general framework for conformal field theories. However, further assumptions are needed to establish the correspondence to the classical theory: Which of the covariant field operators are to be identified with Liouville exponentials? Remember that classically the chiral components $\psi^j_m(x_+)$, $\bar{\psi}^j_m(x_-)$ of a Liouville exponential were given as products of powers of the fundamental fields f_1, f_2, see (2.2.20). These were defined as linear independent solutions of $\partial^2 f = Tf$. One of the basic results obtained by Gervais and Neveu is the following: There exist two operators ψ_1, ψ_2 such that

[6] In the framework of algebraic quantum field theory one may prove existence of exchange relations from general principles of two-dimensional quantum field theory, see [FRS]. However, these results can not be applied in the present approach to Liouville theory, since unitarity of the representations is not assumed a priori, but can in some cases be established a posteriori.

1. ψ_i, $i = 1,2$ satisfy a differential equation of the form
$$\partial^2\psi(z) = \gamma : T(z)\psi(z) :, \tag{3.2.10}$$
where $:(\ldots):$ denotes an appropriate normal ordering.

2. Both operators transform covariantly under conformal transformations according to (3.2.4), where the conformal weight h goes into its classical value of $\frac{1}{2}$ in the classical limit.

3. Acting with ψ_i on states $\xi \in \mathcal{H}_{h_1}$ produces states from only one representation $\mathcal{H}_{h_i(h_1)}$. The decomposition of ψ_i in terms of chiral vertex operators is therefore of the form
$$\psi_i(z) = \sum_{h_1}\psi\binom{h}{h_i(h_1)\, h_1}(z) \qquad i = 1,2. \tag{3.2.11}$$

4. ψ_i satisfy an exchange relation of the form
$$\psi_k(z_1)\psi_l(z_2) = \sum_{mn} B^{mn}_{kl}\psi_m(z_2)\psi_n(z_1), \tag{3.2.12}$$
which reproduces the Poisson brackets of f_i in the classical limit.

These properties justify to view ψ_i as the quantum operators corresponding to the f_i. I will rederive these properties in the present approach that uses irreducible representations instead of Fock spaces as space of states.
By taking operator products of ψ_1, ψ_2, one generates a discrete series of operators $\psi^j_m(z)$ that are to be identified with the quantum analogs of products of powers of f_1, f_2. Covariant operators build from this discrete series will be identified with Liouville exponentials.

Construction of the operators ψ^j_m for all $j \in \mathbb{N}$, $-j \leq m \leq j$ suffices for the construction of Liouville exponential operators. However, it is probable that these do not encompass all operators of interest. For example, Gervais has noted in [Ge2] that the operators corresponding to powers of the metric in models of 2D gravity can not in general be constructed from the ψ^j_m. This motivates me to consider the construction of chiral vertex operators in a more general framework as required for the construction of the ψ^j_m only.

One may wonder that a quantized version of the Liouville field equation is not explicitely required. In considering covariant field operators one has already made an assumption on the dynamics by identifying $L_0 + \bar{L}_0$ with the Hamiltonian: The covariance property (3.2.2) explicitely determines the time evolution just as in the classical theory. It is therefore not surprising that a quantum version of the Liouville equation can be proved a posteriori, as has been shown by Gervais and Schnittger [GS].

3. Chiral vertex operators

The first approach to the construction of chiral vertex operators will exclusively rely on their conformal covariance properties. I will consider covariant operators between Verma-modules, a subclass of which will be found to project to irreducible modules. Conformal covariance uniquely defines (up to normalization) arbitrary matrix elements of operators between irreducible modules. This leads to a unique determination of correlation functions as formal power series with coefficients given by complicated recursion relations. For a certain subclass of chiral vertex operators that correspond to the ψ^j_m one finds operator differential equations which lead to more manageable information on the correlation functions. In the simplest nontrivial case the operator differential equation is equivalent to the one used by Gervais and Neveu.

3.1 Construction

I now want to investigate existence and uniqueness of such operators $\psi_{\mathbb{H}}(z)$, $\mathbb{H} = \binom{h_2}{h_3 h_1}$:

3.1.1 First of all note that it suffices to define the action of $\psi_{\mathbb{H}}(z)$ on the highest weight state v_{h_1}: Given that, simply define its action on arbitrary states recursively via

$$\psi_{\mathbb{H}}(z)L_{-k}\xi_{h_1} = -z^{-k}(z\partial_z + h(1-k))\psi_{\mathbb{H}}(z)\xi_{h_1} + L_{-k}\psi_{\mathbb{H}}(z)\xi_{h_1}. \tag{3.3.1}$$

3.1.2 The vector $\psi_{\mathbb{H}}(z)v_{h_1}$ can be decomposed into L_0 eigenstates:

$$\psi_{\mathbb{H}}(z)v_{h_1} = \sum_{n=0}^{\infty}\xi^{(n)}_{\mathbb{H}}(z), \tag{3.3.2}$$

where $\xi^{(n)}_{\mathbb{H}}(z) \in \mathcal{V}_{h_3}$ and $L_0\xi^{(n)}_{\mathbb{H}}(z) = (h_3+n)L_0\xi^{(n)}_{\mathbb{H}}(z)$. Acting on this expansion with L_0 and using (3.2.4) allows to determine the z-dependence:

$$\xi^{(n)}_{\mathbb{H}}(z) = \zeta^{(n)}_{\mathbb{H}}z^{h_3-h_2-h_1+n}. \tag{3.3.3}$$

Relations between the $\zeta^{(n)}_{\mathbb{H}}$ for different values of n are obtained by calculating the action of L_k on the vectors $\zeta^{(n)}_{\mathbb{H}}$ with the help of (3.2.4) and $L_k v_{h_1} = 0$:

$$L_k\zeta^{(n)}_{\mathbb{H}} = (\Delta + n - k + h(k+1))\zeta^{(n-k)}_{\mathbb{H}} \quad \text{for} \quad k \leq n. \tag{3.3.4}$$

3.1.3 In the following it will be demonstrated that the relations (3.3.4) may be used for a recursive determination of those components of $\zeta^{(n)}_{\mathbb{H}}$, $n>0$ that lie in the irreducible submodule of $\mathcal{V}_{h_3}$. The initial term $\zeta^{(0)}_{\mathbb{H}}$ is proportional to v_{h_3}. Since the factor of proportionality will multiply all $\zeta^{(n)}_{\mathbb{H}}$, it is called the normalization $\mathcal{N}_{\mathbb{H}}$ of $\psi_{\mathbb{H}}(z)$.

In order to investigate the solvability of (3.3.4) first note that L_{-1}, L_{-2} generate all other L_{-k}, $k>0$. It follows that one may write an arbitrary state $\xi^{(n)} \in \mathcal{V}^{(n)}_{h_3}$ as

$$\xi^{(n)} = L_{-1}\xi^{(n-1)} + L_{-2}\xi^{(n-2)} \quad \text{with} \quad \xi^{(i)} \in \mathcal{V}^{(i)}_{h_3}, \quad i = n, n-1, n-2. \tag{3.3.5}$$

In particular, $\mathcal{V}^{(n)}_{h_3}$ has the same dimensionality as $\mathcal{W}^{(n)}_{h_3} := \mathcal{V}^{(n-1)}_{h_3} \oplus \mathcal{V}^{(n-2)}_{h_3}$. Now define a linear map $A^{(n)}$:

$$A^{(n)}: \quad \mathcal{V}^{(n)}_{h_3} \to \mathcal{W}^{(n)}_{h_3} \qquad A^{(n)}(\xi^{(n)}) = L_1\xi^{(n)} + L_2\xi^{(n)}.$$

Solvability of (3.3.4) is equivalent to invertibility of $A^{(n)}$. It is obvious that invertibility of $A^{(n)}$ is equivalent to the existence of nullvectors in $\mathcal{V}^{(n)}_{h_3}$.[7] In all cases where $\mathcal{V}_{h_3}$ is irreducible, (3.3.4) has a unique solution $\zeta^{(n)}_{\mathbb{H}}$ for all n.

3.1.4 Only the case that $\mathcal{V}_{h_3}$ does contain null vectors needs further discussion. Assume that n is the first level where a null vector occurs, so that I may assume $\zeta^{(m)}_{\mathbb{H}}$, $m<n$ to be known. $A^{(n)}$ is still invertible on its image. However, equation (3.3.4) is solvable if and only if $\zeta^{(n-1)}_{\mathbb{H}} + \zeta^{(n-2)}_{\mathbb{H}} \in \mathcal{W}^{(n)}_{h_3}$ has no component in the direct complement of the image of $A^{(n)}$. Since any coefficient of an expansion of $\zeta^{(m)}_{\mathbb{H}}$, $m<n$ with respect to some basis is a function of h_1, h_2, h_3, c only, $A^{(n)}$ is invertible only if the triple (h_1, h_2, h_3) satisfies certain c-dependent restrictions, the fusion rules. These will be determined explicitely below.

3.1.5 Now assume that the fusion rules are satisfied, such that (3.3.4) is solvable. $\zeta^{(n)}_{\mathbb{H}}$ is then uniquely determined up to the coefficient of the null vector itself. After giving it an arbitrary value $\mathcal{N}^{(1)}_{\mathbb{H}}$, the recursion may be continued until the next null vector occurs. If that is the case, additional fusion rules will have to be satisfied. In the case $c = c_{p'p}$ one may have to introduce an infinite set of additional normalization constants $\mathcal{N}^{(i)}_{\mathbb{H}}$, $i = 1, \ldots, \infty$.

3.2 Fusion rules

In order to establish a precise criterion for existence of chiral vertex operators, I will now give a second (equivalent) definition. Roughly speaking, the strategy is the following: As explained above, it suffices to define the vector $\psi_{\mathbb{H}}(z)v_{h_1}$. One may always define candidates for three point functions $<\rho, \psi_{\mathbb{H}}(z)v_{h_1}>$ with arbitrary out-states ρ by writing ρ in terms of Virasoro generators acting on a highest weight state and using $<L_{-n}\sigma, \psi_{\mathbb{H}}(z)v_{h_1}> = <\sigma, [L_n, \psi_{\mathbb{H}}(z)]v_{h_1}>$, covariance of $\psi_{\mathbb{H}}$ and $<v_{h_3}, \psi_{\mathbb{H}}(z)v_{h_1}> =$

[7] A state annihilated by L_1 and L_2 will be annihilated by all L_k, $k>0$.

$z^{h_3-h_2-h_1}$. This amounts to the definition of $\psi_{\mathbb{H}}$ as an operator from $\mathcal{V}_{h_1}$ to the *dual* of $\mathcal{V}_{h_3}$. It turns out that this object may be turned into an operator that maps to $\mathcal{V}_{h_3}$ if and only if decoupling of null vectors holds.

3.2.1 A linear form on $\mathcal{V}_{h_3}$, suggestively denoted by $[\psi_{\mathbb{H}}(z)v_{h_1}]^T$, will be defined as follows: First set $[\psi_{\mathbb{H}}(z)v_{h_1}]^T(v_{h_3}) = z^{h_3-h_2-h_1}$. Extend this definition to arbitrary $\rho \in \mathcal{V}_{h_3}$ by induction on the level of ρ: Assume $[\psi_{\mathbb{H}}(z)v_{h_1}]^T(\rho^{(i)})$ to be defined for any $\rho^{(i)} \in \mathcal{V}_{h_3}^{(i)}$ with $i < n$. As already noted before (3.3.5), there exist unique $\rho^{(n-1)}$, $\rho^{(n-2)}$ such that $\rho^{(n)} = L_{-1}\rho^{(n-1)} + L_{-2}\rho^{(n-2)}$. Then define

$$[\psi_{\mathbb{H}}(z)v_{h_1}]^T(\rho^{(n)}) := \sum_{k=1,2} z^k(z\partial + (k+1)h_2)[\psi_{\mathbb{H}}(z)v_{h_1}]^T(\rho^{(n-k)}). \tag{3.3.6}$$

It is clear that $[\psi_{\mathbb{H}}(z)v_{h_1}]^T(\rho^{(n)})$ depends on z only by $z^{h_3-h_2-h_1+n}$. This concludes the definition of $[\psi_{\mathbb{H}}(z)v_{h_1}]^T$.

3.2.2 Now observe that *if* a chiral vertex operator $\psi_{\mathbb{H}}(z)$ *exists* then one immediately has that

$$[\psi_{\mathbb{H}}(z)v_{h_1}]^T(\rho^{(n)}) = (\rho^{(n)}, \psi_{\mathbb{H}}(z)v_{h_1}), \tag{3.3.7}$$

since both objects obey the recursion relations (3.3.6). A necessary condition for existence of ψ is therefore that

$$[\psi_{\mathbb{H}}(z)v_{h_1}]^T(\mathfrak{n}^{(n)}) = 0, \tag{3.3.8}$$

if $\mathfrak{n}^{(n)}$ is a null vector at level n in $\mathcal{V}_{h_3}$. On the other hand side, if $[\psi_{\mathbb{H}}(z)v_{h_1}]^T$ satisfies (3.3.8), then it is a standard fact from linear algebra that there exists an element $\xi^{(n)} \in \mathcal{V}_{h_3}^{(n)}$ such that

$$[\psi_{\mathbb{H}}(z)v_{h_1}]^T(\rho^{(n)}) = (\rho^{(n)}, \xi^{(n)}), \tag{3.3.9}$$

$\xi^{(n)}$ being unique up to nullvectors[8]. The state $\psi_{\mathbb{H}}(z)v_{h_1}$ is then obtained by summing the $\xi^{(n)}$ as in equation (3.3.2).

3.2.3 In conclusion, I found that the necessary and sufficient condition for an operator ψ to exist is that the linear form $[\psi_{\mathbb{H}}(z)v_{h_1}]^T$ vanishes on null vectors.

3.2.4 The relation to the previous construction of chiral vertex operators is found as follows: Because of

$$(\rho^{(n)}, \xi^{(n)}) = (\rho^{(n-1)}, L_1\xi^{(n)}) + (\rho^{(n-2)}, L_2\xi^{(n)})$$

[8] Let $t: \mathcal{V}^{(n)} \to \mathcal{V}^{(n)T}$, $\xi^{(n)} \to (.,\xi^{(n)})$ be a linear map from $\mathcal{V}^{(n)}$ to its dual $\mathcal{V}^{(n)T}$. May further N be the subspace of $\mathcal{V}^{(n)T}$ that annihilates the null vectors in $\mathcal{V}^{(n)}$. Clearly $Imt \subset N$. But since Imt and N have the same dimensionality they must be equal. Therefore t is invertible on N.

on one hand side and

$$(\rho^{(n)}, \xi^{(n)}) = \sum_{k=1,2} z^k(\Delta + n - k + h(k+1))(\rho^{(n-k)}, \xi^{(n-k)})$$

on the other hand side one finds the recursion relations

$$L_k\xi^{(n)} = z^k(\Delta + n - k + h(k+1))\xi^{(n-k)} \pmod{\mathfrak{n}} \tag{3.3.10}$$

where mod $\mathfrak{n}$ indicates that these equalities hold up to elements of the subspace of $\mathcal{V}_{h_3}$ generated by the null vectors. If one now considers the case that n is the lowest level at which a nullvector occurs, then equations (3.3.10) hold exactly since both hand sides are objects of level less than n. These equations are just the recursion relations used in the previous construction of $\psi_{\mathbb{H}}(z)$. However, now I already know that a solution $\xi^{(n)}$ exists provided eqn. (3.3.8) holds.

3.2.5 Now consider the case that $h_3 = h_{n'n}$, such that $\mathcal{V}_{h_3}$ contains a null vector at level $n'n$. The problem of evaluation of (3.3.8) has essentially been solved by Feigin and Fuchs in [FF2]:[9] What they calculate is the following: There is a simple Virasoro representation on a vector space with basis $\{f_n;\ \ n \in \mathbb{Z}\}$ defined by $L_{-k}f_n := (\mu + n - \lambda(k+1))f_{n+k}$, $\ cf_n := 0$. Now write the null vector $\mathfrak{n}_{n'n} \in \mathcal{V}_{n'n}$ as $\mathfrak{n}_{n'n} = \sigma_{n'n}v_{n'n}$, where $\sigma_{n'n}$ is a polynomial in the L_{-k}'s. What Feigin and Fuchs calculated is $\sigma_{n'n}f_0 = P(\mu, \lambda; n', n; t)$, where $t = -\alpha_+^2$.
Before stating the result, I want to demonstrate that this is precisely what one needs to evaluate eqn. (3.3.8). Consider the vector space of three point functions $(\rho^{(n)}, \psi_{\mathbb{H}}(z)v_{h_1})$, which has as basis $\{z^{h_3-h_2-h_1+n};\ \ n \in \mathbb{N}\}$. The operation

$$(\rho^{(n)}, \psi_{\mathbb{H}}(z)v_{h_1}) \to (L_{-k}\rho^{(n)}, \psi_{\mathbb{H}}(z)v_{h_1}) = (\Delta + n + h_2(k+1))(\rho^{(n)}, \psi_{\mathbb{H}}(z)v_{h_1})$$

defines a Virasoro action on this vector space which may be idenitified with the action on the f_n provided $\mu = h_3 - h_2 - h_1$, $\lambda = -h_2$. The evaluation of (3.3.8) therefore yields a result proportional to $\sigma_{n'n}z^{h_3-h_2-h_1} = P(h_3 - h_2 - h_1, -h_2; n', n; t)z^{h_3-h_2-h_1+n'n}$.

3.2.6 The result of [FF2] may be conveniently written by introducing "spins" $j' = \frac{n'-1}{2}$, $j = \frac{n-1}{2}$ to parametrize the conformal dimensions $h_{n',n}$ as

$$h(j', j) = h_{2j'+1,2j+1} = (j'\alpha_- + j\alpha_+)((j'+1)\alpha_- + (j+1)\alpha_+). \tag{3.3.11}$$

Then $\sigma_{n'n}f_0 = P(h_3 - h_2 - h_1, -h_2; n', n; t) \equiv Q(h_1, h_2; j', j; \alpha_+)$, where

$$Q(h_1, h_2; j', j; \alpha_+) = \prod_{\substack{-j' \leq m' \leq j' \\ -j \leq m \leq j}} \Big\{ (h_2 - h_1)^2 + 2(h_2 + h_1)[m'\alpha_- + m\alpha_+]^2 + h(m', m)h(-m', -m) \Big\}$$

[9] The use of the result of [FF2] is inspired by [FF3], where the first general proof of the minimal model fusion rules was given. However, the construction given in [FF3] of the objects that correspond to the chiral vertex operators is rather different to the more physics-orientated formalism developed here.

Another useful form of this expression is obtained by parametrizing $h_1 = p_1^2 - \alpha_0^2$, $h_2 = (p_1+\delta)^2 - \alpha_0^2$ and introducing $r_{m'm} = m'\alpha_- + m\alpha_+$:

$$Q(h_1,h_2;j',j;\alpha_+) = \prod_{\substack{-j'\le m'\le j'\\ -j\le m\le j}} (\delta + r_{m'm})(\delta - r_{m'm})(\delta + 2p_1 + r_{m'm})(\delta + 2p_1 - r_{m'm}).$$

3.2.7 A chiral vertex operator that maps to $\mathcal{V}_{h(j_3',j_3)}$ will exist whenever $Q(h_1,h_2;j',j;\alpha_+)$ vanishes. This leads to a quadratic equation on h_1,h_2 for any pair m', m. If $h_1 = h(j_1',j_1)$ then the allowed values for h_2 are $h_2 = h(j_1'+m', j_1+m)$, where $-j_3' \le m' \le j_3'$ and $-j_3 \le m \le j_3$. For generic central charge one has only one null vector in $\mathcal{V}(j_3',j_3)$, so that above restriction (fusion rule) is the only constraint on the existence of chiral vertexs operators.

3.2.8 The situation is slightly different if $c = c_{p'p}$. Then the null subspace of $\mathcal{V}(j_3',j_3)$ is generated by two nullvectors. Consider the case $0 \le 2j_3' \le p-2$, $0 \le 2j_3 \le p-2$. One of the null vectors is just $\mathfrak{n}_{n'n}$ the second null vector $\mathfrak{n}_{p'-n',p-n}$ arises due to the degeneracy $h_{n'n} = h_{p'-n',p-n}$. So one of the restrictions on j_1', j_1, j_2', j_2 is the one found above, the other one is obtained by replacing $j_3' \to \frac{p'}{2} - (j_3'+1)$, $j_3 \to \frac{p}{2} - (j_3+1)$. Now it is important to observe that because of $p'\alpha_- + p\alpha_+ = 0$ one has $R(j',j;m',m) = R(\frac{p'}{2} - (j'+1), \frac{p}{2} - (j+1); m', m)$. Therefore the only further consequence of the second nullvector is to restrict m', m to the range

$$\begin{aligned}\max(-j_3', j_3' + 1 - \tfrac{p'}{2}) \le m' \le \min(j_3', \tfrac{p'}{2} - (j_3'+1))\\ \max(-j_3, j_3 + 1 - \tfrac{p}{2}) \le m \le \min(j_3, \tfrac{p}{2} - (j_3+1))\end{aligned} \qquad (3.3.12)$$

3.2.9 The results of this subsection may be summarized as follows: A conformally covariant operator with conformal weight h_2 (see eqn.(3.2.4)) that maps from a Verma module with weight h_1 to a Verma module with weight h_3 exists in the following cases:

1. $h_3 \ne h(j_3',j_3)$: For any values of h_1, h_2
2. $h_3 = h(j_3',j_3)$ and $h_1 \ne h(j_1',j_1)$, $h_2 \ne h(j_2',j_2)$: chiral vertex operator exists if and only if there exist m', m with $-j_3' \le m' \le j_3'$, $-j_3 \le m \le j_3$ such that h_1 and h_2 are solutions of
$$(h_2-h_1)^2 + 2(h_2+h_1)[m'\alpha_- + m\alpha_+]^2 + h(m',m)h(-m',-m) = 0.$$
3. $h_3 = h(j_3',j_3)$ and $h_1 = h(j_1',j_1)$,$h_2 = h(j_2',j_2)$: Further distinguish $c = c_{p'p}$, $c \ne c_{p'p}$:
   (a) $c \ne c_{p'p}$: $\quad j_2' - j_3' \le j_1' \le j_2' + j_3'; \quad j_2 - j_3 \le j_1 \le j_2 + j_3$
   (b) $c = c_{p'p}$: I shall only consider the case $0 \le 2j_3' \le p'-2$, $0 \le 2j_3 \le p-2$, which is the one relevant for the minimal models:
$$\begin{aligned}\max\left(j_2' - j_3', j_2' + j_3' - \tfrac{p'-2}{2}\right) \le j_1' \le \min\left(j_2' + j_3', \tfrac{p'-2}{2} + j_2' - j_3'\right)\\ \max\left(j_2 - j_3, j_2 + j_3 - \tfrac{p-2}{2}\right) \le j_1 \le \min\left(j_2 + j_3, \tfrac{p-2}{2} + j_2 - j_3\right)\end{aligned}$$
   Equivalently one might take
$$\begin{aligned}|j_2' - j_3'| \le j_1' \le \min(j_2' + j_3', p' - 2 - j_2' - j_3')\\ |j_2 - j_3| \le j_1 \le \min(j_2 + j_3, p - 2 - j_2 - j_3),\end{aligned}$$
   since these cases are obtained by $j_1' \to \frac{p'}{2} - (j_1'+1)$, $j_1 \to \frac{p}{2} - (j_1+1)$ and therefore lead to the same set of conformal dimensions.

These restrictions form only a subset of what is known as fusion rules for minimal models. This is due to the fact that I considered the conditions for conformal covariance of operators between *Verma modules* up to now. The conditions for operators to map between *irreducible modules* will be found below.
I would like to point out that there exist more covariant operators than usually considered in the literature. Especially there do exist chiral vertex operators mapping Verma modules without nullvectors to others that do contain null vectors (case 2). This result seems to contradict general folklore.

3.3 Descendant operators

$\psi_{\mathbb{H}}(z)$ is just one member of a whole class of operators $\psi_{\mathbb{H}}(\xi_{h_2}|z)$ which may be labelled by vectors $\xi \in \mathcal{V}_{h_2}$. They are recursively defined by

$$\begin{aligned}\psi_{\mathbb{H}}(v_{h_2}|z) &:= \psi_{\mathbb{H}}(z) \qquad \psi_{\mathbb{H}}(L_{-1}\xi|z) := \partial\psi_{\mathbb{H}}(\xi|z)\\ \psi_{\mathbb{H}}(L_{-n}\xi|z) &:= \tfrac{1}{(n-2)!}(\partial^{n-2}T_<(z)\psi_{\mathbb{H}}(\xi|z) + \psi_{\mathbb{H}}(\xi|z)\partial^{n-2}T_>(z))\end{aligned} \qquad (3.3.13)$$

where $n \ge 2$, and

$$T_<(z) := \sum_{n=0}^{\infty} z^n L_{-n-2} \qquad T_>(z) := \sum_{n=1}^{\infty} z^{-n} L_{n-2}. \qquad (3.3.14)$$

Instead of considering $\psi_{\mathbb{H}}(\xi|z)$ as an operator that maps from $\mathcal{V}_{h_1}$ to $\mathcal{V}_{h_3}$, one may fix an element $\zeta \in \mathcal{V}_{h_1}$ and consider $\psi_{\mathbb{H}}(\ .\ |z)\zeta$ as an operator that maps from $\mathcal{V}_{h_2}$ to $\mathcal{V}_{h_3}$. In the next subsection, a formalism will be presented that puts both points of view on equal footing.

3.3.1 The following theorem gives a convenient characterization of the conformal properties of the descendants (I will omit the subscript $\mathbb{H}$ in the following):

THEOREM

$$[L_n, \psi(\xi|z)] = \sum_{k=-1}^{l(n)} z^{n-k} \binom{n+1}{k+1} \psi(L_k\xi_h|z) \quad where \quad l(n) = \begin{cases} n & for \quad n \geq -1 \\ \infty & for \quad n < -1. \end{cases} \tag{3.3.15}$$

Before giving the proof, I want to explain its implications: The content of the theorem may be summarized even more concise in the following rules:

$$[T_>(u), \psi(\xi|z)] = \psi(T_>(u-z)\xi|z) \tag{3.3.16}$$
$$[\psi(\xi|z), T_<(u)] = \psi(T_>(u-z)\xi|z) \tag{3.3.17}$$
$$\psi(T_<(u-z)\xi|z) = T_<(u)\psi(\xi|z) + \psi(\xi|z)T_>(u), \tag{3.3.18}$$

These formulae allow to write down the complete operator product expansion of $T(u)\psi(\xi|z)$:

$$\begin{aligned} T(u)\psi(\xi|z) &= [T_>(u), \psi(\xi|z)] + T_<(u)\psi(\xi|z) + \psi(\xi|z)T_>(u) \\ &= \sum_{k=-\infty}^{\infty} (u-z)^{-k-2}\psi(L_k\xi|z) \\ &= \psi(T(u-z)\xi|z). \end{aligned} \tag{3.3.19}$$

The sum is finite if ξ contains only finitely many L_{-n} generators. It is now possible to make contact with the more usual formulations of conformal field theories [BPZ]: One has

$$\psi(L_{-n}\xi|z) = \mathrm{Res}_{u-z}[(u-z)^{-n+1}T(u)\psi(\xi|z)] \tag{3.3.20}$$
$$= \oint \frac{du}{2\pi i}(u-z)^{-n+1}T(u)\psi(\xi|z). \tag{3.3.21}$$

In [BPZ], these equations are used to *define* the formalism. In the present formalism they have been *derived* purely algebraically.

3.3.2 The proof of the theorem occupies the rest of this subsection.
As a preliminary note that an alternative basis for $\mathcal{V}_{h_2}$ may be written as follows: Let $\vec{n} = (n_1, \ldots, n_k)$ be a vector of integers with $n_1 \geq 0$, $n_k \geq 0$ and $n_i > 0$ for $i = 2 \ldots k-1$. Then a basis for $\mathcal{V}_{h_2}$ is given by the set of all

$$(L_{-1}^{n_1})(L_{-2}^{n_2})(L_{-1}^{n_3}) \ldots (L_{-1}^{n_{k-1}})(L_{-2}^{n_k})v_{h_2} \tag{3.3.22}$$

It therefore suffices to define ($\xi \in \mathcal{V}_{h_2}$)

$$\begin{aligned} \psi(v_{h_2}|z) &:= \psi(z) \qquad \psi(L_{-1}\xi|z) := \partial\psi(\xi|z) \\ \psi(L_{-2}\xi|z) &:= T_<(z)\psi(\xi|z) + \psi(\xi|z)T_>(z) \end{aligned} \tag{3.3.23}$$

The theorem will be proved for vectors ξ of the form (3.3.22) by induction on the integer s, defined as $s := \sum_{i=1}^{k} n_i$. For $s=0$ one easily recognizes the theorem as the covariant transformation law of $\psi(z)$. Now assume that (3.3.15) holds for $\psi(\xi|z)$. Consider first $[L_n, \psi(L_{-1}\xi|z)]$: By using the definition of $\psi(L_{-1}\xi|z)$ and (3.3.15) this is calculated as:

$$\begin{aligned} [L_n, \psi(L_{-1}\xi|z)] &= \partial\left(\sum_{k=-1}^{l(n)} z^{n-k}\binom{n+1}{k+1}\psi(L_k\xi|z)\right) \\ &= \sum_{k=-1}^{l(n)} z^{n-k-1}(n-k)\binom{n+1}{k+1}\psi(L_k\xi|z) + \sum_{k=-1}^{l(n)} z^{n-k}\binom{n+1}{k+1}\psi(L_{-1}L_k\xi|z) \end{aligned}$$

The first sum may be rewritten by shifting $k' = k+1$ and using $(n-k+1)\binom{n+1}{k} = (k+1)\binom{n+1}{k+1}$ as

$$\sum_{k=0}^{l(n)} z^{n-k}\binom{n+1}{k+1}\psi([L_k, L_{-1}]\xi|z), \tag{3.3.24}$$

so that

$$[L_n, \psi(L_{-1}\xi|z)] = \sum_{k=-1}^{l(n)} z^{n-k}\binom{n+1}{k+1}\psi(L_kL_{-1}\xi|z). \tag{3.3.25}$$

For the computation of $[L_n, \psi(L_{-2}\xi|z)]$ one has to distinguish two cases: $n \geq -1$ and $n < -1$. In the first case use

$$[L_n, T_<(z)] = z^n(z\partial + 2(n+1))T_<(z) + \sum_{m=1}^{n} z^{n-m}(2n-m+2)L_{m-2} + \frac{c}{12}n(n^2-1)z^{n-2}$$
$$[L_n, T_>(z)] = z^n(z\partial + 2(n+1))T_>(z) - \sum_{m=1}^{n} z^{n-m}(2n-m+2)L_{m-2},$$

with the convention that $\sum_{m=1}^{n}(\ldots) = 0$ if $n < 1$ in order to evaluate the right hand side of

$$\begin{aligned} [L_n, \psi(L_{-2}\xi|z)] &= [L_n, T_<(z)]\psi(\xi|z) + \psi(\xi|z)[L_n, T_>(z)] && (3.3.26) \\ &+ T_<(z)[L_n, \psi(\xi|z)] + [L_n, \psi(\xi|z)]T_>(z). && (3.3.27) \end{aligned}$$

By the inductive assumption one recognizes the terms in the second line as

$$\sum_{k=-1}^{n} z^{n-k} \binom{n+1}{k+1} \psi(L_{-2}L_k\xi|z), \tag{3.3.28}$$

while the first line is equal to

$$z^{n+1}\psi(L_{-3}\xi|z) + 2(n+1)z^n\psi(L_{-2}\xi|z) + \frac{c}{12}n(n^2-1)z^{n-2} + S, \tag{3.3.29}$$

where $S := \sum_{m=-1}^{n} z^{n-k}(2n-m+2)[L_{m-2}, \psi(\xi|z)]$ is evaluated as

$$S = \sum_{m=1}^{n} z^{n-m}(2n-m+2) \sum_{k=-1}^{m-2} z^{m-2-k} \binom{m-1}{k+1} \psi(L_k\xi|z) \tag{3.3.30}$$

$$= \sum_{k=1}^{n} z^{n-k} \left\{ \sum_{m=k}^{n} (2n+2-m) \binom{m-1}{k-1} \right\} \psi(L_{k-2}\xi|z). \tag{3.3.31}$$

One may prove by induction that the sum within the curly brackets equals $(k+2)\binom{n+1}{k+1}$, so that

$$S = \sum_{k=1}^{n} z^{n-k} \binom{n+1}{k+1} \psi([L_k, L_{-2}]\xi|z) - \frac{c}{12}n(n^2-1)z^{n-2} \tag{3.3.32}$$

Collecting the different terms one finds

$$[L_n, \psi(L_{-2}\xi|z)] = \sum_{k=-1}^{n} z^{n-k} \binom{n+1}{k+1} \psi(L_k L_{-2}\xi|z), \tag{3.3.33}$$

which was to be proved. The case $n < 1$ proceeds analogously, so I will only list the changes:

$$[L_n, T_<(z)] = z^n(z\partial + 2(n+1))T_<(z) - \sum_{m=n+1}^{0} z^{n-m}(2n-m+2)L_{m-2}$$

$$[L_n, T_>(z)] = z^n(z\partial + 2(n+1))T_>(z) + \sum_{m=n+1}^{0} z^{n-m}(2n-m+2)L_{m-2}$$

$$+\frac{c}{12}n(n^2-1)z^{n-2}$$

$$S = -\sum_{m=n+1}^{0} z^{n-m}(2n-m+2)[L_{m-2}, \psi(\xi|z)]$$

$$= \sum_{k=1}^{\infty} z^{n-k} \left\{ \sum_{m=n+1}^{0} (m-2(n+1)) \binom{m-1}{k+1} \right\} \psi(L_{k-2}\xi|z)$$

The sum in curly brackets obeys a similiar identity as above and is equal to $(k+2)\binom{n+1}{k+1}$. This concludes the proof of the theorem.

3.4 Fusion product

Consider $\psi_{\mathbb{H}}(\xi_2|z)\xi_1$, $\xi_i \in \mathcal{V}_{h_i}$, $i = 1, 2$: Instead of viewing it as the action of an operator on some state one may view it as the result of taking some kind of product of two states:

$$\psi_{\mathbb{H}}(\xi_2|z)\xi_1 \equiv [\xi_2(z)\hat{\otimes}\xi_1(0)]_h \tag{3.3.34}$$

The state ξ_2 is considered to be located at z, ξ_1 at 0. In order to make this more precise I will now introduce the concept of translated states:

3.4.1 Starting from a state $\xi \in \mathcal{V}_h$ one may define a state translated to z by using the translation generator $e^{zL_{-1}}$:

$$\xi(z) := e^{zL_{-1}}\xi. \tag{3.3.35}$$

In fact, translated states are nothing new: One has

$$\xi(z) = \psi\binom{h}{h\ 0}(\xi|z)v_0. \tag{3.3.36}$$

This may be verified by noting that

- one has $v_h(z) = \psi\binom{h}{h\ 0}(z)v_0$ since $v_h(z)$ satisfies $L_k v_h(z) = z^k(z\partial + h(k+1))v_h(z)$, $k \geq 0$, which are the conditions used to define $\psi\binom{h}{h\ 0}(z)v_0$ in sec. 3.1, and that
- $(L_{-1}\xi)(z) = \partial\xi(z)$, $(L_{-2}\xi)(z) = T_<(z)\xi(z)$, from which equation (3.3.36) may be inductively proved for arbitrary ξ.

The conformal properties of translated states may be conveniently summarized by

$$T_>(u)\xi(z) = (T_>(u-z)\xi)(z) \qquad (T_<(u-z)\xi)(z) = T_<(u)\xi(z). \tag{3.3.37}$$

Let me also mention the following important special case of (3.3.37):

$$T_>(u)v_h(z) = \left(\frac{h}{(u-z)^2} + \frac{1}{u-z}\partial\right) v_h(z). \tag{3.3.38}$$

3.4.2 The fusion product of two translated states $\xi_1(z_1) \in \mathcal{V}_{h_1}$ and $\xi_2(z_2) \in \mathcal{V}_{h_2}$ may then be defined in terms of chiral vertex operators as

$$[\xi_2(z_2)\hat{\otimes}\xi_1(z_1)]_h = \psi_{\mathbb{H}}(e^{-z_2 L_{-1}}\xi_2(z_2)|z_2)\xi_1(z_1) = \psi_{\mathbb{H}}(\xi_2|z_2)\xi_1(z_1). \tag{3.3.39}$$

The concept of the fusion product is completely equivalent to that of the chiral vertex operator!

3.4.3 The conformal properties of $\psi(\xi|z)$ derived above may now be rephrased as rules for moving $T(u)$ within fusion products:

$$[\xi(z)\hat{\otimes}T_<(u)\zeta(z')] = T_<(u)[\xi(z)\hat{\otimes}\zeta(z')] + [T_>(u)\xi(z)\hat{\otimes}\zeta(z')] \tag{3.3.40}$$
$$[T_<(u)\xi(z)\hat{\otimes}\zeta(z')] = T_<(u)[\xi(z)\hat{\otimes}\zeta(z')] + [\xi(z)\hat{\otimes}T_>(u)\zeta(z')] \tag{3.3.41}$$
$$T_>(u)[\xi(z)\hat{\otimes}\zeta(z')] = [T_>(u)\xi(z)\hat{\otimes}\zeta(z')] + [\xi(z)\hat{\otimes}T_>(u)\zeta(z')] \tag{3.3.42}$$

3.4.4 On the level of formal power series it is possible to define repeated products such as

$$[[\xi_3(z_3)\hat{\otimes}\xi_2(z_2)]_{h_{23}}\hat{\otimes}\xi_1(z_1)]_h \quad \text{or} \quad [\xi_3(z_3)\hat{\otimes}[\xi_2(z_2)\hat{\otimes}\xi_1(z_1)]_{h_{12}}]_h, \tag{3.3.43}$$

where $\xi_i \in \mathcal{V}_{h_i}$, $i = 1,2,3$. Consider i.e. $[[\xi_3(z_3)\hat{\otimes}\xi_2(z_2)]_{h_{23}}\hat{\otimes}\xi_1(z_1)]_h$, where ξ_i will be assumed to have definite level n_i: By definition, the inner bracket $[\xi_3(z_3)\hat{\otimes}\xi_2(z_2)]_{h_{23}}$ may be written as the formal series

$$[\xi_3(z_3)\hat{\otimes}\xi_2(z_2)]_{h_{23}} = \sum_{n=0}^{\infty}(z_3 - z_2)^{\Delta_{23}+n}\xi_{23}^{(n)}(z_2),$$

where $\Delta_{23} = h_{23} - h_2 - n_2 - h_3 - n_3$ and $\xi_{23}^{(n)} \in \mathcal{V}_{h_{23}}^{(n)}$. Then define

$$\begin{aligned}
&[[\xi_3(z_3)\hat{\otimes}\xi_2(z_2)]_{h_{23}}\hat{\otimes}\xi_1(z_1)]_h := \\
&= \sum_{n=0}^{\infty}(z_3 - z_2)^{\Delta_{23}+n}[\xi_{23}^{(n)}(z_2)\hat{\otimes}\xi_1(z_1)]_h \\
&= \sum_{n=0}^{\infty}(z_3 - z_2)^{\Delta_{23}+n}\sum_{m=0}^{\infty}(z_2 - z_1)^{\Delta_{1(23)}+m-n}\xi^{(n,m)}(z_1) \\
&= (z_3 - z_2)^{\Delta_{23}}\sum_{m=0}^{\infty}(z_2 - z_1)^{\Delta_{1(23)}+m}\sum_{n=0}^{\infty}\left(\frac{z_3 - z_2}{z_2 - z_1}\right)^n \xi^{(n,m)}(z_1)
\end{aligned}$$

where $\Delta_{1(23)} = h - h_{23} - h_1 - n_1$ and $\xi^{(n,m)} \in \mathcal{V}_h^{(m)}$. Note that the sum over n in the last line defines a vector in $\mathcal{V}_h^{(m)}$ if it can be shown to be convergent.

3.4.5 A first hint on the usefulness of using the fusion product language compared to the more common vertex operator language is obtained by rewriting the examples (3.3.43) in terms of chiral vertex operators:

$$[\xi_3(z_3)\hat{\otimes}[\xi_2(z_2)\hat{\otimes}\xi_1(z_1)]_{h_{21}}]_h = \psi\left(\begin{smallmatrix} & h_3 & \\ h & & h_{21}\end{smallmatrix}\right)(\xi_3|z_3)\psi\left(\begin{smallmatrix} & h_2 & \\ h_{21} & & h_1\end{smallmatrix}\right)(\xi_2|z_2)\xi_1(z_1) \tag{3.3.44}$$
$$[[\xi_3(z_3)\hat{\otimes}\xi_2(z_2)]_{h_{32}}\hat{\otimes}\xi_1(z_1)]_h = \sum_n (z_3 - z_2)^{\Delta_{23}+n}\psi\left(\begin{smallmatrix} & h_{32} & \\ h & & h_1\end{smallmatrix}\right)(\xi_{23}^{(n)}|z_2)\xi_1(z_1). \tag{3.3.45}$$

The order $[A[BC]]$ of taking the fusion product therefore simply corresponds to the composition of chiral vertex operators, whereas the expression on the r.h.s of the second line has the form one expects the terms of the operator product expansion of $\psi\left(\begin{smallmatrix} & h_3 & \\ h & & h_{21}\end{smallmatrix}\right)(\xi_3|z_3)\psi\left(\begin{smallmatrix} & h_2 & \\ h_{21} & & h_1\end{smallmatrix}\right)(\xi_2|z_2)$ to have (sum over descendants!).

3.4.6 Correlation functions may now be defined by taking the Shapovalov form with arbitrary "out"-states $\xi \in \mathcal{V}_h$, i.e.

$$(\xi, [\xi_3(z_3)\hat{\otimes}[\xi_2(z_2)\hat{\otimes}\xi_1(z_1)]_{h_{21}}]_h).$$

Such expressions are to be considered as formal series in the variables $z_1, z_3 - z_1, z_2 - z_1$. In order to show convergence to (possibly many-valued) analytic functions one needs more information. This will be discussed later. However, it is important to note that conformal covariance uniquely defines these series up to choice of the normalization constants $\mathcal{N}_{\mathbb{H}}$.

The different orders of multiplication in taking fusion products corresponds to different ways of sewing three-punctured spheres in the more common conformal field theory language (see i.e. [MS]).

3.5 Null vector decoupling

Now consider the special case that either h_1 or h_2 of the conformal weights $\mathbb{H} = \left(\begin{smallmatrix} & h_2 & \\ h_3 & & h_1\end{smallmatrix}\right)$ equals one of the $h(j', j)$. It will turn out that under certain conditions on $\mathbb{H}$, the chiral vertex operators $\psi_{\mathbb{H}}(.|z)$ will have the property to map elements of the null subspace of $\mathcal{V}_{h_1}$ or $\mathcal{V}_{h_2}$ into the null subspace of $\mathcal{V}_{h_3}$. I will call such operators null space preserving. Only those chiral vertex operators that preserve the null subspaces can be used to define chiral vertex operators between irreducible modules. The latter will be seen to obey certain operator differential equations which in the simplest nontrivial case is equivalent to the one used by Gervais and Neveu.

3.5.1 A necessary condition for $\psi_{\mathbb{H}}(.|z)$ to preserve the null space of $\mathcal{V}_{h_1}$ ($\mathcal{V}_{h_2}$) is

$$(v_{h_3}, \psi_{\mathbb{H}}(\mathfrak{n}_2|z)v_{h_1}) = 0 \qquad \left(\ (v_{h_3}, \psi_{\mathbb{H}}(v_{h_2}|z)\mathfrak{n}_1) = 0\ \right), \tag{3.3.46}$$

when $\mathfrak{n}_i, i = 1,2$ are null vectors in $\mathcal{V}_{h_i}$. Similarly as in the evaluation of (3.3.8) one finds that $(v_{h_3}, \psi_{\mathbb{H}}(\mathfrak{n}_2|z)v_{h_1})$ and $(v_{h_3}, \psi_{\mathbb{H}}(v_{h_2}|z)\mathfrak{n}_1)$ are proportional to $Q(h_1, h_3; j_2', j_2; \alpha_+)$ resp. $Q(h_2, h_3; j_1', j_1; \alpha_+)$. In the case that $h_2 = h(j_2', j_2)$ one finds fusion rules

$$j_1' - j_2' \leq j_3' \leq j_1' + j_2' \qquad j_1 - j_2 \leq j_3 \leq j_1 + j_2 \tag{3.3.47}$$

as necessary condition for $\psi_{\mathbb{H}}(.|z)$ to preserve the null subspace of $\mathcal{V}_{h_2}$ and the same condition with $1 \leftrightarrow 2$ in the other case. Again further restrictions arise if $c = c_{p'p}$ from the existence of additional nullvectors in $\mathcal{V}_{h_1}$ or $\mathcal{V}_{h_2}$. They are obtained by replacing in (3.3.12) j_3' and j_3 by j_2' and j_2 or j_1' and j_1 respectively.

3.5.2 These conditions turn out to be sufficient as well: Since $(\xi_3, \psi_{\mathbb{H}}(\mathfrak{n}_2|z)\zeta_1)$ and $(\xi_3, \psi_{\mathbb{H}}(\zeta_2|z)\mathfrak{n}_1)$ can be expressed as some differential operator acting on $(v_{h_3}, \psi_{\mathbb{H}}(\mathfrak{n}_2|z)v_{h_1})$ and $(v_{h_3}, \psi_{\mathbb{H}}(v_{h_2}|z)\mathfrak{n}_1)$ respectively, vanishing of the latter implies vanishing of the former.

In general, the contributions of null vectors in the image of $\psi_{\mathbb{H}}(\mathfrak{n}_2|z)$ or $\psi(.|z)\mathfrak{n}_1$ will not be zero. One may express the property of null space preservation as

$$\psi_{\mathbb{H}}(\mathfrak{n}_2|z) \approx 0 \qquad \text{and} \qquad \psi_{\mathbb{H}}(.|z)\mathfrak{n}_1 \approx 0, \tag{3.3.48}$$

where $\approx$ means equality up to elements of the null subspace.

3.5.3 It is clear, that the whole set of fusion rules, those necessary for existence of CVO's as well as those necessary for null space preservation, will be necessary for the operator to exist on irreducible modules. The complete set of fusion rules in the case $c \neq c_{p'p}$ is

$$|j_1' - j_2'| \leq j_3' \leq j_1' + j_2' \qquad |j_1 - j_2| \leq j_3 \leq j_1 + j_2, \tag{3.3.49}$$

while for $c = c_{p'p}$ it is (I write only the condition on the unprimed j's):

$$|j_1 - j_2| \leq j_3 \leq \min(j_1 + j_2, p - 2 - (j_1 + j_2)) \tag{3.3.50}$$

A triple $\left(\begin{smallmatrix} & (j_2', j_2) & \\ (j_3', j_3) & & (j_1', j_1)\end{smallmatrix}\right)$ will be said to obey the fusion rules (restricted fusion rules) if it satisfies (3.3.49) (resp. (3.3.50)). Under these conditions, the equations (3.3.48) hold as exact equalities. By using the rules (3.3.16)-(3.3.18) for moving $T_<$, $T_>$ one may rewrite (3.3.48) as operator differential equations on $\psi_{\mathbb{H}}(z)$ and $\psi_{\mathbb{H}}(.|z)v_{h_1}$. For example, if $h_2 = h(0, 1/2)$ one finds ($\gamma = \frac{2}{3}(2h_2 + 1) = \alpha_+^2$)

$$\partial^2 \psi_{\mathbb{H}}(z) = \gamma(T_<(z)\psi_{\mathbb{H}} + \psi_{\mathbb{H}}(z)T_>(z)), \tag{3.3.51}$$

if $h_1 = h(0, 1/2)$ one has

$$\partial^2 \psi_{\mathbb{H}}(.|z)v_{h_1} = \gamma(T_<(0)\psi_{\mathbb{H}}(.|z)v_{h_1} + \psi_{\mathbb{H}}(T_>(-z)\ .\ |z)v_{h_1}). \tag{3.3.52}$$

3.5.4 One should note that there exist CVO's between Verma modules that do not preserve the null spaces. The simplest example is obtained by choosing $h_3 = h(-1/2, 0)$, $h_2 = h(1/2, 0)$ and $h_1 = h(0,0) = 0$. In that case $\psi_{\mathbb{H}}$ does not annihilate $L_{-1}v_0$, so that the corresponding chiral vertex operator can not be defined to act between irreducible modules. It nevertheless exists between Verma modules and provides a solution of the Gervais-Neveu operator differential equation (3.3.51)!.
However, there may exist solutions of (3.3.51) that are not chiral vertex operators (even between Verma modules): If $h_3 = h(0,0) = 0$, $h_2 = h(1/2, 0)$ and $h_1 = h(0, -1/2)$ one may construct a formal power series solution of (3.3.51), which due to the failure of the conditions for existence of CVO's can not be covariant with respect to the Virasoro algebra.

3.6 Correlation functions

I now want to describe the additional information one gets when one constructs a correlation function out of CVO's that all have the property of null vector preservation: Differential equations on the conformal blocks. These facts are of course well known since [BPZ], but in the present formalism they may be rigourously derived rather than conjectured as in loc. cit..

3.6.1 Remember that in the present approach correlation functions had been defined as

$$(\xi_\infty, [\ldots[\xi_1(z_1)\hat{\otimes}\ldots[\xi_i(z_i)\hat{\otimes}\xi_{i+1}(z_{i+1})]_{h^1}\ldots]\ldots]_{h^\infty}) \tag{3.3.53}$$

It is useful to have a concise notation for the data involved: Consider all possible multiplications of states ξ_i, $i = 1\ldots n$. They are characterized by the following data:
1) A permutation $\sigma(i)$, $i = 1\ldots n$ of $(1, \ldots, n)$,
2) a complete binary bracketing of $\sigma(1)\ldots\sigma(n)$ such as $(((3,5),((1,4),6)),2)$,
3) The set of tuples (h_i, ξ_i, z_i), where ξ_i is a state in the Verma module of conformal weight h_i and z_i is the position where the state is supposed to be inserted, and finally
4) a set of real numbers h^r, $r = 1, \ldots, n-1$ associated with each pair of brackets which denote the weights of the "intermediate" representations appearing in the multiplication.
Let $\mathfrak{B}_n$ denote the set of all collections of data 1)-2), i.e. of all bracketings $((1,4),\ldots$. The elements of $\mathfrak{B}_n$ will be denoted by $\mathcal{T}, \mathcal{T}'$ etc.. The tuples $(h^1, \ldots, h^{n-1} \equiv h_\infty)$ will be abbreviated as H. One may distinguish the 'external' data (h_i, ξ_i, z_i) from the 'internal' data $\Gamma := (\mathcal{T}, H)$, which parametrize the possible ways to form fusion products of $\xi_i(z_i)$. As a shorthand for the multiplication parametrized by Γ I will use the notation $\prod_\Gamma \xi_i(z_i)$.

3.6.2 The first important observation is that any correlation function $(\xi_\infty, \prod_\Gamma \xi_i(z_i))$

may be expressed as some differential operator acting on $(v_\infty, \prod_\Gamma v_i(z_i))$. To see this write each ξ_i as some polynomial σ_i in the L_{-n} acting on v_i. By using rules (3.3.40)-(3.3.42),(3.3.37) and the defining property of $(.,.)$ one may move one L_{-n} from z_i to all the other z_j, $j \neq i$ and ξ_∞, where it either produces differential operators or reduces the level of the states located at z_j. By iterating this procedure one ultimately ends up with a result that consists of meromorphic differential operators acting on $(v_\infty, \prod_\Gamma v_i(z_i))$. Correlation functions of the form $(v_\infty, \prod_\Gamma v_i(z_i))$ are usually called conformal blocks.

3.6.3 The same procedure may of course be applied if some of the ξ_i are nullvectors $\mathfrak{n}_i$. Now *if all* sub-products within $\prod_\Gamma$ satisfy the fusion rules, then *any* product that contains one or more null vectors $\mathfrak{n}_i$ will produce only elements of the null subspace of $\mathcal{V}_\infty$, so that the corresponding correlation functions vanish. One most conveniently considers the case that one $\xi_i = \mathfrak{n}_i$, all other $\xi_j = v_j$. One thus gets a differential equation on the conformal block. An important example is the case $h_i = h(0,1/2)$ or $h(1/2,0)$, where $\mathfrak{n}_i = (L_{-2} - \frac{3}{2(2h+1)} L_{-1}^2)v_i$. The differential equation one gets is

$$\left\{ \partial_i^2 - \gamma \sum_{j\neq i} \left(\frac{1}{z_i - z_j}\partial_j + \frac{h_j}{(z_i - z_j)^2} \right) \right\} \mathfrak{F}_\Gamma(\{z_i\}) = 0 \qquad (3.3.54)$$

3.6.4 In order to find out how many of these differential equations one needs to completely determine the z_i-dependence, one has to observe that there always are two differential equations following from

$$0 = (L_1 v_\infty, \textstyle\prod_\Gamma v_i(z_i)) = \displaystyle\sum_i \partial_i \mathfrak{F}_\Gamma(\{z_i\}) \qquad (3.3.55)$$

$$h_\infty \mathfrak{F}_\Gamma(\{z_i\}) = (L_0 v_\infty, \textstyle\prod_\Gamma v_i(z_i)) = \displaystyle\sum_i (z_i\partial_i + h_i)\mathfrak{F}(\{z_i\}). \qquad (3.3.56)$$

One therefore does not need to have null vectors in all $\mathcal{V}_{h_i}$, but only in $n-2$ of them in order to fully determine the conformal blocks.

3.6.5 It should be possible to prove by analysis of these differential equations that the formal power series used to define the conformal blocks indeed converge. The necessary results are well known in the case $n = 3$, where (3.3.55) and (3.3.56) allow to reduce the differential equations to one differential equation of fuchsian type:

$$\left(\sum_{r=0}^{k} p_r(z) \partial_z^r \right) \mathfrak{F} = 0, \qquad (3.3.57)$$

where the functions $p_r(z)$ are meromorphic functions with poles of order not greater than $k - r$.

4. Fusion and braiding

The previous discussion was entirely based on conformal covariance properties of the chiral vertex operators, which provide the local information on correlation functions (power series expansions, differential equations). A crucial piece of information that is needed i.e. for the check of locality of the complete field operators is missing: Commutation relations and operator product expansions of chiral vertex operators among themselves. It will be seen that they are of the following form:

$$\psi\begin{pmatrix} h_2 \\ h_\infty \; h_i \end{pmatrix}(\xi_2|z_2)\psi\begin{pmatrix} h_1 \\ h_i \; h_0 \end{pmatrix}(\xi_1|z_1) = \sum_{h_i'} B_{h_i h_i'}\begin{bmatrix} h_2 & h_1 \\ h_\infty & h_0 \end{bmatrix} \psi\begin{pmatrix} h_1 \\ h_\infty \; h_i' \end{pmatrix}(\xi_1|z_1)\psi\begin{pmatrix} h_2 \\ h_i' \; h_0 \end{pmatrix}(\xi_2|z_2)$$

$$\psi\begin{pmatrix} h_2 \\ h_\infty \; h_i \end{pmatrix}(\xi_2|z_2)\psi\begin{pmatrix} h_1 \\ h_i \; h_0 \end{pmatrix}(\xi_1|z_1) = \sum_{h_i'} F_{h_i h_i'}\begin{bmatrix} h_2 & h_1 \\ h_\infty & h_0 \end{bmatrix} \sum_n (z_2 - z_1)^{\Delta_{12}+n} \psi\begin{pmatrix} h_i' \\ h_\infty \; h_0 \end{pmatrix}(\xi_i^{(n)}|z_2),$$

where $\Delta_{12} = h_i' - h_1 - n_1 - h_2 - n_2$ if ξ_1, ξ_2 are states of definite level n_1, n_2 respectively. The matrices B and F will be called braiding and fusion matrices respectively. These matrices depend on the normalization chosen for the chiral vertex operators: If the normalization is relevant I will use the notation $\psi^{\mathcal{N}}\begin{pmatrix} h_2 \\ h_3 \; h_1 \end{pmatrix}$, where the normalization is given by

$$\mathcal{N}\begin{pmatrix} h_2 \\ h_3 \; h_1 \end{pmatrix} = (v_{h_3}, \psi^{\mathcal{N}}\begin{pmatrix} h_2 \\ h_3 \; h_1 \end{pmatrix}(1)v_{h_1}). \qquad (3.4.1)$$

One may of course always take $\mathcal{N} = 1$, but it will turn out to be convenient to use $\mathcal{N}$ with nontrivial dependence on h_1, h_2, h_3. The relation between the fusion matrices for normalization $\mathcal{N} = 1$ and any other choice of $\mathcal{N}$ is given by

$$F^1_{h_i h_i'}\begin{bmatrix} h_2 & h_1 \\ h_\infty & h_0 \end{bmatrix} = \frac{\mathcal{N}\begin{pmatrix} h_2 \\ h_i' \; h_1 \end{pmatrix}\mathcal{N}\begin{pmatrix} h_i' \\ h_\infty \; h_0 \end{pmatrix}}{\mathcal{N}\begin{pmatrix} h_1 \\ h_i \; h_0 \end{pmatrix}\mathcal{N}\begin{pmatrix} h_2 \\ h_\infty \; h_i \end{pmatrix}} F^{\mathcal{N}}_{h_i h_i'}\begin{bmatrix} h_2 & h_1 \\ h_\infty & h_0 \end{bmatrix}. \qquad (3.4.2)$$

The matrix F has a nice interpretation in the language of fusion products: It corresponds to the equivalence transformation that describes associativity of the fusion product. It follows from eqns. (3.3.45), (3.3.44) that the definition of the fusion matrix is equivalent to

$$[\xi_3(z_3)\hat{\otimes}[\xi_2(z_2)\hat{\otimes}\xi_1(z_1)]_{h_{21}}]_h = \sum_{h_{23}} F_{h_{12}h_{23}}\begin{bmatrix} h_3 & h_2 \\ h & h_1 \end{bmatrix} [[\xi_3(z_3)\hat{\otimes}\xi_2(z_2)]_{h_{32}}\hat{\otimes}\xi_1(z_1)]_h$$

To determine the commutativity relations for fusion products requires a choice of the logarithm used to define $(z_2 - z_1)^{h_{21}-h_1-h_2}$. For the principal value one has the relations $\ln(z) = \ln(-z) + i\pi$ for $\arg(z) \in (0,\pi]$, and $\ln(z) = ln(-z) - i\pi$ for $\arg(z) \in (-\pi, 0]$. One therefore has to distinguish two zones: $\mathbb{C}^2_+ := \{(z_1,z_2)\in\mathbb{C}^2\,|\,\arg(z_2-z_1)\in(0,\pi]\}$ and $\mathbb{C}^2_- := \{(z_1,z_2)\in\mathbb{C}^2\,|\,\arg(z_2-z_1)\in(-\pi,0]\}$.

$$[\xi_2(z_2)\hat{\otimes}\xi_1(z_1)]_{h_{21}} = \begin{Bmatrix} e^{i\pi(h_{21}-h_1-h_2)}[\xi_1(z_1)\hat{\otimes}\xi_2(z_2)]_{h_{21}} & \text{in} & \mathbb{C}^2_+ \\ e^{-i\pi(h_{21}-h_1-h_2)}[\xi_1(z_1)\hat{\otimes}\xi_2(z_2)]_{h_{21}} & \text{in} & \mathbb{C}^2_-\,. \end{Bmatrix} \tag{3.4.3}$$

The phase factor will in the following be abbreviated by

$$\Omega\begin{pmatrix} h_{21} \\ h_2\; h_1 \end{pmatrix} = e^{\pi i(h_{21}-h_1-h_2)}. \tag{3.4.4}$$

The definition of the braid matrix reads in fusion product language

$$[\xi_3(z_3)\hat{\otimes}[\xi_2(z_2)\hat{\otimes}\xi_1(z_1)]_{h_{21}}]_h = \sum_{h_{13}} B_{h_{12}h_{13}}\begin{bmatrix} h_3 & h_2 \\ h & h_1 \end{bmatrix}[\xi_2(z_2)\hat{\otimes}[\xi_3(z_3)\hat{\otimes}\xi_1(z_1)]_{h_{31}}]_h$$

It may be calculated by combining associativity and commutativity operations according to the scheme $A(BC)\to(AB)C\to(BA)C\to B(AC)$ in terms of Ω, F:

$$B_{h_{21}h_{31}}\begin{bmatrix} h_3 & h_2 \\ h & h_1 \end{bmatrix} = \sum_{h_{32}} F_{h_{21}h_{32}}\begin{bmatrix} h_3 & h_2 \\ h & h_1 \end{bmatrix}\Omega\begin{pmatrix} h_{32} \\ h_2\; h_3 \end{pmatrix}F^{-1}_{h_{32}h_{31}}\begin{bmatrix} h_2 & h_3 \\ h & h_1 \end{bmatrix}$$

I will therefore take Ω, F as elementary data. Knowledge of these is equivalent to the knowledge of global information on the correlation functions: Their monodromies. Consider i.e. the correlation function

$$(\xi_h, [\xi_3(z_3)\hat{\otimes}[\xi_2(z_2)\hat{\otimes}\xi_1(z_1)]_{h_{21}}]_h)$$

The monodromies when z_1 encircles z_3 or z_2 (or vice versa) are diagonal and given by $\Omega\begin{pmatrix} h_{21} \\ h_2\; h_1 \end{pmatrix}^2$ and $\Omega\begin{pmatrix} h \\ h_3\; h_{21} \end{pmatrix}^2$ respectively. In order to find the monodromy when z_2 circles around z_3, one has to use associativity to express in terms of functions with diagonal monodromy.

I will describe a setup to determine these data by using certain consistency conditions (the Moore-Seiberg polynomial equations) as recursion relations. The initial terms for this recursion are found from the special case where the four point function can be expressed in terms of hypergeometric functions. Such a strategy has been used by Cremmer, Gervais and Roussel in [CGR], where it was called conformal bootstrap. I will consider the case that all conformal weights involved are in the discrete subseries, i.e. $h_i = h(j_i', j_i)$. Due to the subleties in the definition of Liouville theory on states with these conformal weights, it has not yet been considered by Gervais and collaborators.

The main points to be discussed are the following:

1. The very existence of fusion and braiding matrices, or equivalently, of commutativity and associativity operations for the fusion product has to be proved. This is nontrivial if there are fusion rules. It follows if conformal blocks that satisfy all fusion rules form a complete set of solutions to the null vector decoupling equations. Different orders of taking the fusion products lead to different bases in the vector space of solutions to the decoupling equations. Commutativity and associativity operations are special cases of the linear transformations between these bases.

2. A general transformation between one order of taking the fusion product and another may be decomposed into commutativity and associativity operations. If there are two different ways to do this, one gets identities on the representation matrices of these operations. These are the Moore-Seiberg (MS) polynomial equations.

3. In some simple cases one may determine the fusion and braiding matrices explicitely. One may then take some of the MS polynomial equations as recursion relations to determine fusion and braiding matrices in more complicated cases.

4. An important check on these results is done by analyzing whether the fusion and braiding matrices so obtained are indeed consistent with the fusion rules.

4.1 Existence of the fusion matrix

The aim of this section is to outline a way to prove the following theorem:

One has

$$[\xi_3(z_3)\hat{\otimes}[\xi_2(z_2)\hat{\otimes}\xi_1(z_1)]_{h_{21}}]_h \doteq \sum_{h_{23}} F_{h_{21}h_{32}}\begin{bmatrix} h_3 & h_2 \\ h & h_1 \end{bmatrix}[[\xi_3(z_3)\hat{\otimes}\xi_2(z_2)]_{h_{32}}\hat{\otimes}\xi_1(z_1)]_h,$$

where $\doteq$ means equality up to elements of the null subspace, provided the following assumptions hold:

1. *$h_i = h(j_i', j_i)$, $i = 1,2,3$, $\xi_i \in \mathcal{H}_{h_i}$.*
2. *The triples $\begin{pmatrix} h_{21} \\ h_2\; h_1 \end{pmatrix}$, $\begin{pmatrix} h \\ h_3\; h_{21} \end{pmatrix}$ satisfy the fusion rules (restricted fusion rules for $c = c_{p'p}$).*

Then the triples $\begin{pmatrix} h_{32} \\ h_3\; h_2 \end{pmatrix}$, $\begin{pmatrix} h \\ h_{32}\; h_1 \end{pmatrix}$ will also satisfy the fusion rules (restricted for $c = c_{p'p}$).

I will not give the proof in full detail. Rather I will try to explain the main issues involved: First, the theorem is equivalent to the validity of

$$(\xi_h, [\xi_3(z_3)\hat{\otimes}[\xi_2(z_2)\hat{\otimes}\xi_1(z_1)]_{h_{21}}]_h) = \sum_{h_{23}} F_{h_{21}h_{32}}\left[\begin{smallmatrix} h_3 & h_2 \\ h & h_1 \end{smallmatrix}\right](\xi_h, [[\xi_3(z_3)\hat{\otimes}\xi_2(z_2)]_{h_{32}}\hat{\otimes}\xi_1(z_1)]_h) \tag{3.4.5}$$

for arbitrary ξ_h that can be generated by Virasoro action on v_h. By using the rules (3.3.40)-(3.3.42) for moving the Virasoro generators one may express the correlation function on both hand sides of (3.4.5) as the same meromorphic differential operator acting on the correlators with $\xi_i = v_i \equiv v_{h_i}$, $\xi_h = v_h$. It therefore suffices to prove

$$\mathcal{G}^{h_{21}}_{(12)3} = \sum_{h_{23}} F_{h_{21}h_{32}}\left[\begin{smallmatrix} h_3 & h_2 \\ h & h_1 \end{smallmatrix}\right]\mathcal{G}^{h_{32}}_{1(23)}, \tag{3.4.6}$$

where

$$\mathcal{G}^{h_{21}}_{(12)3}(z_1,z_2,z_3) := (v_h, [v_3(z_3)\hat{\otimes}[v_2(z_2)\hat{\otimes}v_1(z_1)]_{h_{21}}]_h) \tag{3.4.7}$$

$$\mathcal{G}^{h_{32}}_{1(23)}(z_1,z_2,z_3) := (v_h, [[v_3(z_3)\hat{\otimes}v_2(z_2)]_{h_{32}}\hat{\otimes}v_1(z_1)]_h). \tag{3.4.8}$$

For $c = c_{p'p}$ both $\mathcal{G}^{h_{12}}_{(12)3}$ and $\mathcal{G}^{h_{23}}_{1(23)}$ are simultaneous solutions of eight differential equations: From the decoupling of the null vectors in $\mathcal{V}_{h_i}$ one gets a partial differential equation of order $(2j_i'+1)(2j_i+1)$ and another of order $(p-2j_i'-1)(p-2j_i-1)$ for each $i=1,2,3$, but in addition one has two equations (3.3.55), (3.3.56) expressing projective invariance of the correlation functions. These latter two equations may be used to prove that $\mathcal{G}^{h_{21}}_{(12)3}$ and $\mathcal{G}^{h_{32}}_{1(23)}$ are of the form

$$\mathcal{G}(z_1,z_2,z_3) = (z_3-z_1)^{h-h_1-h_2-h_3}F\left(\frac{z_2-z_1}{z_3-z_1}\right). \tag{3.4.9}$$

Each of the null vector decoupling equations reduces to an ordinary differential equation of Fuchsian type on the function $F(z)$. This differential equation has poles at 0,1 only. Solutions to differential equations of this kind are uniquely determined up to normalization by demanding diagonal monodromy around 0 (or 1) and by specifying the leading singularity at 0 (resp. 1). In order to get $\mathcal{G}^{h_{21}}_{(12)3}$ one has to take the solutions with diagonal monodromy around 0, for $\mathcal{G}^{h_{32}}_{1(23)}$ around 1.

The crucial fact, from which the theorem follows, is that one gets a *complete* set of solutions to the six null vector decoupling equations by varying h_{21} (resp. h_{32}) in the range allowed by the fusion rules. One has to exclude the possibility that there exist solutions to the differential equations that can not be identified with conformal blocks. If completeness holds, then the theorem is just a consequence of the fact that each solution may be expanded with respect to any complete basis of solutions.

The basic strategy for the proof of completeness is the following: Each differential equation implies restrictions on the exponents of the singularities of its solutions, what has to be proved is that these restrictions are nothing but the fusion rules.

It is useful to consider the following four-point functions:

$$\mathcal{G}^{h_{21}}_{(12)3}(z_1,z_2,z_3,z_4) := (v_0, [v_h(z_4)\hat{\otimes}[v_3(z_3)\hat{\otimes}[v_2(z_2)\hat{\otimes}v_1(z_1)]_{h_{21}}]_h]_0) \tag{3.4.10}$$

$$\mathcal{G}^{h_{32}}_{1(23)}(z_1,z_2,z_3,z_4) := (v_0, [v_h(z_4)\hat{\otimes}[[v_3(z_3)\hat{\otimes}v_2(z_2)]_{h_{32}}\hat{\otimes}v_1(z_1)]_h]_0) \tag{3.4.11}$$

These satisfy eight differential equations from null vector decoupling and three differential equations expressing projective invariance of the solutions (let $h_4 \equiv h$):

$$\sum_{i=1}^{4} z_i^n\left(z_i\frac{\partial}{\partial z_i} + (n+1)h_i\right)\mathcal{G}(z_1,z_2,z_3,z_4) = 0; \qquad n\in\{-1,0,1\}. \tag{3.4.12}$$

These three equations determine $\mathcal{G}$ to be of the form

$$\mathcal{G}(z_1,z_2,z_3,z_4) = \prod_{i>j}(z_i-z_j)^{\Delta-h_i-h_j}F\left(\frac{(z_2-z_1)(z_4-z_3)}{(z_3-z_1)(z_4-z_2)}\right), \tag{3.4.13}$$

where $\Delta = \frac{1}{3}\sum_{i=1}^4 h_i$. Any of the eight null vector decoupling equations leads to an ordinary fuchsian differential equation on $F(z)$. It is easy to see that

$$\lim_{z_4\to\infty} z_4^{2h_4}\mathcal{G}(z_1,z_2,z_3,z_4) = \mathcal{G}(z_1,z_2,z_3). \tag{3.4.14}$$

Now consider

$$(v_0, [[v_h(z_4)\hat{\otimes}v_3(z_3)]_{h_{21}}\hat{\otimes}[v_2(z_2)\hat{\otimes}v_1(z_1)]_{h_{21}}]_0). \tag{3.4.15}$$

This function is also of the form

$$\prod_{i>j}(z_i-z_j)^{\Delta-h_i-h_j}\tilde{F}\left(\frac{(z_2-z_1)(z_4-z_3)}{(z_3-z_1)(z_4-z_2)}\right),$$

where $\tilde{F}$ satisfies the same differential equations, also has diagonal monodromy around 0 and has the same exponent at 0. Therefore $F=\tilde{F}$.

To see how to find the restrictions on h_{21} that follow from the differential equations, consider the simple example where $h_i = h(0,1/2)$ for $i = 1,\dots 4$. One half of the null vector decoupling equations is then

$$\left(\left(\frac{\partial}{\partial z_i}\right)^2 - \gamma \sum_{j\neq i}\left(\frac{1}{z_i - z_j}\frac{\partial}{\partial z_j} + \frac{h_i}{(z_i - z_j)^2}\right)\right) \mathcal{G}(z_1, z_2, z_3, z_4) = 0.$$

The restrictions on the exponent of the singularity at $z_1 \to z_2$ may be obtained by taking the limit $z_3, z_4 \to \infty$ in (DE i) for $i = 1, 2$. The differential equations (DE i) go over into the differential equations on $(v_{h_{21}}, [v_2(z_2)\hat{\otimes}v_1(z_1)]_{h_{21}})$ in this limit. These in turn imply polynomial equations on the exponents which are equivalent to the fusion rules that restrict h_{21}, see sec. 3.5. Similarly one may take the limit $z_1, z_2 \to \infty$ to identify the restrictions on the exponent as $z_3 \to z_4$ that follow from (DE 3), (DE 4) with the fusion rules on h_{12} that are necessary for null vector decoupling in $(v_{h_{43}}, [v_4(z_4)\hat{\otimes}v_3(z_3)]_{h_{43}})$.

In order to treat the general case one does not need the explicit forms of the null vector decoupling equations, but only the following fact: The differential equations on the four point functions go over into those on $(v_{h_{21}}, [v_2(z_2)\hat{\otimes}v_1(z_1)]_{h_{21}})$ (resp. $(v_{h_{43}}, [v_4(z_4)\hat{\otimes}v_3(z_3)]_{h_{43}})$) for $z_4, z_3 \to \infty$ (resp. $z_1, z_2 \to \infty$). It follows from the derivation of the fusion rules for $\left(\begin{smallmatrix} h_{21} \\ h_2\ h_1 \end{smallmatrix}\right)$ that they are always equivalent to the null vector decoupling equations on $(v_{h_{21}}, [v_2(z_2)\hat{\otimes}v_1(z_1)]_{h_{21}})$, and correspondingly for $\left(\begin{smallmatrix} h_{43} \\ h_4\ h_3 \end{smallmatrix}\right)$.

One thereby finds that the conformal blocks provide a complete set of solutions to the null vector decoupling equations. The theorem follows.

4.2 Polynomial equations

The data F and Ω satisfy certain identities. To derive these, introduce the following conformal blocks to each permutation (ijk) of (123) :

$$\mathcal{G}^{h_{ji}}_{(ij)k} := (v_h, [v_k(z_k)\hat{\otimes}[v_j(z_j)\hat{\otimes}v_i(z_i)]_{h_{ji}}]_h) \tag{3.4.16}$$

$$\mathcal{G}^{h_{kj}}_{i(jk)} := (v_h, [[v_k(z_k)\hat{\otimes}v_j(z_j)]_{h_{kj}}\hat{\otimes}v_i(z_i)]_h) \tag{3.4.17}$$

These are analytic functions on the universal cover of $\mathbb{A} := \mathbb{P}^3/z_i = z_j; i,j = 1,2,3$ of the following form

$$\mathcal{G}^{h_{ji}}_{(ij)k} = (z_j - z_i)^{\Delta_{ij}}(z_k - z_i)^{\Delta_{(ij)k}} H^{h_{ji}}_{(ij)k}\left(\frac{z_j - z_i}{z_k - z_i}\right) \tag{3.4.18}$$

$$\mathcal{G}^{h_{kj}}_{i(jk)} = (z_k - z_j)^{\Delta_{kj}}(z_j - z_i)^{\Delta_{i(jk)}} H^{h_{kj}}_{i(jk)}\left(\frac{z_k - z_j}{z_j - z_i}\right), \tag{3.4.19}$$

where $\Delta_{ij} = h_{ij} - h_i - h_j$, $\Delta_{(ij)k} = h - h_{ij} - h_k$, $\Delta_{i(jk)} = h - h_{jk} - h_i$ and the functions $H(z)$ are holomorphic and single-valued in a neighborhood of 0. Note that the logarithms used to define $(z_i - z_j)^{\Delta_{ij}}$ etc. are all taken as the principal values, i.e. to be real for $z_i - z_j \in \mathbb{R}^+$. Consider the region in $\mathbb{C}^3$ where (z_1, z_2), (z_1, z_3), (z_2, z_3) are all in $\mathbb{C}^2_+$. One then has the following relations between the functions $\mathcal{G}$:
(DE i)

$$\mathcal{G}^{h_{21}}_{(12)3} = \Omega\left(\begin{smallmatrix} h_{21} \\ h_2\ h_1 \end{smallmatrix}\right)\mathcal{G}^{h_{21}}_{(21)3} \qquad \mathcal{G}^{h_{32}}_{1(23)} = \Omega\left(\begin{smallmatrix} h_{32} \\ h_3\ h_2 \end{smallmatrix}\right)\mathcal{G}^{h_{32}}_{1(32)} \tag{3.4.20}$$

$$\mathcal{G}^{h_{21}}_{(12)3} = \Omega\left(\begin{smallmatrix} h \\ h_{21}\ h_3 \end{smallmatrix}\right)\mathcal{G}^{h_{21}}_{3(12)} \qquad \mathcal{G}^{h_{32}}_{1(23)} = \Omega\left(\begin{smallmatrix} h \\ h_1\ h_{32} \end{smallmatrix}\right)\mathcal{G}^{h_{32}}_{(23)1} \tag{3.4.21}$$

In addition one has the associativity relations

$$\mathcal{G}^{h_{ji}}_{(ij)k} = \sum_{h_{kj}} F_{h_{ji}h_{kj}}\left[\begin{smallmatrix} h_k & h_j \\ h & h_i \end{smallmatrix}\right]\mathcal{G}^{h_{kj}}_{i(jk)} \tag{3.4.22}$$

$$\mathcal{G}^{h_{kj}}_{i(jk)} = \sum_{h_{ji}} F^{-1}_{h_{kj}h_{ji}}\left[\begin{smallmatrix} h_k & h_j \\ h & h_i \end{smallmatrix}\right]\mathcal{G}^{h_{ji}}_{(ij)k} \tag{3.4.23}$$

Now the expression of $\mathcal{G}_{(12)3}$ in terms of $\mathcal{G}_{3(12)}$ may be computed in two ways: Either by using (3.4.21) or by a sequence of operations that may be symbollically written as $(12)3 \to 1(23) \to 1(32) \to (13)2 \to (31)2 \to 3(12)$. By the linear independence of the $\mathcal{G}^{h_{21}}_{3(12)}$ for different h_{21} one gets the following identity:

$$\Omega\left(\begin{smallmatrix} h \\ h_3\ h_{21} \end{smallmatrix}\right) = \sum_{h_{32}h_{21}} F_{h_{21}h_{32}}\left[\begin{smallmatrix} h_3 & h_2 \\ h & h_1 \end{smallmatrix}\right]\Omega\left(\begin{smallmatrix} h_{32} \\ h_3\ h_2 \end{smallmatrix}\right)F^{-1}_{h_{32}h_{31}}\left[\begin{smallmatrix} h_2 & h_3 \\ h & h_1 \end{smallmatrix}\right]\Omega\left(\begin{smallmatrix} h_{31} \\ h_3\ h_1 \end{smallmatrix}\right)F_{h_{31}h_{21}}\left[\begin{smallmatrix} h_2 & h_1 \\ h & h_3 \end{smallmatrix}\right].$$

Similarly one gets

$$\Omega\left(\begin{smallmatrix} h \\ h_{12}\ h_3 \end{smallmatrix}\right) = \sum_{h_{21}h_{31}} F^{-1}_{h_{32}h_{21}}\left[\begin{smallmatrix} h_3 & h_2 \\ h & h_1 \end{smallmatrix}\right]\Omega\left(\begin{smallmatrix} h_{21} \\ h_2\ h_1 \end{smallmatrix}\right)F_{h_{21}h_{31}}\left[\begin{smallmatrix} h_3 & h_1 \\ h & h_2 \end{smallmatrix}\right]\Omega\left(\begin{smallmatrix} h_{31} \\ h_3\ h_1 \end{smallmatrix}\right)F^{-1}_{h_{31}h_{32}}\left[\begin{smallmatrix} h_1 & h_3 \\ h & h_2 \end{smallmatrix}\right].$$

The inverse of F may be calculated in terms of Ω, F by representing $(12)3 \to 1(23)$ as the sequence of moves $(12)3 \to (21)3 \to 3(21) \to (32)1 \to 1(32) \to 1(23)$. The result is simply

$$F^{-1}_{h_{21}h_{32}}\left[\begin{smallmatrix} h_3 & h_2 \\ h & h_1 \end{smallmatrix}\right] = F_{h_{21}h_{32}}\left[\begin{smallmatrix} h_1 & h_2 \\ h & h_3 \end{smallmatrix}\right] \tag{3.4.24}$$

A further important identity may be derived by considering fusion products of four highest weight states. $[v_4\hat{\otimes}[v_3\hat{\otimes}[v_2\hat{\otimes}v_1]]]$ may be expressed in terms of $[[[v_4\hat{\otimes}v_3]\hat{\otimes}v_2]\hat{\otimes}v_1]$ in two ways: Either by $4(3(21)) \to (43)(21) \to ((32)2)1$ or by $4(3(21)) \to 4((32)1) \to (4(32))1 \to ((43)2)1$. This leads to the identity

$$F_{h_{321}h_{43}}\left[\begin{smallmatrix} h_4 & h_3 \\ h & h_{21} \end{smallmatrix}\right]F_{h_{21}h_{432}}\left[\begin{smallmatrix} h_{43} & h_2 \\ h & h_1 \end{smallmatrix}\right] = \sum_{h_{32}} F_{h_{21}h_{32}}\left[\begin{smallmatrix} h_3 & h_2 \\ h_{321} & h_1 \end{smallmatrix}\right]F_{h_{321}h_{432}}\left[\begin{smallmatrix} h_4 & h_{32} \\ h & h_1 \end{smallmatrix}\right]F_{h_{32}h_{43}}\left[\begin{smallmatrix} h_4 & h_3 \\ h_{432} & h_2 \end{smallmatrix}\right]. \tag{3.4.25}$$

It is useful to write $h_i = h(j_i', j_i)$ and abbreviate the tuple (j_i', j_i) as J_i. I will then write

$$F_{J_{21}J_{32}}\begin{bmatrix} J_3 & J_2 \\ J & J_1 \end{bmatrix} \quad \text{instead of} \quad F_{h_{12}h_{23}}\begin{bmatrix} h_3 & h_2 \\ h_{123} & h_1 \end{bmatrix}.$$

If one then considers eqn. (3.4.25) in the special cases $J_1 = (0, 1/2)$, $J_{21} = (j_2', j_2 + 1/2)$ and $J_1 = (1/2, 0)$, $J_{21} = (j_2' + 1/2, j_2)$, one finds that it allows to express the fusion matrices with $J_1 = (j_1', j_1 + 1/2)$ (or $J_1 = (j_1' + 1/2, j_1)$) in terms of those with $J_1 = (i_1', i_1)$, $i_1' \leq j_1'$, $i_1 \leq j_1$. Equation (3.4.25) therefore uniquely determines $F_{J_{21}J_{32}}\begin{bmatrix} J_3 & J_2 \\ J & J_1 \end{bmatrix}$ in terms of $F_{J_{21}J_{32}}\begin{bmatrix} J_3 & J_2 \\ J & (0,1/2) \end{bmatrix}$ and $F_{J_{21}J_{32}}\begin{bmatrix} J_3 & J_2 \\ J & (1/2,0) \end{bmatrix}$. These are to be determined next.

4.3 Fusion matrix for spin 1/2

For the following discussion it is useful to parametrize the conformal dimensions involved as $h(J) = \gamma J(J+1) - J$, where J is not assumed to be an integer. The parametrization of the discrete series $h(j', j)$ is recovered by identifying $J = j - \alpha_-^2 j'$.
Consider the following conformal blocks:

$$\mathcal{G}^{\pm}_{(12)3}(z, z', z'') := (v_\infty, [v_1(z')\hat{\otimes}[v(z)\hat{\otimes}v_0(z'')]_{J_0 \pm 1/2}]_{J_\infty}) \tag{3.4.26}$$

$$\mathcal{G}^{\pm}_{1(23)}(z, z', z'') := (v_\infty, [[v_1(z')\hat{\otimes}v(z)]_{J_1 \pm 1/2}\hat{\otimes}v_0(z'')]_{J_\infty}), \tag{3.4.27}$$

where v_i, $i = 0, 1, \infty$ have conformal dimensions $h_i = h(J_i)$ and v has conformal dimension $h(0, 1/2)$. The equation describing the decoupling of the null vector in $\mathcal{V}_{h(0,1/2)}$ reads

$$\left(\gamma^{-1}\partial_z^2 - \frac{1}{z - z'}\partial_{z'} - \frac{1}{z - z''}\partial_{z''} - \frac{h_1}{(z - z')^2} - \frac{h_0}{z - z''}\right)\mathcal{G}^{\pm} = 0. \tag{3.4.28}$$

In addition one has the equations expressing projective invariance, eqns. (3.3.55),(3.3.56), which now read

$$(z\partial_z + z'\partial_{z'} + z''\partial_{z''} + h_1 + h_0 + h - h_\infty)\mathcal{G}^{\pm} = 0 \tag{3.4.29}$$

$$(\partial_z + \partial_{z'} + \partial_{z''})\mathcal{G}^{\pm} = 0 \tag{3.4.30}$$

By inserting these into (3.4.28) one finds an ordinary differential equation. It suffices to consider it in the case $z' = 1$, $z'' = 0$ since the z', z''-dependence may be found from (3.4.9). $\mathcal{G}^{\pm}(z) \equiv \mathcal{G}^{\pm}(z, 1, 0)$ satisfies

$$\left(\gamma^{-1}\partial_z^2 + \left(\frac{1}{z-1} + \frac{1}{z}\right)\partial_z - \frac{h_1}{(z-1)^2} - \frac{h_0}{z^2} + \frac{h_1 + h_0 + h - h_\infty}{z(z-1)}\right)\mathcal{G}_i^{\pm}(z) = 0. \tag{3.4.31}$$

The solutions $\mathcal{G}^{\pm}_{(12)3}(z)$ have diagonal monodromy around 0, whereas the monodromy of $\mathcal{G}^{\pm}_{1(23)}$ is diagonal around 1. To find $\mathcal{G}^{\pm}_{(12)3}(z)$, make the ansatz $\mathcal{G}_{(12)3}(z) = z^a(1-z)^b g(z)$. If a and b are taken as solutions of

$$a(a + \gamma - 1) = \gamma h_0 \qquad b(b + \gamma - 1) = \gamma h_1 \tag{3.4.32}$$

then $g(z)$ has to be solution of the hypergeometric differential equation

$$z(z-1)g'' + [2(\gamma + a + b)z - (\gamma + 2a)]g' + (2ab + \gamma(a + b + h + h_0 + h_1 - h_\infty))g = 0. \tag{3.4.33}$$

There is a unique solution of (3.4.33) that is holomorphic at $z = 0$. This is the hypergeometric function $F(A, B; C; z)$, where A, B, C are determined as solutions of

$$AB = 2ab + \gamma(a + b + h + h_1 + h_0 - h_\infty) \tag{3.4.34}$$

$$1 + A + B = 2(a + b + \gamma) \qquad C = 2a + \gamma \tag{3.4.35}$$

Now observe that (3.4.32) is solved by $a_\pm = h(J_0 \pm 1/2) - h(J_0) - h$ and $b_\pm = h(J_1 \pm 1/2) - h(J_1) - h$, as it must be in order to give the expected exponents. In order to get $\mathcal{G}^{+}_{(12)3}$ one therefore has to choose a_+, while a_- yields $\mathcal{G}^{-}_{(12)3}$. The general solution of (3.4.34) (3.4.35) may be written as

$$A = \tfrac{1}{2} + s_0\alpha_0 + s_1\alpha_1 + s_\infty\alpha_\infty \qquad\qquad C = 1 - 2s_0\alpha_0$$
$$B = \tfrac{1}{2} + s_0\alpha_0 + s_1\alpha_1 - s_\infty\alpha_\infty$$

where $\alpha_i = \gamma J_i - \frac{\gamma - 1}{2}$ and $s_i \in \{-, +\}$ for $i \in \{0, 1, \infty\}$. The signs s_1 and s_∞ are irrelevant because of $F(A, B; C; z) = F(B, A; C; z)$ and $F(A, B; C; z) = (1-z)^{C-A-B}F(C-A, C-B; C; z)$. $s_0 = +$ corresponds to $\mathcal{G}^{+}_{(12)3}$, and $s_0 = -$ to $\mathcal{G}^{-}_{(12)3}$. One obtains the following expressions for the A, B, C:

$$A_1^+ = \gamma(J_0 + J_1 + J_\infty + \tfrac{3}{2}) - 1 \qquad\qquad C_1^+ = \gamma(2J_0 + 1)$$
$$B_1^+ = \gamma(J_0 + J_1 - J_\infty + \tfrac{1}{2}) \tag{3.4.37}$$

$$A_1^- = \gamma(-J_0 + J_1 + J_\infty + \tfrac{1}{2}) \qquad\qquad C_1^- = 2 - \gamma(2j_0 + 1)$$
$$B_1^- = \gamma(-J_0 + J_1 - J_\infty - \tfrac{1}{2}) + 1 \tag{3.4.38}$$

The differential equation (3.4.28) depends on J_i only via the conformal dimensions h_i, which are unchanged under $J_i \to -J_i - 1 + \gamma^{-1}$. One may check from the explicit expressions that $J_1 \to -J_1 - 1 + \gamma^{-1}$ and $J_\infty \to -J_\infty - 1 + \gamma^{-1}$ leave $\mathcal{G}^{\pm}_{(12)3}$ unchanged, whereas $J_0 \to -J_0 - 1 + \gamma^{-1}$ exchanges $\mathcal{G}^{+}_{(12)3}$ and $\mathcal{G}^{-}_{(12)3}$. In the case $c = c_{p'p}$ one has a further symmetry of the parametrization of solutions under $j_i' \to p/2 - (j_i' + 1)$, $j_i \to p/2 - (j_i + 1)$.

In the case of $\mathcal{G}^{\pm}_{1(23)}$ one may use the ansatz $\mathcal{G}^{\pm}_{1(23)} = z^{a_+}(1-z)^{b_\pm} h(1-z)$ to find a hypergeometric differential equation for $h(t)$. However, now it is the sign s_0 that is irrelevant, and $s_1 = +$ for $\mathcal{G}^{+}_{1(23)}$, $s_1 = -$ for $\mathcal{G}^{-}_{1(23)}$. The expressions for $A_2^{\pm}$ are obtained from those of $A_1^{\pm}$ simply by exchanging J_1 with J_0.

$\mathcal{G}^{\pm}_{(12)3}$ and $\mathcal{G}^{\pm}_{1(23)}$ are related by analytic continuation. In order to expand $\mathcal{G}^{\pm}_{(12)3}$ in the basis $\mathcal{G}^{\pm}_{1(23)}$ one may use the identity

$$
\begin{aligned}
F(A,B;C;z) = {} & \frac{\Gamma(C)\Gamma(C-B-A)}{\Gamma(C-A)\Gamma(C-B)} F(A,B;A+B-C+1;1-z) \qquad (3.4.39) \\
& + \frac{\Gamma(C)\Gamma(A+B-C)}{\Gamma(A)\Gamma(B)} (1-z)^{C-A-B} F(C-A,C-B;C-A-B+1;1-z).
\end{aligned}
$$

In this way one finds that

$$
\mathcal{G}^{s_0}_{(12)3}(z) = \sum_{s_1 \in \{+,-\}} F_{s_0 s_1} \mathcal{G}^{s_1}_{1(23)}(z), \tag{3.4.40}
$$

where

$$
\begin{aligned}
F_{++} &\equiv F_{J_0+\frac{1}{2},J_1+\frac{1}{2}}\begin{bmatrix} J_1 \, 1/2 \\ J_\infty \, J_0 \end{bmatrix} = \frac{\Gamma(\gamma(2J_0+1))\Gamma(1-\gamma(2J_1+1))}{\Gamma(\gamma(J_0-J_1-J_\infty-\frac{1}{2})+1)\Gamma(\gamma(J_0-J_1+J_\infty+\frac{1}{2}))} \\
F_{+-} &\equiv F_{J_0+\frac{1}{2},J_1-\frac{1}{2}}\begin{bmatrix} J_1 \, 1/2 \\ J_\infty \, J_0 \end{bmatrix} = \frac{\Gamma(\gamma(2J_0+1))\Gamma(-1+\gamma(2J_1+1))}{\Gamma(\gamma(J_0+J_1+J_\infty-\frac{3}{2})-1)\Gamma(\gamma(J_0+J_1-J_\infty+\frac{1}{2}))} \\
F_{-+} &\equiv F_{J_0-\frac{1}{2},J_1+\frac{1}{2}}\begin{bmatrix} J_1 \, 1/2 \\ J_\infty \, J_0 \end{bmatrix} = \frac{\Gamma(2-\gamma(2J_0+1))\Gamma(1-\gamma(2J_1+1))}{\Gamma(\gamma(-J_0-J_1-J_\infty-\frac{3}{2})+2)\Gamma(\gamma(-J_0-J_1+J_\infty-\frac{1}{2})+2)} \\
F_{--} &\equiv F_{J_0-\frac{1}{2},J_1-\frac{1}{2}}\begin{bmatrix} J_1 \, 1/2 \\ J_\infty \, J_0 \end{bmatrix} = \frac{\Gamma(2-\gamma(2J_0+1))\Gamma(\gamma(2J_1+1))}{\Gamma(\gamma(-J_0+J_1+J_\infty+\frac{1}{2}))\Gamma(\gamma(-J_0+J_1-J_\infty-\frac{1}{2})+1)}
\end{aligned}
$$

4.4 Consistency with fusion rules

From the theorem on the existence of the fusion matrix it is clear that its matrix elements can be nonvanishing only if all necessary fusion rules are satisfied. Having obtained explicit expressions one should check whether these do indeed have this property. Problematic are the cases where one of the two values $J_0 \pm 1/2$ (or $J_1 \pm 1/2$) violates, the other respects the fusion rules. One then has one solution to (3.4.28) which corresponds to a conformal block (called the physical solution), whereas the other solution can not be represented as expectation value of CVO's between irreducible representations (unphysical solution).[10] The question is whether fusion and braiding might mix physical and unphysical solutions.

I will first consider the case that $c \neq c_{p'p}$. The case that $J_0 + 1/2$ violates the fusion rules can only occur if $J_0 - 1/2$ equals the upper limit of the range for J_i that is given by the fusion rules for the triple $\left({}_{J_1}{}^{J_i}{}_{J_\infty} \right)$. This is the case if and only if

$$
J_0 - 1/2 = J_1 + J_\infty \qquad \text{(Case } -+)
$$

Similarly $J_0 - 1/2$ violates the fusion rules iff $J_0 + 1/2$ equals the lower limit of the range for J_i given by the fusion rules for $\left({}_{J_1}{}^{J_i}{}_{J_\infty} \right)$. The corresponding cases are

$$
\begin{aligned}
J_0 + 1/2 &= J_1 - J_\infty \qquad &&\text{(Case } +-) \\
J_0 + 1/2 &= J_\infty - J_1 \qquad &&\text{(Case } ++)
\end{aligned}
$$

Now observe that each of these cases corresponds to a case where one of the two values $J_1 \pm 1/2$ is forbidden by the fusion rules. This correspondence is given by the notation (Case $s_0 s_1$) with $s_0 \in \{\pm\}$, $s_1 \in \{\pm\}$: (Case $s_0 s_1$) denotes the case where $J_0 + s_0 1/2$ and $J_1 + s_1 1/2$ both correspond to the *physical* solution.
By inserting the relations between J_0, J_1, J_∞ that occur in the various cases into the explicit expressions for the fusion matrix elements, one finds factors $\Gamma(-n)$ in the numerators of two matrix elements for each case. The vanishing matrix elements are:

$$
\begin{aligned}
&(\text{Case} -+): \quad F_{++} = 0, F_{--} = 0 \qquad && (\text{Case} ++): \quad F_{+-} = 0, F_{-+} = 0 \\
&(\text{Case} +-): \quad F_{++} = 0, F_{--} = 0
\end{aligned}
$$

For $c = c_{p'p}$ there are more cases to be considered due to the more restrictive fusion rules. The conclusion that physical and unphysical solutions do not mix may be reached by using $p - \gamma^{-1}p' = 0$ in a case by case check completely analogous to the one above. Alternatively one may use the symmetry of the space of solutions to (3.4.28) under $j_i' \to p/2 - (j_i' + 1)$, $j_i \to p/2 - (j_i + 1)$ to relate the new cases that occur for $c = c_{p'p}$ to those analyzed above.

[10] The unphysical solutions can probably be represented in terms of chiral vertex operators between Verma modules, see remarks at the end of sec. 3.5.

This result may also be formulated as follows: (3.4.28) defines a two-dimensional vector bundle over the punctured Riemann sphere. This bundle splits into two subbundles with one-dimensional monodromies for the special values of J_0, J_1, J_∞ considered above. This situation may be compared to that in the WZNW models, where the vector bundle is defined by the Knizhnik-Zamolodchikov equations, but only a certain subbundle, which may be characterized by a certain algebraic condition, can be identified with the conformal blocks.

4.5 Fusion matrix for half-integer spins

The explicit form of the fusion matrix for the case that $h_3 = h(j_3', j_3)$, $h_2 = (j_2', j_2)$, with j_2, j_3 half-integer has been determined by Cremmer, Gervais and Roussel in [CGR]. I will write their result only in the case $j_3' = 0$, $j_2' = 0$, since the general case looks similiar. It is useful to parametrize all other conformal dimensions in terms of spins as $h_i = \gamma j_i(j_i + 1)$. As introduced previously, F^1 will be the fusion matrix of the chiral vertex operators with normalization $\mathcal{N} = 1$. It is given by

$$F^1_{j_{21}j_{32}}\begin{bmatrix} j_3 & j_2 \\ j & j_1 \end{bmatrix} = \frac{g\begin{pmatrix} & j_3 & \\ j_{32} & & j_2 \end{pmatrix} g\begin{pmatrix} & j_{32} & \\ j & & j_1 \end{pmatrix}}{g\begin{pmatrix} & j_2 & \\ j_{21} & & j_1 \end{pmatrix} g\begin{pmatrix} & j_3 & \\ j & & j_{21} \end{pmatrix}} \left\{ \begin{matrix} j_1 & j_2 \\ j_3 & j \end{matrix} \middle| \begin{matrix} j_{21} \\ j_{32} \end{matrix} \right\}_q, \tag{3.4.41}$$

with explicitely known constants $g\begin{pmatrix} & j_2 & \\ j_3 & & j_1 \end{pmatrix}$. These constants give the normalization one has to choose in order to find chiral vertex operators with fusion matrix given by q-Racah-Wigner coefficients, see (3.4.2).

The proof of (3.4.41) in [CGR] starts by proving the formula for $j_3 = 1/2$. There one has explicit expressions both for F^1 and the Racah-Wigner coefficients, which allow to uniquely determine the constants $g\begin{pmatrix} & j_2 & \\ j_3 & & j_1 \end{pmatrix}$. Then one only has to observe that both hand sides of (3.4.41) satisfy the recursion relation (3.4.25) to conclude that (3.4.41) holds for any half-integer j_3. Below I will rederive this result from a free-field representation of the chiral vertex operators.

The fact that the fusion matrices for any half-integer spins will be consistent with the fusion rules (restricted fusion rules for $c = c_{p'p}$) follows from the result for spin 1/2 obtained in sec. 4.4 and the recursion relation (3.4.25).

4.6 Concluding remarks

The main new point here is the proof that the chiral vertex operators that satisfy the fusion rules form an algebra that is closed under fusion and braiding. This has been shown in section 4.1 and verified from the explicit form of the fusion matrix in 4.4 for the case of spin 1/2. In particular, in the case of $c = c_{p'p}$ one finds that the explicit expression (3.4.41) will involve the *restricted* q-Racah-Wigner coefficients that are nonvanishing only if the additional fusion rules for this case are satisfied.

I would like to emphasize that the constructive definition of chiral vertex operators allowed me to *prove* the Moore-Seiberg polynomial equations rather than to assume them as conditions for consistency.

Even if one is not willing to adopt the present framework for constructing quantum Liouville theory, but assumes the Gervais-Neveu operator differential equations for spin 1/2 and validity of the polynomial equations, one will be led to the conclusion that Liouville theory contains the minimal models as a closed subsector. Whatever extra states the space of states might contain besides a subspace isomorphic to the irreducible module, if one considers only operators that satisfy the fusion rules the extra states will decouple in any correlation function with external states generated by the Virasoro algebra.

5. Free field construction of chiral vertex operators

Free field realizations provide a very useful way to obtain explicit information on chiral vertex operators. They yield integral representations for the conformal blocks as well as technical means for an explicit calculation of fusion and braiding matrices.

The basic requirement is again that of conformal covariance. Simple examples for covariant operators are given by normal ordered exponentials of the free field, the vertex operators $V_\alpha(z)$. A crucial observation for the construction of more general covariant operators is that the vertex operators $S_\pm(z) = V_{\alpha_\pm}(z)$, corresponding to the two values $\alpha_\pm$ of α for which $h(\alpha) = 1$, transform into a total derivative under the Virasoro algebra. $S_\pm$ are usually called screening charges. For more general vertex operators one may therefore try the following ansatz:

$$V_\alpha^{r'r}(z) = \int du_1 \dots du_{r'} dv_1 \dots dv_r S_-(u_1) \dots S_-(u_{r'}) S_+(v_1) \dots S_+(v_r) V_\alpha(z). \qquad (3.5.1)$$

The result of commuting with Virasoro generators will in general contain contributions from the boundary of the integration region besides those terms expected in a covariant transformation law. The construction of a set of contours for the integration over the variables $u_1, \dots, u_{r'}, v_1, \dots, v_r$ such that no boundary terms appear in the Virasoro transformation law is the main problem in this approach.

The approach to be followed here is based on the work of Felder and Wieczerkowski [FW], who found a remarkable quantum group strucure on spaces of multiple loops on the punctured complex plane. Construction of contours for which the boundary contributions in the Virasoro transformation law cancel will turn out to be equivalent to the Clebsch Gordan problem on the quantum group level. It is this structure that makes the complicated contour manipulations needed for the computation of fusion and braiding matrices manageable.

The new feature of the present treatment is the development of a suitable vertex operator calculus that allows to utilize the results of [FW] for definition of chiral vertex operators and calculation of their fusion and braiding matrices. It will turn out that the formulation in terms of (Fock space) fusion products will again have advantages in making conformal properties and the structures related to the quantum group $\mathcal{U}_q(sl(2))$ more transparent.

5.1 Vertex operators

5.1.1 One of the ingredients for the construction of chiral vertex operators are the vertex operators $V_\alpha(z)$. These are defined in terms of oscillators a_n with $[a_n, a_m] = \delta_{n+m,0}$ and an operator $e^{\alpha q}$ that shifts the zero-mode momentum a_0: $e^{\alpha q} a_0 = (a_0 - 2\alpha) e^{\alpha q}$.

$$V_\alpha(z) := e^{\alpha q} e^{\alpha \varphi_<(z)} e^{\alpha \varphi_>(z)}, \qquad (3.5.2)$$

where

$$\varphi_<(z) = \sum_{n=1}^\infty \frac{a_{-n}}{n} z^n \qquad \varphi_>(z) = a_0 \ln(z) - \sum_{n=1}^\infty \frac{a_n}{n} z^{-n}. \qquad (3.5.3)$$

They map the Fock space $\mathcal{F}_\beta$ to $\mathcal{F}_{\beta+\alpha}$ and are Virasoro-covariant:

$$[L_n, V_\alpha(z)] = z^n (z\partial + h_\alpha(n+1)) V_\alpha(z) \qquad h_\alpha = \alpha^2 - 2\alpha\alpha_0 \qquad (3.5.4)$$

5.1.2 One may recursively define a set of descendant operators by

$$V(v_\alpha|z) := V_\alpha(z) \qquad V(a_{-n}\xi|z) := \frac{1}{(n-1)!} (\partial^{n-1} J_<(z) V(\xi|z) + V(\xi|z) \partial^{n-1} J_>(z)), \qquad (3.5.5)$$

where $\xi \in \mathcal{F}_\alpha$ and

$$J_>(z) = \partial\varphi_>(z) = \sum_{n=0}^\infty a_n z^{-n-1} \qquad J_<(z) = \partial\varphi_<(z) = \sum_{n=1}^\infty a_{-n} z^{n-1}. \qquad (3.5.6)$$

This defines an action of the Heisenberg algebra on the space of vertex operators. If I want to emphasize the latter point of view I will also use the notation

$$(a_{-n_1} \dots a_{-n_k} V_\alpha)(\xi|z) := V_\alpha(a_{-n_1} \dots a_{-n_k} \xi|z). \qquad (3.5.7)$$

5.1.3 The operator product expansion of two vertex operators $V_\alpha(z)$ is obtained as follows: First of all one has

$$V_\alpha(z) V_\beta(z') = (z - z')^{2\alpha\beta} : V_\alpha(z) V_\beta(z') : \qquad (3.5.8)$$

$$= (z - z')^{2\alpha\beta} e^{(\alpha+\beta)q} \exp(\alpha\varphi_<(z) + \beta\varphi_<(z')) \exp(\alpha\varphi_>(z) + \beta\varphi_>(z')). \qquad (3.5.9)$$

A nice way to write the operator product expansion is the following:

$$V_\alpha(z) V_\beta(z') = V_{\alpha+\beta}(V_\alpha(z - z') v_\beta | z'), \qquad (3.5.10)$$

where $V_\alpha(z - z') v_\beta$ is to be considered as a formal power series with coefficients given in terms of Schur polynomials in the oscillators:

$$V_\alpha(z - z') v_\beta = \sum_{n=0}^\infty z^{2\alpha\beta+n} S_n \left(\left\{ \frac{\alpha a_{-k}}{k} \right\} \right) v_\beta \qquad (3.5.11)$$

$$S_n(\{x_k\}) = \sum_{k_1 + 2k_2 + \dots = n} \prod_i \frac{x_i^{k_i}}{k_i!}. \qquad (3.5.12)$$

In order to prove equation (3.5.10) expand

$$: V_\alpha(z)V_\beta(z') := \sum_{n=0}^{\infty}(z-z')^n\mathcal{O}_n(z'), \qquad \mathcal{O}_n(z') = \frac{1}{n!}(\partial_z^n : V_\alpha(z)V_\beta(z') :)_{z=z'}. \quad (3.5.13)$$

It is easy to see that the $\mathcal{O}_n(z')$ satisfy the following recursion relations:

$$(n+1)\mathcal{O}_{n+1}(z') = \sum_{l=0}^{n}\frac{n-l+1}{(n-l)!}\Big((\partial^{n-l}J_<(z'))\mathcal{O}_l(z') + \mathcal{O}_l(z')(\partial^{n-l}J_>(z'))\Big),$$

with initial term $: V_{\alpha+\beta}(z') :$. Comparing with the recursion relation satisfied by the Schur polynomials, $(n+1)S_{n+1} = \sum_{l=0}^{n}(n-l+1)x_{n-l+1}S_l$, one finds

$$\mathcal{O}_n(z') = \left(S_n\left(\left\{\frac{\alpha a_{-k}}{k}\right\}\right)V_{\alpha+\beta}\right)(z'). \quad (3.5.14)$$

The claim follows from (3.5.5), (3.5.7).□

5.1.4 The braiding of two vertex operators $V_\alpha(z)$ and $V_\beta(z')$ is uniquely defined if one defines $(z-z')^{2\alpha\beta}$ by the principal value of the logarithm. One gets (see section 3.4.3):

$$V_\alpha(z)V_\beta(z') = \begin{Bmatrix} e^{2i\pi\alpha\beta}V_\beta(z')V_\alpha(z) & (z'z) \in \mathbb{C}_+^2 \\ e^{-2i\pi\alpha\beta}V_\beta(z')V_\alpha(z) & (z'z) \in \mathbb{C}_-^2. \end{Bmatrix} \quad (3.5.15)$$

5.1.5 In the rest of this subsection I will demonstrate that the vertex operators $V_\alpha(\xi|z)$ have the conformal transformation properties of chiral vertex operators. The Virasoro transformation properties of descendant operators may be summarized as

$$[L_n, V_\alpha(\xi|z)] = V_\alpha(\mathcal{L}_n(z)\xi|z) \quad \text{where} \quad \mathcal{L}_n(z) = \sum_{k=-1}^{l(n)}\binom{n+1}{k+1}z^{n-k}L_k. \quad (3.5.16)$$

The proof is analogous to that of equation (3.3.15). However, now it is a nontrivial fact that the definition (3.5.5) leads to

$$V(v_\alpha|z) = V_\alpha(z) \qquad V(L_{-1}\xi|z) = \partial V(\xi|z) \quad (3.5.17)$$
$$V(L_{-n}\xi|z) = \tfrac{1}{(n-2)!}(\partial^{n-2}T_<(z)V(\xi|z) + V(\xi|z)\partial^{n-2}T_>(z)), \quad (3.5.18)$$

where $\xi \in \mathcal{F}_\alpha$. I will scetch how this is proved:
First, one may again use that it suffices to consider L_{-1}, L_{-2}. The next step is to reduce the problem to the case that $\xi = v_\alpha$ in (3.5.17),(3.5.18):

Let ξ be of the form $\xi = \prod_i a_{-n_i}v_\alpha$. $V(L_{-1}\xi|z)$ may then be calculated by using $[L_{-1}, a_{-n}] = na_{-n-1}$ to move L_{-1} to the right. In order to evaluate the term $V(\prod_i a_{-n_i}L_{-1}v_\alpha|z)$ use (3.5.5) to express it in terms of $V(L_{-1}v_\alpha|z)$. Assuming the validity of (3.5.17) for $\xi = v_\alpha$ it is easy to see that one gets the same terms as by direct evaluation of the r.h.s. of (3.5.17).
$V(L_{-2}\xi|z)$ may be treated similarly: On the l.h.s. use $[L_{-2}, a_{-n}] = na_{-n-2}$ to move L_{-2} to the right. For evaluation of the r.h.s. of (3.5.18) one should rewrite $V(\xi|z)$ in terms of $J_<$, $J_>$ and V_α and then use the relations

$$[T_<(z), \partial^{n-1}J_<(z)] = \frac{1}{n+1}\partial^{n+1}J_<(z) \qquad [\partial^{n-1}J_>(z), T_>(z)] = \frac{1}{n+1}\partial^{n+1}J_>(z),$$

which may be proved by direct calculation. With their help one may move $T_<$, ($T_>$) to the right (left) until they directly multiply V_α. The terms from evaluation of both sides coincide provided (3.5.18) holds for $\xi = v_\alpha$.

The proof that (3.5.17),(3.5.18) holds for $\xi = v_\alpha$ is done by direct calculation. This is trivial for (3.5.17). To prove (3.5.18) one may proceed as follows:

$$\begin{aligned} V(L_{-2}v_\alpha|z) &= V\left(\left(\tfrac{1}{2}a_{-1}^2 + \tfrac{1}{2}a_0a_{-2} + \alpha_0a_{-2}\right)v_\alpha|z\right) \\ &= \tfrac{1}{2}J_<J_<V_\alpha + J_<V_\alpha J_> + \tfrac{1}{2}V_\alpha J_>J_> + (\alpha+\alpha_0)(\partial J_<V_\alpha + V_\alpha\partial J_>). \end{aligned}$$

It is easy to show that

$$J_<V_\alpha J_> = (J_<J_>)_<V_\alpha + V_\alpha(J_<J_>)_> - \partial J_<V_\alpha - V_\alpha\partial J_>,$$

where $J_<J_> = (J_<J_>)_< + (J_<J_>)_>$ and $(J_<J_>)_<$ denotes the holomorphic part of $J_<J_>$. The proof is finished by using the following identities:

$$\begin{aligned} T_> &= \tfrac{1}{2}J_>J_> + (J_<J_>)_> + \alpha_0\partial J_> \\ T_< &= \tfrac{1}{2}J_<J_< + (J_<J_>)_< + \alpha_0\partial J_<. \end{aligned}$$

5.2 Screened vertex operators

The aim of this section is to introduce a set of auxilliary Fock space operators, called screened vertex operators (SVO), and derive their essential properties. The chiral vertex operators will later be constructed as certain composites of SVO's. However, one should notice that the SVO's are nonlocal, Virasoro non-covariant objects. Their physical meaning is therefore obscure at least. On the other hand, the space of these SVO's carries an interesting quantum group structure, first observed in [GoSi], which originates from a quantum group structure on spaces of multiple loops as studied in [FW].

5.2.1 There are two values of α, denoted $\alpha_\pm$ that are distinguished by the property that

$h(\alpha_\pm) = 1$. The corresponding vertex operators $V_{\alpha_\pm}(z)$ are called screening charges and denoted $S_\pm(z)$. Their Virasoro transformation law reads

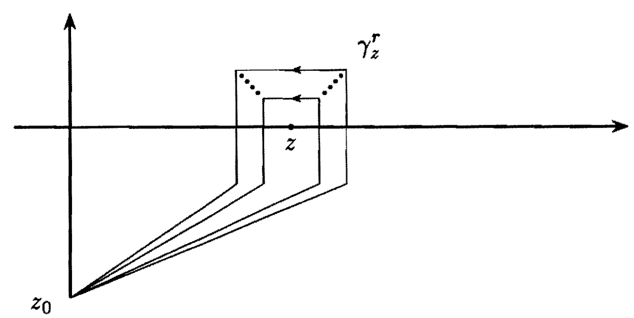
$$[L_n, S_\pm(z)] = \partial(z^{n+1} S_\pm(z)). \tag{3.5.19}$$

I will only consider one screening charge, say $S(z) \equiv S_+(z)$ in the following. The screened vertex operators are defined to be of the form

$$V^r_\alpha(z) := \int_{\gamma^r_z} du_r \dots du_1 S(u_r) \dots S(u_1) V_\alpha(z), \tag{3.5.20}$$

where the contour γ^r_z will be chosen as follows: Fix an arbitrary reference point z_0. For definiteness I will choose $z_0 = -i$.

Figure 1:



This operator will be well-defined in the following sense: Taking the matrix elements between two arbitrary Fock-space states one gets a well-defined integral: Let $\xi_i \in \mathcal{F}_{\alpha_i}$, $\xi_f \in \mathcal{F}_{\alpha_i+\alpha+r\alpha_+}$ be two states of level n_i and n_f respectively. The matrix element $< \xi_f, V^r_\alpha(z)\xi_i >$ is then given as

$$\begin{aligned} &< \xi_f, V^r_\alpha(z)\xi_i > \\ &= \int_{\gamma^r_z} du_1 \dots du_r \prod_{n>m} (u_n - u_m)^{2\alpha_+^2} \prod_m (u_m - z)^{2\alpha\alpha_+} u_m^{2\alpha_i\alpha_+} P_{n_f n_i}(u_1, \dots, u_r; z), \end{aligned}$$

where the meromorphic factor $P_{n_f n_i}(u_1, \dots, u_r; z)$ is given by

$$\begin{aligned} P_{n_f n_i}(u_1, \dots, u_r; z) = &\sum_{n_> = \max(0, n_i - n_f)}^{n_i} \; \sum_{n_1 + \dots + n_k = n_f - n_i + n_>} \; \sum_{m_1 + \dots + m_l = n_>} \\ &< \xi_f, \prod_{s=1}^k u_s^{n_s} S_{n_s}\left(\left\{\tfrac{\alpha_+ a_{-n}}{n}\right\}\right) \prod_{t=1}^l u_t^{-m_t} S_{m_t}\left(\left\{\tfrac{\alpha_+ a_n}{-n}\right\}\right) \xi_i > . \end{aligned}$$

The definition of the integrand is completed by requiring it to be real for $\{u_1, \dots, u_r, z\} \subset \mathbb{R}_+$ and $u_r > u_{r-1} > \dots u_1 > z$.

5.2.2 It will be useful to write $\alpha = \frac{1-n}{2}\alpha_+$ with $n \in \mathbb{R}$ in the following. Correspondingly I will write V_n, V^r_n instead of V_α, V^r_α. I will need three essential properties of $V^r_n(z)$: Let $q = e^{\pi i \alpha_+^2}$, $[x] = \frac{q^x - q^{-x}}{q - q^{-1}}$. Then

$$\begin{aligned} &1) \quad [L_k, V^r_n(\xi|z)] = V^r_n(\mathcal{L}_k(z)\xi|z) - [n-r](q^r - q^{-r}) V^{r-1}_n(\xi|z) S(z_0) z_0^{k+1}. \\ &2) \quad V^r_n(\xi|z) = V^{r-n}_{-n}(Q_n\xi|z) \quad \text{for} \quad n \in \mathbb{N}, \quad r \geq n. \\ &\quad\;\; \text{Especially} \quad V^r_n(\xi|z) = 0 \quad \text{if} \quad Q_n\xi = 0. \\ &3) \quad \frac{\partial}{\partial z_0} V^r_n(\xi|z) = -[n-r](q^r - q^{-r}) V^{r-1}_n(\xi|z) S(z_0) \end{aligned} \tag{3.5.21}$$

Proof: 1)

$$\begin{aligned} [L_k, V^r_n(\xi|z)] = V^r_n(\mathfrak{L}_k(z)\xi|z) + &\sum_{i=1}^r \int_{\gamma^{j-1}_z} du_r \dots \check{du}_i \dots du_1 S(u_r) \dots S(u_{i+1}) S(z_0) z_0^{k+1} \times \\ &\times \left(e^{4i\pi\alpha_+^2(i-1+\frac{1-n}{2})} - 1\right) S(u_{i-1}) \dots S(u_1) V_n(\xi|z), \end{aligned}$$

where the checkmark $\check{du}$ denotes omission of that factor. In order to move $S(z_0)$ to the right use (3.5.15) in $\mathbb{C}^2_-$. One gets

$$\begin{aligned} [L_k, V^r_n(\xi|z)] = V^r_n(\mathfrak{L}_k(z)\xi|z) + &\sum_{i=1}^r \left(e^{4\pi i\alpha_+^2(i-1+\frac{1-n}{2})} - 1\right) e^{-2\pi i\alpha_+^2(i-1+\frac{1-n}{2})} \times \\ &\times \int_{\gamma^{r-1}_z} du_{r-1} \dots du_1 S(u_{r-1}) \dots S(u_1) V_n(\xi|z) S(z_0) z_0^{k+1}, \end{aligned}$$

where the sum is easily evaluated as $-[n-r](q^r - q^{-r})$.

2) Use equation (3.5.10) to perform the operator product expansions of n of the r screening charges with the vertex operator V_n. The result may be written as

$$\int_{\gamma^{r-n}_z} du_r \dots du_{n+1} S(u_r) \dots S(u_{r-n}) V_{-n} \left(\int_{\gamma^n_z} du_n \dots du_1 S(u_n - z) \dots S(u_1 - z) \xi | z \right).$$

Since here $\xi \in \mathcal{F}_{1,n}$, the integration contour closes. One may therefore shift the integration variables to get the BRS-operator Q.

3) One has

$$\frac{\partial}{\partial z_0} V_n^r(\xi|z) = \sum_{i=1}^{r} \int_{\gamma_z^{j-1}} du_r \dots \check{du}_i \dots du_1 S(u_r) \dots S(u_{i+1}) S(z_0) \times$$
$$\times \left(e^{4i\pi\alpha_+^2(i-1+\frac{1-n}{2})} - 1 \right) S(u_{i-1}) \dots S(u_1) V_n(\xi|z),$$

so that the rest of the calculation is the same as in 1). □

5.3 Composition of screened vertex operators

In principle one could define composition of SVO's by summing over intermediate states. However, in order to avoid the questions of convergence of the resulting series it is convenient to directly define the composed operators by first normal-ordering the expressions

$$S(.) \dots S(.) V(\xi_1|z_1) S(.) \dots S(.) V(\xi_2|z_2) \dots\dots S(.) \dots S(.) V(\xi_n|z_n) \qquad (3.5.22)$$

before performing the integrations.
In the following only those compositions of SVO's will be of interest where all SVO's share the same normalization point z_0.
It is the aim of this section to derive three properties of the compostition of SVO's that are crucial for the identification of the quantum group structure on the space of SVO's and for identifying those combinations of SVO's that will form CVO's. These properties are:
1) Virasoro transformation law:

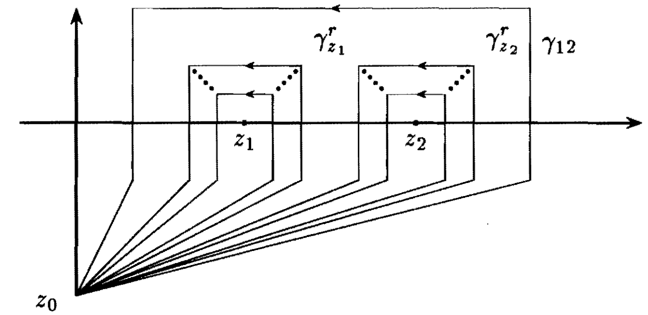
$$\begin{aligned} &[L_k, V_{n_2}^{r_2}(\xi_2|z_2) V_{n_1}^{r_1}(\xi_1|z_1)] = \\ &V_{n_2}^{r_2}(\mathcal{L}_k(z_2)\xi_2|z_2) V_{n_1}^{r_1}(\xi_1|z_1) + V_{n_2}^{r_2}(\xi_2|z_2) V_{n_1}^{r_1}(\mathcal{L}_k(z_1)\xi_1|z_1) \\ &-[n_2 - r_2](q^{r_2} - q^{-r_2}) q^{n_1-1-2r_1} V_{n_2}^{r_2-1}(\xi_2|z_2) V_{n_1}^{r_1}(\xi_1|z_1) z_0^{k+1} S(z_0) \\ &-[n_1 - r_1](q^{r_1} - q^{-r_1}) V_{n_2}^{r_2}(\xi_2|z_2) V_{n_1}^{r_1-1}(\xi_1|z_1) z_0^{k+1} S(z_0) \end{aligned} \qquad (3.5.23)$$

2) Screening

$$\int_{\gamma_{12}} du S(u) V_{n_2}^{r_2}(\xi_2|z_2) V_{n_1}^{r_1}(\xi_1|z_1)$$
$$= V_{n_2}^{r_2+1}(\xi_2|z_2) V_{n_1}^{r_1}(\xi_1|z_1) + q^{-(n_2-1-2r_2)} V_{n_2}^{r_2}(\xi_2|z_2) V_{n_1}^{r_1+1}(\xi_1|z_1),$$

where the contour γ_{12} is defined as follows:

Figure 2:



3) Braiding: For $(z_2, z_1) \in \mathbb{C}_+^2$ one has the following equality:

$$V_{n_2}^{r_2}(\xi_2|z_2) V_{n_1}^{r_1}(\xi_1|z_1) = \sum_{k=0}^{r_2} (\mathcal{R}_{n_2 n_1})_{r_2,r_1}^{r_2-k,r_1+k} V_{n_1}^{r_1+k}(\xi_1|z_1) V_{n_2}^{r_2-k}(\xi_2|z_2) \qquad (3.5.24)$$

$$(\mathcal{R}_{n_2 n_1})_{r_2,r_1}^{r_2-k,r_1+k} = q^{\frac{1}{2}k(k-1)} q^{\frac{1}{2}(1-n_1+2(r_1+k))(1-n_2+2(r_2-k))} \frac{1}{[k]!} \prod_{l=0}^{k-1} [r_2 - l][n_2 - r_2 + l]$$

One may already observe the appearance of quantum group R-matrix and co-product, which will be elaborated upon in the next subsection. But first I will turn to the proofs:

1) This is easily proved by using the Virasoro transformation properties of $V_{n_i}^{r_i}(\xi_i|z_i)$, $i = 1, 2$ and equation (3.5.15) in $\mathbb{C}_-^2$.

2) One has to split γ_{12} into the sum of a contour around z_1 and one around z_2. In order to move $S(u)$ to the right of $V_{n_2}^{r_2}(\xi_2|z_2)$ one has to use (3.5.15) in $\mathbb{C}_+^2$. This yields the phase factor $q^{-(n_2-1-2r_2)}$.

3) The following proof follows a similiar calculation in [FW], which had to be adapted to the present formalism.
The left hand side of (3.5.24) is defined as the integral

$$I[(\gamma_2)^{r_2}(\gamma_1)^{r_1}] := \int_{\gamma_2^{r_2}} du_1 \dots du_{r_2} \int_{\gamma_1^{r_1}} dv_1 \dots dv_{r_1}$$
$$S(u_{r_2}) \dots S(u_1) V_{n_2}(\xi_2|z_2) S(v_1) \dots S(v_{r_1}) V_{n_1}(\xi_1|z_1) \qquad (3.5.25)$$

over the following contours:

Figure 3:

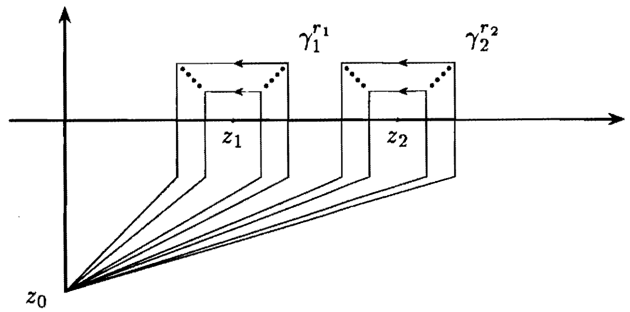
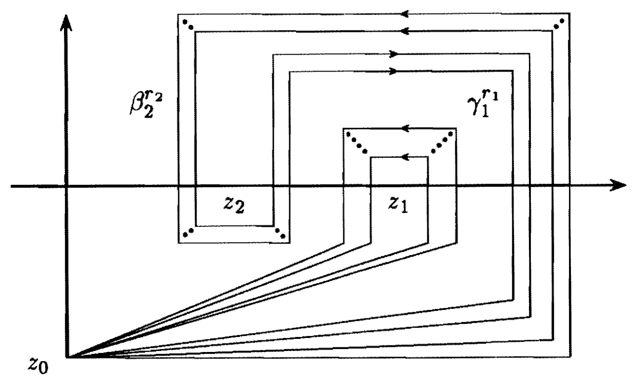


Figure 3 depicts the situation where the integrand of (3.5.25) is defined to be real, namely $z_1, z_2 \in \mathbb{R}$, $z_1 < v_1 < \dots z_{r_1} < z_2 < u_1 < \dots u_{r_2}$. In order to compare left and right hand sides of (3.5.24) it is convenient to consider the case that $z_1, z_2 \in \mathbb{R}$, but $z_2 < z_1$, where now the l.h.s. of (3.5.24) is to be understood as the analytic continuation of (3.5.25) in $\{(z_1, z_2) \in \mathbb{C}_+^2\}$. The result of this analytic continuation is an integral over the following contours:

Figure 4:



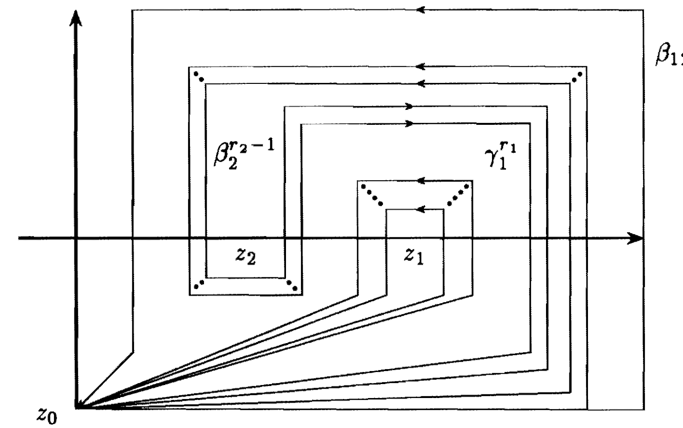
Introduce

$$I[(\beta_2)^{r_2}(\gamma_1)^{r_1}] := \int_{\beta_2^{r_2}} du_1 \dots du_{r_2} \int_{\gamma_1^{r_1}} dv_1 \dots dv_{r_1}$$
$$S(u_{r_2}) \dots S(u_1) S(v_{r_1}) \dots S(v_1) V_{n_1}(\xi_1|z_1) V_{n_2}(\xi_2|z_2) \qquad (3.5.26)$$

In order to relate $I[(\gamma_2)^{r_2}(\gamma_1)^{r_1}]$ and $I[(\beta_2)^{r_2}(\gamma_1)^{r_1}]$ one has to move $V_{n_2}(\xi_2|z_2)$ to the right by using (3.5.15) in $\mathbb{C}_+^2$:

$$I[(\gamma_2)^{r_2}(\gamma_1)^{r_1}] = q^{\frac{1}{2}(1-n_2)(1-n_1+2r_1)} I[(\beta_2)^{r_2}(\gamma_1)^{r_1}] \qquad (3.5.27)$$

Now deform the β_2-loops into a sum of a γ_1 and a β_{12}-loop as indicated in the following diagram:
Figure 5:



The term $I[(\beta_2)^{r_2-1}(\gamma_1)^{r_1+1}]$ appears with factors of

- $q^{2(1-n_2)}$ due to the monodromy of $S(v_{r_2})$ around $V_{n_2}(\xi_2|z_1)$ and
- $q^{+2(r_2-1)}$ due to braiding of $S(u_{r_2})$ with $S(u_1), \dots, S(u_{r_2-1})$ in $\mathbb{C}_+^2$.

One therefore obtains

$$I[(\beta_2)^{r_2}(\gamma_1)^{r_1}] = I[\beta_{12}(\beta_2)^{r_2-1}(\gamma_1)^{r_1}] + q^{2(r_2-n_2)} I[(\gamma_1)^{r_1+1}(\beta_2)^{r_2-1}], \qquad (3.5.28)$$

where $I[\beta_{12}(\beta_2)^{r_2-1}(\gamma_1)^{r_1}]$ is defined with the same integrand as $I[(\beta_2)^{r_2}(\gamma_1)^{r_1}]$, but u_{r_2} is integrated over β_{12} instead of β_2.
For iteration of this procedure, consider the possible ways to produce a term $I[(\beta_{12})^{r_2-k}(\gamma_1)^{r_1+k}]$. There is one such term for each choice of labels $0 \le i_1 < \dots < i_k \le r_2 - 1$ of those curves within $(\beta_2)^{r_2}$ that are to produce a curve γ_1. Each term appears with a factor $q^{2\sum_{l=1}^k (1-n_2)} q^{2\sum_{l=1}^k i_l}$. It follows that

$$I[(\beta_2)^{r_2}(\gamma_1)^{r_1}] = \sum_{k=0}^{r_2} (-1)^k q^{2k(1-n_2)} \natural_k(r_2)\, I[(\beta_{12})^{r_2-k}(\gamma_1)^{r_1+k}], \qquad (3.5.29)$$

with

$$\natural_k(r_2) = \sum_{0 \le i_1 < \dots < i_k \le r_2 - 1} q^{2\sum_{l=1}^k i_l} = q^{k(r_2-1)} \begin{bmatrix} r_2 \\ k \end{bmatrix}, \qquad (3.5.30)$$

where the last equality is known as Gauss' formula.

In the next step consider $I[(\beta_{12})^{r_2-k}(\gamma_1)^{r_1+k}]$ and decompose the β_{12}-loops into γ_1- and γ_2-loops. Decomposing one β_{12}-loop yields

$$I[(\beta_{12}^{r_2-k}(\gamma_1)^{r_1+k}] = I[(\beta_{12})^{r_2-k-1}(\gamma_1)^{r_1+k+1}] + q^{1-n_1+2(r_1+k)} I[(\beta_{12})^{r_2-k-1}(\gamma_1)^{r_1+k}(\gamma_2)],$$

where

$$I[(\beta_{12})^{r_2-k-1}(\gamma_1)^{r_1+k}(\gamma_2)] = \int_{\beta_{12}^{r_2-k-1}} du_1 \dots du_{r_2-k-1} \int_{\gamma_1^{r_1+k}} dv_1 \dots dv_{r_1+k} \int_{\gamma_2} dw$$
$$S(u_{r_2-k-1}) \dots S(u_1)S(v_1) \dots S(v_{r_1})V_{n_1}(\xi_1|z_1)S(w)V_{n_2}(\xi_2|z_2). \qquad (3.5.31)$$

This is to be iterated until no β_{12}-curve is left over: A generic contribution to $I[(\gamma_1)^{r_1+k+l}(\gamma_2)^{r_2-k-l}]$ is obtained by choosing labels $0 \leq i_1 < \dots < i_{r_2-k-l} \leq r_2-k-1$. Moving $S(u_{i_1+1}), S(u_{i_2+1}), \dots, S(u_{i_{r_2-k-l}})$ to the right of $V_{n_1}(\xi_1|z_1)$ produces a factor $q^{2\sum_{s=1}^{r_2-k-l}(i_s-(s-1))}q^{(r_2-k-l)(1-n_1+2(r_1+k))}$. As before the sum over ordered (r_2-k-l)-tuples can be performed with Gauss' formula, so the result is

$$I[(\beta_{12})^{r_2-k}(\gamma_1^{r_1+k}] = \sum_{l=0}^{r_2-k} q^{(r_2-k-l)(1-n_1+2(r_1+k)+l)} \begin{bmatrix} r_2-k \\ l \end{bmatrix} I[(\gamma_1)^{r_1+k+l}(\gamma_2)^{r_2-k-l}].$$

Putting everything together one gets

$$I[(\beta_2)^{r_2}(\gamma_1)^{r_1}] = \sum_{l=0}^{r_2}\sum_{k=0}^{r_2-l}(-1)^k q^{k(r_2-2n_2+1)}q^{(r_2-k-l)(1-n_1+2(r_1+k)+l)} \times$$
$$\times \begin{bmatrix} r_2 \\ k \end{bmatrix}\begin{bmatrix} r_2-k \\ l \end{bmatrix} I[(\gamma_1)^{r_1+k+l}(\gamma_2)^{r_2-k-l}]$$

By reordering the summations this is rewritten as

$$I[(\beta_2)^{r_2}(\gamma_1)^{r_1}] = \sum_{k=0}^{r_2}\left\{\sum_{l=0}^{k}(-1)^l q^{-l(2r_2-2n_2+k+1)}\begin{bmatrix} k \\ l \end{bmatrix}\right\}$$
$$(-1)^k q^{k(r_2-2n_2+1)}q^{(r_2-k)(1-n_1+2(r_1+k))}\begin{bmatrix} r_2 \\ k \end{bmatrix} I[(\gamma_1)^{r_1+k}(\gamma_2)^{r_2-k}]$$

The sum in curly brackets may be evaluated by

$$\left\{\dots\right\} = (-1)^k q^{k(n_2-r_2)}q^{\frac{1}{2}k(k-1)}\prod_{l=0}^{k-1}[n_2-(r_2-l)]. \qquad (3.5.32)$$

Inserting these formulae into (3.5.27) proves the claim. □

5.4 Quantum group structure

Consider the space $\mathfrak{S}$ of all compositions of SVO's with a common normalization point z_0, a basis being given by all monomials $V_{n_s}^{r_s}(\xi_s|z_s)\dots V_{n_1}^{r_1}(\xi_1|z_1)$. The algebraic structures that can be consistently defined to act on $\mathfrak{S}$ are severely restricted by the existence of the braid relation (3.5.24): Both sides of (3.5.24) must transform the same way. By relating the matrix appearing in (3.5.24) to the universal R-matrix of the quantum group $\mathcal{U}_q(sl(2))$ it will be shown that it is consistent to define a $\mathcal{U}_q(sl(2))$-action on $\mathfrak{S}$. The main advantage of introducing this structure comes from the following observation: The additional terms that appear in the commutation relations of screened vertex operators (and their products) besides those expected for covariant operators are proportional to the action of the quantum group generator E. This observation will be crucial for the construction of chiral vertex operators in the next subsection.

5.4.1 Define operations $\bar{E}, \bar{F}, \bar{H}$ to act on the $V_n^r(\xi|z)$ by

$$\bar{F}(V_n^r(\xi|z)) = V_n^{r+1}(\xi|z)$$
$$\bar{E}(V_n^r(\xi|z)) = \frac{[r][n-r]}{q-q^{-1}}V_n^{r-1}(\xi|z) \qquad \bar{H}(V_n^r(\xi|z)) = (n-1-2r)V_n^r(\xi|z). \qquad (3.5.33)$$

It follows that $\bar{E}$, $\bar{F}$ and $\bar{H}$ satisfy the following algebra:

$$[\bar{H},\bar{E}] = 2\bar{E} \qquad [\bar{E},\bar{F}] = q^{\bar{H}} - q^{-\bar{H}} \qquad (3.5.34)$$
$$[\bar{H},\bar{F}] = -2\bar{F}$$

The universal enveloping algebra generated by $\bar{E},\bar{F},\bar{H}$ will be denoted $\overline{\mathcal{U}_q(sl(2))}$ since the generators $\bar{E},\bar{F},\bar{H}$ will turn out to be related to the generators E,F,G of $\mathcal{U}_q(sl(2))$ as defined in the appendix.

An action of $\overline{\mathcal{U}_q(sl(2))}\otimes\overline{\mathcal{U}_q(sl(2))}$ is defined on products of screened vertex operators by

$$(X\otimes Y)(V_{n_2}^{r_2}(\xi_2|z_2)V_{n_1}^{r_1}(\xi_1|z_1)) = X(V_{n_2}^{r_2}(\xi_2|z_2))Y(V_{n_1}^{r_1}(\xi_1|z_1)) \quad \text{for all } X,Y\in\overline{\mathcal{U}_q(sl(2))}.$$

5.4.2 These definitions allow to rewrite the braid relation (3.5.24) as

$$V_{n_2}^{r_2}(\xi_2|z_2)V_{n_1}^{r_1}(\xi_1|z_1) = (P_{21}\mathcal{R})(V_{n_2}^{r_2}(\xi_2|z_2)V_{n_1}^{r_1}(\xi_1|z_1)), \qquad (3.5.35)$$

where P_{21} exchanges $V_{n_2}^{r_2}(\xi_2|z_2)$ and $V_{n_1}^{r_1}(\xi_1|z_1)$ and $\mathcal{R}$ is defined as

$$R = \sum_{k=0}^{\infty}\frac{q^{\frac{1}{2}k(k-1)}}{[k]!}q^{\frac{1}{2}\bar{H}\otimes\bar{H}}\bar{E}^k\otimes\bar{F}^k. \qquad (3.5.36)$$

$\mathcal{R}$ is related to the R-matrix R as defined in the appendix by $\mathcal{R} = R_{q^{-1}}$ if one identifies the generators as follows:

$$\bar{E} = q^{-\frac{H}{2}}(1-q^{-2})^{\frac{1}{2}}E \qquad \bar{F} = q^{\frac{H}{2}}(1-q^{-2})^{\frac{1}{2}}F \qquad \bar{H} = H. \qquad (3.5.37)$$

In appendix A it was shown that for two vectors v_1 and v_2 from $\mathcal{U}_q(sl(2))$-representations H_1 and H_2 the tensor product $v_1 \otimes v_2 \in H_1 \otimes H_2$ and the braided tensor product $B(v_1 \otimes v_2) \in H_2 \otimes H_1$ transform with the same representation matrix, *provided* the action of $\mathcal{U}_q(sl(2))$ on tensor products is defined by means of the co-product Δ_q. $P_{21}\mathcal{R}$ acts on $V^{r_2}_{n_2}(\xi_2|z_2)V^{r_1}_{n_1}(\xi_1|z_1)$ just as the $\mathcal{U}_q(sl(2))$ braiding operation B defined by means of $R_{q^{-1}}$. It is therefore consistent to define the action of $\mathcal{U}_q(sl(2))$ on $\mathfrak{S}$ by the co-product $\Delta_{q^{-1}}$.
The action of $\Delta_{q^{-1}}$ on $\bar{E}$, $\bar{F}$ and $\bar{H}$ is given by

$$\Delta_{q^{-1}}(\bar{E}) = \bar{E} \otimes q^{\bar{H}} + 1 \otimes \bar{E} \qquad \Delta(\bar{H})_{q^{-1}} = \bar{H} \otimes 1 + 1 \otimes \bar{H} \qquad (3.5.38)$$
$$\Delta_{q^{-1}}(\bar{F}) = \bar{F} \otimes 1 + q^{-\bar{H}} \otimes \bar{F}$$

5.4.3 Now it is easy to see that (3.5.23) leads to

$$\begin{aligned}
&[L_k, V^{r_s}_{n_s}(\xi_s|z_s)\dots V^{r_1}_{n_1}(\xi_1|z_1)] = \\
&+\left(\sum_{i=1}^{s} V^{r_s}_{n_s}(\xi_s|z_s)\dots V^{r_i}_{n_i}(\mathcal{L}_k(z_i)\xi_i|z_i)\dots V^{r_1}_{n_1}(\xi_1|z)\right) \\
&-\bar{E}(V^{r_s}_{n_s}(\xi_s|z_s)\dots V^{r_1}_{n_1}(\xi_1|z_1))z_0^{k+1}S(z_0)
\end{aligned}$$

Any linear combination of products of screened vertex operators that is annihalated by the generator E will yield a covariantly transforming operator.

5.5 Construction of chiral vertex operators

In section 3 it was demonstrated that the conformal properties of chiral vertex operators could be nicely reformulated in the language of fusion products. Now it turns out to be convenient to start by defining Fock-space fusion products such that the required conformal properties hold. As in sec. 3.4 these will be sets of bilinear maps

$$[.\bar{\otimes}.]_{n_{12}}: \quad \mathcal{F}_{n_1} \otimes \mathcal{F}_{n_2} \to \mathcal{F}_{n_{12}}, \qquad (3.5.39)$$

that depend parametrically on the points z_1, z_2 where the states are considered to be inserted. The fusion product of two states $\xi_1 \in \mathcal{F}_{n_1}$, $\xi_2 \in \mathcal{F}_{n_2}$, considered to be located at z_1 and z_2 respectively, will be denoted as

$$[\xi_1(z_1)\bar{\otimes}\xi_2(z_2)]_{n_{12}}. \qquad (3.5.40)$$

The main properties to satisfy are the following rules for moving Virasoro generators (cf. (3.3.40)-(3.3.42)):

$$\begin{aligned}
[\xi_1(z_1)\bar{\otimes}T_<(u)\xi_2(z_2)]_{n_{12}} &= T_<(u)[\xi_1(z_1)\bar{\otimes}\xi_2(z_2)]_{n_{12}} + [T_>(u)\xi_1(z_1)\bar{\otimes}\xi_2(z_2)]_{n_{12}} \\
[T_<(u)\xi_1(z_1)\bar{\otimes}\xi_2(z_2)]_{n_{12}} &= T_<(u)[\xi_1(z_1)\bar{\otimes}\xi_2(z_2)]_{n_{12}} + [\xi_1(z_1)\bar{\otimes}T_>(u)\xi_2(z_2)]_{n_{12}} \\
T_>(u)[\xi_1(z_1)\bar{\otimes}\xi_2(z_2)]_{n_{12}} &= [\xi_1(z_1)\bar{\otimes}T_>(u)\xi_2(z_2)]_{n_{12}} + [T_>(u)\xi_1(z_1)\bar{\otimes}\xi_2(z_2)]_{n_{12}} \\
T_>(u)\xi(z) = (T_>(u-z)\xi)(z) &\qquad (T_<(u-z)\xi)(z) = T_<(u)\xi(z)
\end{aligned}$$

5.5.1 In order to define $[\xi_1(z_1)\bar{\otimes}\xi_2(z_2)]_{n_{12}}$ one may start by considering vectors of the form

$$V^{r_1}_{n_1}(\xi_1|z_1)V^{r_2}_{n_2}(\xi_2|z_2)v_0. \qquad (3.5.45)$$

By using equations (3.5.16),(3.5.17),(3.5.18) and (3.5.15) one finds by similiar calculations as used in the proof of the conformal properties of screened vertex operators that the rules (3.5.41)-(3.5.44) hold up to additional terms proportional to

$$E\left(V^{r_1}_{n_1}(\xi_1|z_1)V^{r_2}_{n_2}(\xi_2|z_2)\right)v_0.$$

In order to get rid of these terms one has to choose appropriate linear combinations of states (3.5.45). This is nothing but the problem of finding the highest weight states in a Clebsch-Gordan decomposition of the tensor product of $\mathcal{U}_q(sl(2))$-representations.

5.5.2 It is convenient to rescale the screened vertex operators such that one may apply the standard q-Clebsch-Gordan coefficients as introduced in the appendix for the formation of $\bar{E}$-invariant combinations. To find the appropriate factors one may note that $V^r_n(\xi|z)$ behaves with respect to the quantum group as $\bar{F}^r v_n$, where v_n is the highest weight state of the Verma module V_n (see appendix). $\bar{F}^r v_n$ is related to the standard basis e^j_m of H_j by

$$e^j_m = \lambda^j_m V^r_n; \qquad n = 2j+1; \qquad r = j-m, \qquad (3.5.46)$$

where λ^j_m is explicitely given by

$$e^j_m = \lambda^j_m v^r_n := \frac{q^{-\frac{1}{2}r(r-1)}}{(1-q^{-2})^{\frac{r}{2}}} q^{\frac{1}{2}(j-m)(j+m+1)}([2j-r+1|2r][r]!)^{-\frac{1}{2}} v^r_n. \qquad (3.5.47)$$

Therefore define

$$e^j_m(\xi|z) := \lambda^j_m V^r_n(\xi|z); \qquad n = 2j+1; \qquad r = j-m. \qquad (3.5.48)$$

The braiding matrix of vertex operators $e^{j_1}_{m_1}(\xi_1|z_1)$ and $e^{j_2}_{m_2}(\xi_2|z_2)$ is now given by the matrix elements $(R^{j_1 j_2}_{q^{-1}})^{n_1 n_2}_{m_1 m_2}$ of the universal $\mathcal{U}_q(sl(2))$ R-matrix in the basis $e^{j_1}_{m_1} \otimes e^{j_2}_{m_2}$. Moreover, since $\bar{E}(\dots) = 0$ is equivalent to $E(\dots) = 0$ one may take the usual q-Clebsch-Gordan coefficients to define $\bar{E}$-invariants:

$$[\xi_2(z_2)\bar{\otimes}\xi_1(z_1)]_{j_{21}} := \sum_{m=j-j_1}^{j_2} \begin{pmatrix} j_2 & j_1 \\ m & j-m \end{pmatrix}\!\Big|\, j\Big)_{q^{-1}} e^{j_2}_{j-m}(\xi_2|z_2)e^{j_1}_{m}(\xi_1|z_1)v_0. \qquad (3.5.49)$$

It is not difficult to translate the notion of a Fock space fusion product back to obtain the more common chiral vertex operators. Explicitely it may be written as

$$V\begin{pmatrix} j_2 \\ j\ j_1 \end{pmatrix}(\xi|z) = \sum_m \left(\begin{matrix} j_2 & j_1 \\ j-m & m \end{matrix} \middle| j \right)_{q^{-1}} \lambda_m^{j_1} e_{j-m}^{j_2}(\xi|z) \int_{\gamma_0^{j_1-m}} dv_1 \dots dv_{j_1-m} S(v_1)\dots S(v_{j_1-m}). \tag{3.5.50}$$

5.5.3 The screened vertex operators have a nontrivial dependence of the point z_0 used to define the contours. It is important to note that this dependence also disappears upon forming chiral vertex operators. It is easy to see that

$$\frac{\partial}{\partial z_0} V_{n_1}^{r_1}(\xi_1|z_1) V_{n_2}^{r_2}(\xi_2|z_2) v_0 = E(V_{n_1}^{r_1}(\xi_1|z_1) V_{n_2}^{r_2}(\xi_2|z_2)) v_0, \tag{3.5.51}$$

so that any linear combination that is annihilated by E is automatically z_0-independent.

5.5.4 The normalization of this operator is defined as

$$< v_j, [\xi_2(z_2)\bar{\otimes}\xi_1(z_1)] >=: \mathcal{N}\begin{pmatrix} j_2 \\ j\ j_1 \end{pmatrix}(z_2 - z_1)^{\Delta(j)-\Delta(j_2)-\Delta(j_1)} \tag{3.5.52}$$

The normalization may be explicitely calculated by noting that the corresponding integral depends analytically on the parameter α_+^2 (except for possible poles at rational values of α_+^2). It will turn out that for real $\alpha_+^2 < 0$ the normalization may be expressed in terms of Gamma functions, which possess an analytic continuation to $\alpha_+^2 > 0$. For $\alpha_+^2 < 0$ the operator product of the screening charge with the vertex operators $V_{2j_2+1}(\xi|z_2)$

$$S(u) V_{2j_2+1}(\xi|z_2) = (u - z_2)^{-2j_2\alpha_+^2} : S(u) V_{2j_2+1}(\xi|z_2) : \tag{3.5.53}$$

becomes nonsingular. One may therefore take the normalization point z_0 to z_2 and shrink the contours γ_{z_2} to zero length. In this limit only the term with $m = j - j_2$ contributes in (3.5.50). The resulting vertex operator is proportional to the "screened vertex operators" as defined by Felder in [Fel1]. The integral defining its normalization may then be expressed in terms of the integrals computed by Dotsenko and Fateev in [DF]. The result is

$$\mathcal{N}\begin{pmatrix} j_2 \\ j\ j_1 \end{pmatrix} = \left(\begin{matrix} j_2 & j_1 \\ j_2 & j-j_2 \end{matrix} \middle| j \right)_{q^{-1}} \lambda_{2j_1+1}^{j-j_2} \mathcal{N}_{\text{Felder}}, \tag{3.5.54}$$

where

$$\begin{aligned} \mathcal{N}_{\text{Felder}} = (1-q^2)^{-\frac{r}{2}} q^{r(r+1)} q^{-r(2j_1+1)} [r]! [2j_1 - r + 1|2j_1] \times \\ \times \prod_{i=0}^{r-1} \frac{\Gamma(1-(2j_1-i)\alpha_+^2)\Gamma(1-(2j_2-i)\alpha_+^2)}{\Gamma(2-(j_1+j_2+j+1-i)\alpha_+^2)} \end{aligned} \tag{3.5.55}$$

5.6 Operator differential equations

The construction of the previous subsection can be applied for any *real* values of j, j_1, j_2 such that $j_1 + j_2 - j$ is an integer. However, the chiral vertex operators $V_{\mathbb{J}}(\xi|z)$, $\mathbb{J} = \begin{pmatrix} j_2 \\ j\ j_1 \end{pmatrix}$ for j_2 half-integer have special properties: They satisfy certain operator differential equations, which are prescisely of the form of differential equations following from null vector decoupling in Verma-modules. In the present case these operator differential equations arise in the following way:

5.6.1 Assume the null vector of the Verma-module to be written in the form $\sigma_{n'n}\bar{v}_{n'n}$, where $\bar{v}_{n'n}$ is the highest weight state of the Verma-module and $\sigma_{n'n}$ is a polynomial in the Virasoro generators. The Fock spaces $\mathcal{F}_{n'n}$ for $n > 0$, $n' > 0$ are distinguished by the property that the action of the polynomials $\sigma_{n'n}$ on the highest weight state of the Fock space vanishes. This means that one has (now $n' = 1, n = 2j_2 + 1$!)

$$V_{\mathbb{J}}(\sigma_{1,2j_2+1} v_{1,2j_2+1}|z) \equiv 0. \tag{3.5.56}$$

By using the equations

$$\begin{aligned} V_{\mathbb{J}}(L_{-1}\xi|z) &= \partial V_{\mathbb{J}}(\xi|z) \\ V_{\mathbb{J}}(L_{-n}\xi|z) &= \tfrac{1}{(n-2)!}(\partial^{n-2} T_<(z) V_{\mathbb{J}}(\xi|z) + V_{\mathbb{J}}(\xi|z)\partial^{n-2} T_>(z)), \end{aligned}$$

which follow from above rules for moving T within fusion products, one may express (3.5.56) as a differential equation on $V_{\mathbb{J}}(z)$, just as in section 3.5. In the simplest case $j_2 = 1/2$ one gets the well known equation

$$\partial^2 V_i(z) = \gamma(T_<(z) V_i(z) + V_i(z) T_>(z)), \tag{3.5.57}$$

where

$$V_1 \equiv \sum_{j_1} V\begin{pmatrix} 1/2 \\ j_1+1/2\ j_1 \end{pmatrix}, \qquad V_2 \equiv \sum_{j_1} V\begin{pmatrix} 1/2 \\ j_1-1/2\ j_1 \end{pmatrix}. \tag{3.5.58}$$

5.6.2 These operator differential equations imply differential equations for the conformal blocks constructed from such chiral vertex operators, which coincide with those obtained from null vector decoupling.

> *The correlation functions of free field vertex operators $V_{\mathbb{J}}(z)$ for j_2 half-integer therefore coincide with the conformal blocks as constructed from chiral vertex operators between irreducible modules in section 3.*

This result already implies that the extra states that a Fock module contains besides the irreducible submodule must decouple in correlation functions.

5.6.3 One should note that $V\begin{pmatrix} 1/2 \\ j_1-1/2\, j_1\end{pmatrix}$ cannot exist for $j_1 = 0$: Since $v_0 \equiv v_{1,1}$, $L_{-1}v_0 = 0$ one would have

$$0 = < v_{-1,0}, V\begin{pmatrix} 1/2 \\ j_1-1/2\, j_1\end{pmatrix}(z)L_{-1}v_0 > \propto \partial z^{h(\alpha_{-1,0})-h(\alpha_{1,2})} \neq 0, \tag{3.5.59}$$

which can hold only if $V\begin{pmatrix} 1/2 \\ j_1-1/2\, j_1\end{pmatrix}$ vanishes identically on $\mathcal{F}_{1,1} \equiv \mathcal{F}_0$.

5.7 Composition of chiral vertex operators

As in section 3 one may write Fock space fusion products such as $[\xi_2(z_2)\bar{\otimes}\xi_1(z_1)]_{j_{21}}$ as an expansion in powers of $z_2 - z_1$: Let ξ_i, $i = 1,2$ be energy eigenstates such that $L_0\xi_i = h(j_i) + n_i$. Then one has

$$[\xi_2(z_2)\bar{\otimes}\xi_1(z_1)]_{j_{21}} = \sum_{n=0}^{\infty}(z_2 - z_1)^{h(j_{12})+n-h(j_1)-n_1-h(j_2)-n_2}\xi_{12}^{(n)}, \tag{3.5.60}$$

where $L_0\xi_{12}^{(n)} = h(j_{12}) + n$. The series is to be considered as formal. In this sense it is possible to form iterated fusion products such as

$$[\xi_3(z_3)\bar{\otimes}[\xi_2(z_2)\bar{\otimes}\xi_1(z_1)]_{j_{21}}]_j \equiv V\begin{pmatrix} j_3 \\ j\, j_{21}\end{pmatrix}(\xi_3|z_3)V\begin{pmatrix} j_2 \\ j_{21}\, j_1\end{pmatrix}(\xi_2|z_2)V\begin{pmatrix} j_1 \\ j_1\, 0\end{pmatrix}(\xi_1|z_1)v_0$$

$$[[\xi_3(z_3)\bar{\otimes}\xi_2(z_2)]_{j_{32}}\bar{\otimes}\xi_1(z_1)]_j \equiv V\begin{pmatrix} j_{32} \\ j\, j_1\end{pmatrix}\left(V\begin{pmatrix} j_3 \\ j_{32}\, j_2\end{pmatrix}(\xi_3|z_3 - z_2)v_{j_2}\Big|z_2\right)V\begin{pmatrix} j_1 \\ j_1\, 0\end{pmatrix}(\xi_1|z_1)v_0,$$

where I also presented the chiral vertex operator notation of the same object.
On the other hand side, it is possible to give an alternative definition of multiple fusion products as

$$\{\xi_3(z_3)\bar{\otimes}\{\xi_2(z_2)\bar{\otimes}\xi_1(z_1)\}_{j_{21}}\}_j \tag{3.5.61}$$
$$= \sum_{m_1+m_2+m_3=j}\begin{pmatrix} j_3 & j_{21} \\ m_3 & m_1+m_2\end{pmatrix}\Big|j\Big)_{q^{-1}}\begin{pmatrix} j_2 & j_1 \\ m_2 & m_1\end{pmatrix}\Big|j_{21}\Big)_{q^{-1}} e^{j_3}_{m_3}(\xi_3|z_3)e^{j_2}_{m_2}(\xi_2|z_2)e^{j_1}_{m_1}(\xi_1|z_1)v_0$$

$$\{\{\xi_3(z_3)\bar{\otimes}\xi_2(z_2)\}_{j_{32}}\bar{\otimes}\xi_1(z_1)\}_j \tag{3.5.62}$$
$$= \sum_{m_1+m_2+m_3=j}\begin{pmatrix} j_{32} & j_1 \\ m_2+m_3 & m_1\end{pmatrix}\Big|j\Big)_{q^{-1}}\begin{pmatrix} j_3 & j_2 \\ m_3 & m_2\end{pmatrix}\Big|j_{32}\Big)_{q^{-1}} e^{j_3}_{m_3}(\xi_3|z_3)e^{j_2}_{m_2}(\xi_2|z_2)e^{j_1}_{m_1}(\xi_1|z_1)v_0,$$

where I used curly brackets to distinguish them from the objects defined before. These objects will be maps

$$\{.\bar{\otimes}\{.\bar{\otimes}.\}_{j_{21}}\}_j : \quad \mathcal{F}_{j_1}\otimes\mathcal{F}_{j_2}\otimes\mathcal{F}_{j_3} \to \mathcal{F}_j \tag{3.5.63}$$
$$\{\{.\bar{\otimes}.\}_{j_{32}}\bar{\otimes}.\}_j : \quad \mathcal{F}_{j_1}\otimes\mathcal{F}_{j_2}\otimes\mathcal{F}_{j_3} \to \mathcal{F}_j, \tag{3.5.64}$$

defined again in the sense that taking matrix elements with arbitrary L_0 eigenstates in $\mathcal{F}_j$ produces well-defined integrals.
The products (3.5.61)(3.5.62) will satisfy all conformal properties one expects multiple fusion products to have, since they are defined as $\bar{E}$-invariant linear combinations of screened vertex operators.
I now want to present a formal argument that the multiple products with curly brackets are the same as the corresponding products with square brackets:

First one may rewrite the definition (3.5.62) as follows:

$$\{\{\xi_3(z_3)\bar{\otimes}\xi_2(z_2)\}_{j_{32}}\bar{\otimes}\xi_1(z_1)\}_j =$$
$$= \sum_{m_1+m_{23}=j}\begin{pmatrix} j_{32} & j_1 \\ m_{32} & m_1\end{pmatrix}\Big|j\Big)_{q^{-1}}([j_{32}-m_{32}]![j_{32}+m_{32}+1|2j])^{1/2}\times$$
$$\times F^{j_{32}-m_{32}}\left(\sum_{m_2+m_3=j_{32}}\begin{pmatrix} j_3 & j_2 \\ m_3 & m_2\end{pmatrix}\Big|j_{32}\Big)_{q^{-1}} e^{j_3}_{m_3}(\xi_3|z_3)e^{j_2}_{m_2}(\xi_2|z_2)\right)e^{j_1}_{m_1}(\xi_1|z_1)$$

Now consider the factor $[\widehat{\xi_3\bar{\otimes}\xi_2}]_{j_{32}} := \sum_{m_2+m_3=j_{32}}\begin{pmatrix} j_3 & j_2 \\ m_3 & m_2\end{pmatrix}\Big|j_{32}\Big)_{q^{-1}} e^{j_3}_{m_3}(\xi_3|z_3)e^{j_2}_{m_2}(\xi_2|z_2)$:

$$[\widehat{\xi_3\bar{\otimes}\xi_2}]_{j_{32}} = \int_{C_{z_2,z_3}} du_1\ldots du_r S(u_1)\ldots S(u_r)V_{j_3}(\xi_3|z_3)V_{j_2}(\xi_2|z_2)$$
$$= \int_{C_{z_2,z_3}} du_1\ldots du_r V_{j_{23}}\Big(S(u_1-z_2)\ldots S(u_r-z_2)V_{j_3}(\xi_3|z_3-z_2)\xi_2\Big|z_2\Big),$$

where in the last step I have repeatedly used eqn. (3.5.10). For the following step it is important that $[\widehat{\xi_3\bar{\otimes}\xi_2}]_{j_{23}}$ is z_0-independent. This fact allows me to shift the integration variables $u_1\ldots u_r$ to get

$$[\widehat{\xi_3\bar{\otimes}\xi_2}]_{j_{23}} = V_{j_{23}}\left(\int_{C_{z_3-z_2,0}} du_1\ldots du_r S(u_1)\ldots S(u_r)V_{j_3}(\xi_3|z_3-z_2)\xi_2\Big|z_2\right)$$
$$= V_{j_{23}}\left(\left[\xi_3(z_3-z_2)\bar{\otimes}\xi_2(0)\right]_{j_{23}}\Big|z_2\right). \tag{3.5.66}$$

By reinserting this result into (3.5.65) one finds

$$\{\{\xi_3(z_3)\bar{\otimes}\xi_2(z_2)\}_{j_{32}}\bar{\otimes}\xi_1(z_1)\}_j =$$
$$= \sum_{m_1+m_{23}=j}\begin{pmatrix} j_{32} & j_1 \\ m_{32} & m_1\end{pmatrix}\Big|j\Big)_{q^{-1}} e^{j_{23}}_{m_{23}}\left(\left[\xi_3(z_3-z_2)\bar{\otimes}\xi_2(0)\right]_{j_{23}}\Big|z_2\right)e^{j_1}_{m_1}(\xi_1|z_1)$$

$$= \Big[\big[\xi_3(z_3)\bar{\otimes}\xi_2(z_2)\big]_{j_{32}}\bar{\otimes}\xi_1(z_1)\Big]_j,$$

where the last step is nothing but the definition of the iterated product.□

I previously classified this argument as formal. So what is missing for a rigorous proof? The crucial point will be a proof of convergence of the formal power series that define the matrix elements of iterated fusion products (denoted by square brackets). In the case that the j's involved are all half-integers this will be possible by exploiting the differential equations that the corresponding CVO's satisfy. The arguments will then be analogous to those given in sect. 3: One shows that the matrix elements satisfy the differential equations in the sense of formal series, and convergence of these series follows from the theory of differential equations.
Without proof of convergence one may still take equations (3.5.61)(3.5.62) as definitions for multiple fusion products. Then one has to conjecture that factorization holds, i.e. that correlation functions can be expanded as sums over inermediate states.

5.8 Braiding and Fusion for generic c

As before fusion and brading matrices are identified with the linear transformations between different orders of taking repeated fusion products.

$$[\xi_3(z_3)\bar{\otimes}[\xi_2(z_2)\bar{\otimes}\xi_1(z_1)]_{j_{21}}]_j = \sum_{j_{32}} F_{j_{21}j_{32}}\begin{bmatrix} j_3 & j_2 \\ j & j_1 \end{bmatrix} [[\xi_3(z_3)\bar{\otimes}\xi_2(z_2)]_{j_{32}}\bar{\otimes}\xi_1(z_1)]_j \tag{3.5.67}$$

$$[\xi_3(z_3)\bar{\otimes}[\xi_2(z_2)\bar{\otimes}\xi_1(z_1)]_{j_{21}}]_j = \sum_{j_{31}} B_{j_{21}j_{31}}\begin{bmatrix} j_3 & j_2 \\ j & j_1 \end{bmatrix} [\xi_2(z_2)\bar{\otimes}[\xi_3(z_3)\bar{\otimes}\xi_1(z_1)]_{j_{31}}]_j \tag{3.5.68}$$

Given now the representations (3.5.61),(3.5.62) it is easy to calculate the matrices F and B. The fact that

$$F_{j_{21}j_{32}}\begin{bmatrix} j_3 & j_2 \\ j & j_1 \end{bmatrix} = \left\{\begin{matrix} j_1 & j_2 \\ j_3 & j \end{matrix}\middle|\begin{matrix} j_{21} \\ j_{32} \end{matrix}\right\}_q \tag{3.5.69}$$

is then an immediate consequence of the definition of q-RW coefficients. The matrix B is then for $(z_3, z_2) \in \mathbb{C}^2_+$ calculated as

$$\begin{aligned}
&[\xi_3(z_3)\bar{\otimes}[\xi_2(z_2)\bar{\otimes}\xi_1(z_1)]_{j_{21}}]_j = \\
&= \sum_{m_1+m_2+m_3=j}\ \sum_{n_2+n_3=j-m_1} \left(\begin{matrix} j_2 & j_1 \\ m_2 & m_1 \end{matrix}\middle| j_{21}\right)_{q^{-1}} \left(\begin{matrix} j_3 & j_{21} \\ m_3 & m_1+m_2 \end{matrix}\middle| j\right)_{q^{-1}} (R^{j_3j_2}_{q^{-1}})^{n_3n_2}_{m_3m_2} \times \\
&\quad \times e^{j_2}_{m_2}(\xi_2|z_2)e^{j_3}_{m_3}(\xi_3|z_3)e^{j_1}_{m_1}(\xi_1|z_1) \\
&= \sum_{j_{31}} (-)^{j_1+j-j_{31}-j_{21}} q^{\frac{1}{2}(c_{j_1}+c_j-c_{j_{31}}-c_{j_{21}})} \left\{\begin{matrix} j_2 & j_1 \\ j_3 & j \end{matrix}\middle|\begin{matrix} j_{21} \\ j_{31} \end{matrix}\right\}_{q^{-1}} \times \\
&\quad \times \sum_{m_1+m_2+m_3=j} \left(\begin{matrix} j_3 & j_1 \\ m_3 & m_1 \end{matrix}\middle| j_{31}\right)_{q^{-1}} \left(\begin{matrix} j_2 & j_{31} \\ m_2 & m_1+m_3 \end{matrix}\middle| j\right)_{q^{-1}} e^{j_2}_{m_2}(\xi_2|z_2)e^{j_3}_{m_3}(\xi_3|z_3)e^{j_1}_{m_1}(\xi_1|z_1) \\
&= \sum_{j_{31}} (-)^{j_1+j-j_{31}-j_{21}} q^{\frac{1}{2}(c_{j_1}+c_j-c_{j_{31}}-c_{j_{21}})} \left\{\begin{matrix} j_2 & j_1 \\ j_3 & j \end{matrix}\middle|\begin{matrix} j_{21} \\ j_{31} \end{matrix}\right\}_{q^{-1}} \times \\
&\quad \times [\xi_2(z_2)\bar{\otimes}[\xi_3(z_3)\bar{\otimes}\xi_1(z_1)]_{j_{21}}]_j,
\end{aligned}$$

where the product of R-matrix and Clebsch-Gordan coefficient has been evaluated with the help of eqn. (6.1.40) of appendix A.

5.9 Braiding and fusion for rational c

I shall now consider the case that $c = c_{p,p}$. I found previously (sect. 4) that the minimal model chiral vertex operators, defined to map between modules with conformal weights $h(j'j)$ with $0 \leq 2j' + 1 \leq p'$, $0 \leq 2j + 1 \leq p$ and to satisfy the restricted fusion rules, form an algebra that is closed under the fusion and braiding operations. It is an old conjecture, going back to the work [DF], that this algebra is faithfully represented by the Fock space chiral vertex operators considered in this section. This question is nontrivial due to the extra states in Fock modules as compared to irreducible Virasoro modules. The results obtained in sect. 5.6 already imply that the conjecture is true since minimal model conformal blocks coincide with those constructed via free field representation.

However, in comparing the result of section 4 that fusion and braiding matrices for $c = c_{p'p}$ are given by the *restricted* q-Racah-Wigner coefficients with the explicit form found in the previous subsection, one finds an apparent paradox: The sums in (3.5.67),(3.5.68) are generically performed over those values of j_{31}, j_{32} allowed by the unrestricted fusion rules. Consideration of the behaviour of q-Clebsch-Gordan and q-Racah-Wigner coefficients for $c = c_{p'p}$ (see below) reveals that it does not lead to a truncation of the sum to values j_{31}, j_{32} consistent with the restricted fusion rules. There must be a reason for the "unphysical" chiral vertex operators to vanish.

It turns out that the resolution to this question is closely related to the mechanism which prevents that "extra" states of Fock modules contribute in conformal blocks. The explanation of this mechanism is based on the work of Felder that classified the extra states as unphysical w.r.t. a suitable BRST-cohomology. Crucial for a decoupling mechanism analogous to that in gauge theories is the property of BRST-invariance of Fock space chiral vertex operators, which ensures that physical and unphysical subspaces are mapped into themselves. Substantial evidence for this property has been obtained in [FeSi][11].
In the present language of fusion products the property of BRST-invariance may be

[11] It seems that recently this approach has been extended to a complete proof, see [Si]. The necessary relations on spaces of multiple loops are derived by algebro-geometric techniques.

formulated as follows:

$$Q_{12}[\xi_2(z_2)\hat{\otimes}\xi_1(z_1)]_{j_{12}} = [(Q_2\xi_2)(z_2)\hat{\otimes}\xi_1(z_1)]_{j_{12}} + [(\xi_2(z_2)\hat{\otimes}(Q_1\xi_1(z_1)]_{j_{12}}, \qquad (3.5.70)$$

where Q_{12}, Q_1, Q_2 are the BRST-Operators defined on the Fock-modules $\mathcal{F}_{j_{12}}$, $\mathcal{F}_{j_1}$ and $\mathcal{F}_{j_2}$ respectively.
In this subsection I will assume BRST-invariance as formulated in (3.5.70 in order to show that the fusion and braiding matrices are indeed given by the restricted q-Racah-Wigner coefficients that satisfy the additional fusion rules at $c = c_{p'p}$.

I will only consider the braiding matrix explicitely. The range of values for j_{31} that respects the orinary but violates the restricted fusion rules is

$$\max(p-1-j_1-j_3, p-1-j-j_2) \leq j_{31} \leq \min(j_1+j_3, j+j_2). \qquad (3.5.71)$$

One has to consider the behaviour of the normalization factors, of the Clebsch-Gordan and the Racah-Wigner coefficients at $q^p = -1$: What matters are the possible zeroes or poles due to $[p] = 0$. The necessary checks are slighty tedious: One needs the general explicit expression for Clebsch-Gordan coefficients

$$\begin{aligned} \left(\begin{smallmatrix} j_1 & j_2 \\ m_1 & m_2 \end{smallmatrix}\middle| j\right)_q &= q^{-\frac{1}{2}(j_1+j_2-j)(j_1+j_2+j+1)} q^{m_1 j_2 - m_2 j_1} ([2j+1])^{\frac{1}{2}} \Delta(j_1 j_2 j) \times \\ &\times \Big([j_1-m_1]![j_1+m_1]![j_2+m_2]![j_2-m_2]![j-m]![j+m]!\Big)^{\frac{1}{2}} \times \\ &\times \sum_n \frac{(-1)^n}{[n]!} q^{n(j+j_1+j_2+1)} \Big([j_1-m_1-n]![j_2+m_2-n]! \times \\ &\times [j_1+j_2-j-n]![j-j_1-m_2+n]![j-j_2+m_1+n]!\Big)^{-1}, \end{aligned}$$

where

$$\begin{aligned} &\max(0, j_1+m_2-j, j_2-m_1-j) \leq n \leq \min(j_1-m_1, j_2+m_2, j_1+j_2-j) \\ &\Delta(j_1 j_2 j) = ([j_1+j_2-j]![j+j_1-j_2]![j+j_2-j_1]!)^{\frac{1}{2}} ([j_1+j_2+j+1]!)^{-\frac{1}{2}}, \end{aligned}$$

as well as the Racah-Wigner coefficients

$$\begin{aligned} &\left\{\begin{smallmatrix} j_1 & j_2 \\ j_3 & j \end{smallmatrix}\middle|\begin{smallmatrix} j_{12} \\ j_{23} \end{smallmatrix}\right\}_q = \Delta(j_1 j_2 j_{12})\Delta(j_2 j_3 j_{23})\Delta(j_{12} j_3 j)\Delta(j_1 j_{23} j)([2j_{12}+1])^{\frac{1}{2}}([2j_{23}+1])^{\frac{1}{2}} \times \\ &\times \sum_n (-1)^n [n+1]! \times \\ &\times \Big([n-j_1-j_2-j_3]![n-j_1-j_{23}-j]![n-j_{12}-j_3-j]![n-j_2-j_3-j_{23}]! \times \\ &\times [j_1+j_2+j_3+j-n]![j_1+j_3+j_{12}+j_{23}-n]![j_2+j+j_{12}+j_{23}-n]!\Big)^{-1}, \end{aligned}$$

where the range of summation is such that none of the arguments of [.] becomes negative. Proofs for these expressions can be found in [HH].

It becomes necessary to distinguish three cases:

1. $\max(p-1-j_1-j_3, p-1-j-j_2) \leq j_{31} < \frac{p-1}{2}$:
   In this range of values of j_{31} the product of normalization factors vanishes, whereas the braid matrix remains nonsingular.
2. $j_{31} = \frac{p-1}{2}$:
   The braid matrix vanishes due to the factor $([2j_{31}+1])^{\frac{1}{2}}$ in the Racah-Wigner coefficients.
3. $\frac{p-1}{2} < j_{31} \leq \min(j_1+j_3, j+j_2)$:
   Consider the terms appearing on the r.h.s. of eqn. (3.5.68): By using (3.5.66) this is rewritten as

$$\sum_{m_1+m_2+m_3=j} \left(\begin{smallmatrix} j_2 & j_{31} \\ m_2 & m_1+m_3 \end{smallmatrix}\middle| j\right)_q e^{j_2}_{m_2}(\xi_2|z_2) e^{j_{31}}_{m_1+m_3} \left([\xi_3(z_3-z_1)\hat{\otimes}\xi_1(0)]_{j_{31}}|z_1\right). \qquad (3.5.72)$$

With a little work one may check from the explicit expression that the Clebsch-Gordan coefficients vanish for $m \geq p - j_{31}$. Therefore consider

$$\sum_{m<p-j_{31}} \left(\begin{smallmatrix} j_2 & j_{31} \\ j-m & m \end{smallmatrix}\middle| j\right)_q e^{j_2}_{j-m}(z_2) e^{j_{31}}_m(\xi_{31}|z_1). \qquad (3.5.73)$$

It follows from equation (3.5.21) that for each term of the sum

$$e^{j_{31}}_m(\xi_{31}|z_1) = e^{p-1-j_{31}}_m(Q_{2j_{31}+1-p}\xi_{31}|z_1).$$

Under above assumption that the fusion product $[\xi_3(z_3-z_1)\hat{\otimes}\xi_1(0)]_{j_{31}}$ has the property of BRS-invariance, eqn. (3.5.70), one concludes that these contributions also vanish if $Q\xi_1 = 0$ and $Q\xi_3 = 0$.

Therefore the fusion- and braiding matrices in the $c = c_{p'p}$ case are indeed given by the restricted Racah-Wigner coefficients one expects for the corresponding data of the minmal models.

SECTION IV: QUANTUM GROUP STRUCTURE

The observation that leads to the search for a quantum group structure in Liouville theory or minimal models is the fact that the structure constants of operator product expansions and exchange relations, the fusion and braiding matrices, are given by q-6j symbols. As explained previously, they could be expressed in terms of the elementary data $\Omega\begin{pmatrix} j_{12} \\ j_1 j_2 \end{pmatrix}$ and $F_{j_{12}j_{23}}\begin{bmatrix} j_1 j_2 \\ j_3 j_\infty \end{bmatrix}$. It is useful to compare their occurence in correlation functions to the meaning in the representation theory of the quantum group:

1. Braiding of fusion products:

$$[v_{j_1}(z_1)\hat{\otimes} v_{j_2}(z_2)]_{j_{12}} = q^{\pm\frac{1}{2}(j_{12}(j_{12}+1)-j_1(j_1+1)-j_2(j_2+1))}[v_{j_2}(z_2)\hat{\otimes} v_{j_1}(z_1)]_{h(j_{12})} \tag{4.1}$$

Quantum group braiding in a Clebsch-Gordan basis (see Appendix A):

$$B^{\pm 1}(e_m^{j_{12}}(j_1 j_2)) = q^{\pm\frac{1}{2}(j_{12}(j_{12}+1)-j_1(j_1+1)-j_2(j_2+1))} e_m^{j_{12}}(j_2 j_1) \tag{4.2}$$

2. Fusion:

$$[v_{j_3}(z_3)\hat{\otimes}[v_{j_2}(z_2)\hat{\otimes} v_{j_1}(z_1)]_{j_{12}}]_j = \sum_{j_{23}} \left\{ \begin{matrix} j_1 & j_2 \\ j_3 & j \end{matrix} \middle| \begin{matrix} j_{12} \\ j_{23} \end{matrix} \right\}_q [[v_{j_3}(z_3)\hat{\otimes} v_{j_2}(z_2)]_{j_{23}} \hat{\otimes} v_{j_1}(z_1)]_j \tag{4.3}$$

Definition of q-6j symbols:

$$e_m^{j_{12}j}(j_1 j_2|j_3) = \sum_{j_{23}} \left\{ \begin{matrix} j_1 & j_2 \\ j_3 & j \end{matrix} \middle| \begin{matrix} j_{12} \\ j_{23} \end{matrix} \right\}_q e_m^{j_{23}j}(j_1|j_2 j_3), \tag{4.4}$$

where $e_m^{j_{12}j}$ and $e_m^{j_{23}j}$ are defined in appendix A.

The observation is that the equivalence transformations representing associativity and commutativity of fusion products resp. quantum group tensor products are represented by the same matrices.[1] More abstractly, one may view fusion and braiding operations as generating an algebra with relations given by the polynomial equations, which may be called moduli algebra. Then the observation is that both the vector space of conformal blocks and the vector space of invariants in tensor products of quantum group representations form equivalent representations of this moduli algebra.[2]

1. Covariant objects

The occurrence of $\mathcal{U}_q(sl(2))$ Racah-Wigner coefficients shows a connection to the representation theory of the quantum group $\mathcal{U}_q(sl(2))$. One may hope that $\mathcal{U}_q(sl(2))$ is realized as an additional symmetry of the theory. In order to exploit the information that is provided by such a symmetry it is natural look for objects on which the symmetry may act covariantly. These may be useful for the following two reasons:

- They could be the key to a deeper understanding of the reasons for these structures to occur.
- At least they should provide a transparent formalism to handle the existing information.

I now want to present two proposals due to Babelon/Gervais and Moore/ Reshetikhin/ Mack/ Schomerus that do not yet help with the first point, but provide solutions of the second point, albeit being complementary in their applicability.

1.1 BABELON/GERVAIS' PROPOSAL

The operators used by Gervais and collaborators are reconstructed from chiral vertex

[1] The proper mathematical formulation will probably be a statement on the functorial equivalence of two tensor categories, but I'm not enough used to that language to phrase it that way.

[2] There should be a connection of this point of view to attempts to recover conformal field theory from Chern-Simons theory: In a Hamiltonian approach to Chern-Simons theory as developed by Alekseev, Grosse and Schomerus in [AGS] one indeed constructs something as a moduli algebra by quantization of the moduli space of flat connections on a Riemann surface. It would be interesting to see whether this moduli algebra is related to what I called moduli algebra.

operators as follows:

$$V^j_M(z) = \sum_{J_0} V\begin{pmatrix} & j & \\ J_0-M & & J_0 \end{pmatrix}(z), \tag{4.1.1}$$

where the variable J_0 is related to the momentum variable ϖ used by Gervais et al. by $\varpi = 2J_0 + \varpi_0$, $\varpi_0 \equiv 1 - \alpha_-^2 \equiv 1 + \pi/h$. The operators $V^j_M(z)$ will be assumed to be normalized by

$$(v_{J_0-M}, V^j_M(1) v_{J_0}) = 1. \tag{4.1.2}$$

It was shown in [Ba][Ge1] that there exist momentum dependent coefficients $C^{jM}_m(\varpi)$ such that the linear combinations

$$\xi^j_m := \sum_{M=-j}^{j} C^{jM}_m(\varpi) V^j_M(z) \tag{4.1.3}$$

obey exchange relations with coefficients given by the matrix elements of the universal $\mathcal{U}_q(sl(2))$-R-matrix:

$$\xi^{j_1}_{m_1}(z_1)\xi^{j_2}_{m_2}(z_2) = \sum_{n_1 n_2} (R^{j_1 j_2})^{n_1 n_2}_{m_1 m_2} \xi^{j_2}_{n_2}(z_2)\xi^{j_1}_{n_1}(z_1). \tag{4.1.4}$$

The point about this specific form of exchange relation is that it is *necessary* for a covariant transformation law of operators to be consistent with the existence of a braid relation: One may simply define a $\mathcal{U}_q(sl(2))$ action on the space spanned by the ξ^j_m by:

$$E\xi^j_m = ([j-m][j+m+1])^{1/2}\xi^j_{m+1} \qquad F\xi^j_m = ([j+m][j-m+1])^{1/2}\xi^j_{m-1} \qquad H\xi^j_m = 2m\xi^j_m$$

This $\mathcal{U}_q(sl(2))$ action is extended to products of field operators by identifying them with tensor products of $\mathcal{U}_q(sl(2))$ modules, i.e.

$$E(\xi^{j_1}_{m_1}(z_1)\xi^{j_2}_{m_2}(z_2)) = (E\xi^{j_1}_{m_1}(z_1))(q^{-H/2}\xi^{j_2}_{m_2}(z_2)) + (q^{H/2}\xi^{j_1}_{m_1}(z_1))(E\xi^{j_2}_{m_2}(z_2)) \tag{4.1.5}$$

Under this identification one may recognize the r.h.s. of (4.1.4) as the result of the braiding operation B defined in appendix A on $\xi^{j_1}_{m_1}(z_1)\xi^{j_2}_{m_2}(z_2)$. The equality (4.1.4) is consistent with above $\mathcal{U}_q(sl(2))$ action *only if* the representation matrices of the $\mathcal{U}_q(sl(2))$ action on both sides of (4.1.4) turn out to be the same. But this is just what was proved in appendix A.

The existence of exchange relations with $\mathcal{U}_q(sl(2))$ R-matrix is a necessary condition for $\mathcal{U}_q(sl(2))$ to be a symmetry of the theory. However, their occurrence does not imply that the space of states as defined up to now carries a $\mathcal{U}_q(sl(2))$ representation, as would be necessary to establish $\mathcal{U}_q(sl(2))$ as a symmetry in the ordinary sense[3].

[3] see the work [MaS1] for the discussion of a generalized notion of symmetry in quantum theories that encompasses quantum group symmetries

The purpose of the rest of this subsection will be to analyze the compatibility of this result with the additional fusion rules that are relevant when considering the action of operators V^j_M on modules with conformal dimensions $h_{n'n}$. In order to get a qualitative understanding of the problems it suffices to consider the simplest case $j_1 = 1/2 = j_2$:

Let $V_\pm(z) = V^{1/2}_{\pm 1/2}$. The exchange relations of the operators $V_\pm$ may be found in two ways: Either by using the relation between fusion and braiding matrix and the explicit expressions for the latter found in chapter III.4, or by reducing the differential equations satisfied by the correlators

$$< v_{J_\infty}, V_\pm(z_1)V_\pm(z_2)v_{J_0} > \tag{4.1.6}$$

to the hypergeometric differential equation and using (for $|\arg(-z)| < \pi$)

$$\begin{aligned} F(a,b;c;z) = & \frac{\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)}(-z)^{-a}F(a, 1-c+a; 1-b+a; \frac{1}{z}) \\ & + \frac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)}(-z)^{-b}F(b, 1-c+b; 1-a+b; \frac{1}{z}) \end{aligned}$$

One finds $(k,l,m,n = +,-)$

$$V_k(z_1)V_l(z_2) = \sum_{mn} B^{mn}_{kl} V_n(z_2)V_m(z_1) \qquad \text{for} \quad |\arg(-\frac{z_1}{z_2})| < \pi, \tag{4.1.7}$$

where the nonvanishing matrix elements are given by

$$B^{++}_{++}(\varpi) = B^{--}_{--}(\varpi) = e^{-i\pi\gamma/2}$$

$$B^{+-}_{+-}(\varpi) = B^{-+}_{-+}(-\varpi) = \frac{\Gamma(1+\gamma\varpi)\Gamma(-\gamma\varpi)}{\Gamma(\gamma)\Gamma(1-\gamma)} e^{i\pi\gamma(\frac{1}{2}+\varpi)}$$

$$B^{-+}_{+-}(\varpi) = B^{+-}_{-+}(-\varpi) = \frac{\Gamma(1+\gamma\varpi)\Gamma(\gamma\varpi)}{\Gamma(\gamma(1+\varpi))\Gamma(1+\gamma(\varpi-1))} e^{i\pi\gamma/2}$$

The problematic cases are those where an operator ψ_+ acts on a state from the module with $J_0 = 0 \leftrightarrow \varpi = \varpi_0$. Remember that solutions $V_\pm$ to the operator differential equation $\partial^2 V_\pm = \gamma : TV_\pm :$ could not be realized as covariant operators on a Fock space or an irreducible module build on a highest weight state with conformal weight 0 (sects. III.3.5,III.5.6). Since $B^{-+}_{+-}(\varpi_0) = 0$ it is consistent with the existence of the braid relation (4.1.7) to use covariant operators only, i.e. to demand $V_+(z)v_{\varpi_0} = 0$. This may be realized by choosing a ϖ-dependent normalization that vanishes at $\varpi = \varpi_0$.

Now consider the definition of the operators ξ. For $j_1 = j_2 = 1/2$ it reads explicitely $(\xi_\pm := \xi^{1/2}_{\pm 1/2})$

$$\xi_- = e^{-i\pi\gamma\varpi/2}\Gamma(\gamma\varpi)V_- + e^{i\pi\gamma(\varpi-1)/2}\Gamma(-\gamma\varpi)V_+ \tag{4.1.8}$$

$$\xi_+ = e^{i\pi\gamma(\varpi+1)/2}\Gamma(\gamma\varpi)V_- + e^{-i\pi\gamma\varpi/2}\Gamma(-\gamma\varpi)V_+ \tag{4.1.9}$$

With some calculations one finds that one has

$$\xi_k(z_1)\xi_l(z_2) = \sum_{mn} R^{mn}_{kl}\xi_m(z_2)\xi_l(z_1) \qquad \text{for} \quad |\arg(-\frac{z_1}{z_2})| < \pi, \tag{4.1.10}$$

where the matrix elements R^{mn}_{kl} are indeed those of the universal $\mathcal{U}_q(sl(2))$ R-matrix ($q = e^{i\pi\gamma}$):

$$\begin{aligned} R^{++}_{++} &= R^{--}_{--} = q^{-1/2} \\ R^{-+}_{+-} &= R^{+-}_{-+} = q^{1/2} \end{aligned} \qquad R^{+-}_{+-} = q^{1/2}(1-q^2) \tag{4.1.11}$$

The problem is now that one finds the universal $\mathcal{U}_q(sl(2))$ R-matrix in (4.1.10) only if the operators $V_\pm$ are normalized according to (4.1.2) for *any* ϖ, i.e. also for $\varpi = \varpi_0$. But that is incompatible with Virasoro-covariance. One therefore has the choice between $\mathcal{U}_q(sl(2))$-compatible braiding but Virasoro-noncovariance on the module with $\varpi = \varpi_0$, or Virasoro-covariance of $V_\pm$ on the total space of states but having to give up the construction (4.1.8), (4.1.9) of operators with $\mathcal{U}_q(sl(2))$ R-matrix braiding.[4]
I therefore conclude that the transformation to fields ξ^j_m is not suitable to construct $\mathcal{U}_q(sl(2))$-covariant objects in the case of minimal models or, more generally, if one wants to consider Liouville theory on modules with conformal dimensions $h_{n'n}$ as proposed in the present work.

1.2 Proposal of Moore/Reshetikhin, Mack/Schomerus

In the following only that part of the theory will be considered where all states and operators have spins $J = j - \alpha_-^2 j'$, $j, j' \in \frac{1}{2}\mathbb{N}$. States and operators with $j' = 0$ form a closed subsector of the theory which will be the one to be considered explicitely in the following.
The basic idea is now to change the definition of the space of states

$$\text{from} \quad \mathcal{H} = \bigoplus_{j=0}^{m(c)} \mathcal{H}_j \qquad \text{into} \quad \mathcal{H} = \bigoplus_{j=0}^{m(c)} \bigoplus_{-j\leq m\leq j} \mathcal{H}_{j,m}, \tag{4.1.12}$$

where each $\mathcal{H}_{j,m}$ is a copy of $\mathcal{H}_j \equiv \mathcal{H}_{h(0,j)}$, and $m(c) = \infty$ if $c \neq c_{p'p}$, $m(c) = \frac{p-2}{2}$ if $c = c_{p'p}$.
One has thus by hand introduced degrees of freedom on which the quantum group may act. This is clearly artificial and unnecessary for a consistent formulation of the theory but will be seen to lead to a nice formulation of the quantum group structure.

[4] This does not exclude the possibility that a modification of (4.1.8), (4.1.9) exists that works even if V_+ is normalized to vanish at $\varpi = \varpi_0$. I am rather sceptical on that.

First of all one may trivially extend the definition of CVO's to the enlarged space of states by

$$\psi_{m_2m_1}\binom{j}{j_2 j_1} := \bar{P}^{j_2}_{m_2}\psi\binom{j}{j_2 j_1}P^{j_1}_{m_1}, \tag{4.1.13}$$

where $P^{j_1}_{m_1} : \mathcal{H} \to \mathcal{H}_{j_1}$ projects to $\mathcal{H}_{j_1,m_1}$ and identifies with $\mathcal{H}_{j_1}$, whereas $\bar{P}^{j_2}_{m_2} : \mathcal{H}_{j_2} \to \mathcal{H}$ is the obvious imbedding of $\mathcal{H}_{j_2}$ in $\mathcal{H}$.
If one denotes an arbitrary state in $\mathcal{H}$ by $\xi^j_m \in \mathcal{H}_{j,m}$ one has an obvious realization $\mathcal{U}(\zeta)$, $\zeta \in \mathcal{U}_q(sl(2))$ on $\mathcal{H}$: $\mathcal{U}(E)\xi^j_m = ([j-m][j+m+1])^{1/2}\xi^j_{m+1}$ etc..

A natural definition for the notion of a covariant transformation law for operators Γ^j_m has been given in [MaS1]: Let the co-product be written as $\Delta(\rho) = \sum_\sigma \rho^1_\sigma \otimes \rho^2_\sigma$, with $\rho, \rho^1_\sigma, \rho^2_\sigma \in \mathcal{U}_q(sl(2))$. $\mathcal{U}(\xi)$ will then be said to act covariantly on the operator Γ^j_m if

$$\mathcal{U}(\xi)\Gamma^j_m = \sum_\sigma \Gamma^j_{m'}\Pi^j_{m'm}(\rho^1_\sigma)\mathcal{U}(\rho^2_\sigma), \tag{4.1.14}$$

where $\Pi^j_{m'm}$ is the representation matrix in the spin j representation. This definition is natural in the sense that the action of Γ^j_m on a state ψ in a representation V of $\mathcal{U}_q(sl(2))$ transforms as the tensor product representation $V_j \otimes V$.

One may now define covariant field operators as

$$\Gamma^j_m(z) := \sum_{j_1,j_2} \sum_{\substack{m_1,m_2 \\ m_1+m=m_2}} \left(\begin{smallmatrix} j & j_1 \\ m & m_1 \end{smallmatrix}\middle| j_2\right)_q \psi_{m_2m_1}\binom{j}{j_2j_1}(z). \tag{4.1.15}$$

By using equation (6.1.9) of appendix A one may check that this object indeed transforms covariantly in the sense of (4.1.14) ($d^{j\pm}_m = ([j\mp m][j\pm m+1])^{1/2}$):

$$\begin{aligned} &\mathcal{U}(E)\Gamma^j_m(z) \\ &= \sum_{j_1j_2} \sum_{\substack{m_1,m_2 \\ m_1+m=m_2}} \left(\begin{smallmatrix} j & j_1 \\ m & m_1 \end{smallmatrix}\middle| j_2\right)_q d^{j_2+}_{m_2}\psi_{m_2+1,m_1}\binom{j}{j_2j_1} \\ &= \sum_{j_1j_2} \sum_{\substack{m_1,m_2 \\ m_1+m=m_2-1}} \left(\begin{smallmatrix} j & j_1 \\ m & m_1 \end{smallmatrix}\middle| j_2\right)_q d^{j_2+}_{m_2-1}\psi_{m_2m_1}\binom{j}{j_2j_1} \\ &= \sum_{j_1j_2} \sum_{\substack{m_1,m_2 \\ m_1+m=m_2-1}} \left(\left(\begin{smallmatrix} j & j_1 \\ m+1 & m_1 \end{smallmatrix}\middle| j_2\right)_q d^{j+}_m q^{-m_1} + \left(\begin{smallmatrix} j & j_1 \\ m & m_1+1 \end{smallmatrix}\middle| j_2\right)_q d^{j_1+}_{m_1} q^m\right)\psi_{m_2m_1}\binom{j}{j_2j_1} \\ &= d^{j+}_m\Gamma^j_{m+1}(z)\mathcal{U}(q^{-H/2}) + q^m\Gamma^j_m(z)\mathcal{U}(E) \\ &= \Gamma^j_{m'}(z)\Pi^j_{m'm}(E)\mathcal{U}(q^{-H/2}) + \Gamma^j_{m'}(z)\Pi^j_{m'm}(q^{H/2})\mathcal{U}(E) \end{aligned}$$

For the check that the Γ^j_m have a braid relation in terms of the universal $\mathcal{U}_q(sl(2))$ R-matrix one has to distinguish two cases: $c = c_{p'p}$ and $c \neq c_{p'p}$. In the second case one may use the explicit form of the braiding matrix for the chiral vertex operators ψ in terms of the q-Racah-Wigner coefficients together with identity (6.1.40) of appendix A to prove

$$\Gamma^{j_1}_{m_1}(z_1)\Gamma^{j_2}_{m_2}(z_2) = \sum_{m'_1 m'_2} (R^{j_1 j_2})^{m'_1 m'_2}_{m_1 m_2} \Gamma^{j_2}_{m'_2}(z_2)\Gamma^{j_1}_{m'_1}(z_1). \tag{4.1.16}$$

I will not discuss the subtleties that arise at $c = c_{p'p}$, since references [MaS1][MaS2] contain a clear exposition of these matters. The additional fusion rules lead to the fact that the appropriate quantum group is a truncation of $\mathcal{U}_q(sl(2))$ in which coassociativity has to be weakened to quasi-coassociativity, i.e. coassociativity up to an equivalence transformation.
Let me however point out that the field theoretic input that is needed to apply the constructions of Mack and Schomerus to arbitrary minimal models has been obtained in this work: The braiding matrices of chiral vertex operators were found to be given by *restricted* q-Racah-Wigner coefficients which are compatible with the additional fusion rules for $c = c_{p'p}$.

In comparison with the proposal of Babelon and Gervais one observes that the two approaches are complimentary in their applicability: Both provide constructions of operators with exchange relations that are compatible with a covariant transformation law under $\mathcal{U}_q(sl(2))$. The proposal of Babelon and Gervais works for generic conformal weights (i.e. $h \neq h_{n'n}$) only. It is not clear whether $\mathcal{U}_q(sl(2))$ can be realized as a symmetry since the space of states does not seem to carry a $\mathcal{U}_q(sl(2))$ representation. On the contrary, the approach of Mack and Schomerus works on the modules with $h = h_{n'n}$.

1.3 Discussion

To my opinion, the reason for structures related to quantum groups to occur are not yet fully understood. A natural quantum group realization on Virasoro representation spaces has not been found, so that it is questionable whether there is an action on the space of states at all. An action of the quantum group on the space of states can be introduced by defining the space of states in redundant manner: The degrees of freedom on which $\mathcal{U}_q(sl(2))$ acts are introduced by hand. In that case one has the problem of finding a natural interpretation for the additional degrees of freedom.

A possible "explanation" for the quantum group structure is provided by the free field representation: There is a natural quantum group strucure on the space of multiple loops on a punctured sphere [FW], which leads to the q-6j braiding and fusion matrices via free field construction of chiral vertex operators.

2. Reconstruction of correlation functions from quantum group structure

Up to now the quantum group structures appeared as a consequence of conformal invariance: The latter led to a definition of correlation functions as power series expansions. Conformal symmetry provides the local information on correlation functions by specifying arbitrary derivatives with respect to the locations of insertions. The quantum group structure appeared in considering the global properties of conformal blocks: Their monodromies were given in terms of objects well known from the representation theory of $\mathcal{U}_q(sl(2))$.
I consider it to be an interesting observation that it is in a certain sense possible to go backwards: To reconstruct correlation functions from the global information given in terms of quantum group objects. This constitutes a kind of Riemann problem of reconstructing holomorphic functions from their monodromy representation. Its solution is unique up to meromorphic factors which are themselves uniquely fixed once one also knows the integer parts of the leading exponents in operator product expansions.
This fact supports to view the quantum group structure as an essential piece of information on the theory. It suggests to view the rôles of conformal symmetry and quantum group structure as being dual to each other: Both being (almost) sufficient to determine the conformal blocks, but the former describing their local structure, whereas the latter encode their global structure.

2.1 Formulation of the problem

Consider the example of four point functions with one point at infinity:

$$\mathcal{G}^{J_{21}}_{(12)3}\begin{bmatrix} J_3 & J_2 \\ J & J_1 \end{bmatrix}(z_1, z_2, z_3) := (v_J, [v_{J_3}(z_3)\hat{\otimes}[v_{J_2}(z_2)\hat{\otimes}v_{J_1}(z_1)]_{J_{12}}]_J), \tag{4.2.1}$$

where all J's are assumed to be half-integers. These may be considered to form a basis for a vector space with dimensionality given by the range of j_{12}. There are further canonical choices of bases associated with different orders of performing the products: Let (ijk) be any permutation of (123).

$$\mathcal{G}^{J_{ij}}_{(ij)k}\begin{bmatrix} J_k & J_j \\ J & J_i \end{bmatrix} = (v_J, [v_{J_k}(z_3)\hat{\otimes}[v_{J_j}(z_2)\hat{\otimes}v_{J_i}(z_1)]_{J_{ij}}]_J)$$
$$\mathcal{G}^{J_{jk}}_{i(jk)}\begin{bmatrix} J_k & J_j \\ J & J_i \end{bmatrix} = (v_J, [[v_{J_k}(z_3)\hat{\otimes}v_{J_j}(z_2)]_{J_{jk}}\hat{\otimes}v_{J_i}(z_1)]_J).$$

The aim is to reconstruct these functions from the following information:

1. Analyticity:
   The funtions $\mathcal{G}$ are analytic on the universal cover of the complement of the hyperplanes $z_i = z_j$, $i \neq j \in \{1, 2, 3\}$ in $\mathbb{C}^3$.

2. Projective invariance: Let $\Gamma = (ij)k$ or $\Gamma = i(jk)$.

$$\mathcal{G}_\Gamma^{J_{ij}}\begin{bmatrix} J_k & J_j \\ J & J_i \end{bmatrix} = (z_k - z_i)^{\Delta(J)-\Delta(J_i)-\Delta(J_j)-\Delta(J_k)} \mathcal{F}_\Gamma^{j_{ij}}\begin{bmatrix} J_k & J_j \\ J & J_i \end{bmatrix}\left(\frac{z_j - z_i}{z_k - z_i}\right) \tag{4.2.2}$$

3. Asymptotics:

$$\begin{aligned} \mathcal{G}_{(ij)k} &\approx (z_j - z_i)^{\Delta_{ij}}(z_k - z_i)^{\Delta_{(ij)k}} \quad \text{for} \quad |z_j - z_i| \ll |z_k - z_i| \\ \mathcal{G}_{i(jk)} &\approx (z_k - z_j)^{\Delta_{jk}}(z_k - z_i)^{\Delta_{i(jk)}} \quad \text{for} \quad |z_k - z_j| \ll |z_k - z_i|, \end{aligned}$$

where ($\Delta(J) = \gamma J(J+1) - J$)

$$\Delta_{ij} = \Delta(J_{ij}) - \Delta(J_i) - \Delta(J_j) \qquad \Delta_{(ij)k} = \Delta(J) - \Delta(J_{ij}) - \Delta(J_k)$$

and the first line means that $\mathcal{G}_{(ij)k}(z_j - z_i)^{-\Delta_{ij}}(z_k - z_i)^{-\Delta_{(ij)k}}$ is a single valued analytic function in a neighborhood of the indicated asymptotic zone.

4. Braiding: Consider the case that $(z_i, z_j), (z_i, z_k) \in \mathbb{C}_+^2$. In this region one has

$$\mathcal{G}_{(ij)k}^{J_{ij}} = \mathcal{G}_{(ji)k}^{J_{ij}} e^{\pi i \Delta_{ij}} \qquad \mathcal{G}_{(ij)k}^{J_{ij}} = \mathcal{G}_{k(ij)}^{J_{ij}} e^{\pi i \Delta_{(ij)k}}. \tag{4.2.3}$$

5. Fusion:

$$\mathcal{G}_{(ij)k}^{J_{ij}} = \sum_{J_{jk}} \begin{Bmatrix} J_i & J_j \\ J_k & J \end{Bmatrix}\!\!\begin{matrix} J_{ij} \\ J_{jk} \end{matrix}\Big\}_q \mathcal{G}_{i(jk)}^{J_{jk}} \tag{4.2.4}$$

Note that these requirements are not all indedendent, since i.e. the fourth is a consequence of the third.
This is a variant of the Riemann monodromy problem since the data on braiding and fusion allow to determine all monodromies. The assumption of projective invariance allows in the present case to reduce the problem to the case of one complex variable.

2.2 Solution by Knizhnik-Zamolodchikov equations

I now want to explain why a general theorem of Drinfeld ensures existence of a solution to the problem under consideration. In order to prepare for its statement I will have to discuss the following "universal" version of the KZ-equations:

$$\frac{\partial G}{\partial z_i} = \tilde{h} \sum_{j \neq i} \frac{t^{ij}}{z_i - z_j} G. \tag{4.2.5}$$

where G takes values in $\mathcal{U}_\mathfrak{g} \otimes \mathcal{U}_\mathfrak{g} \otimes \mathcal{U}_\mathfrak{g}$, $\mathfrak{g} = sl(2)$. t^{ij} is defined as

$$t^{ij} = t^{ji} = \frac{1}{2} H^i H^j + E^i F^j + F^i E^j, \tag{4.2.6}$$

where H, E, F are the usual generators of $sl(2)$ and $\xi^3 = \xi \otimes \mathrm{id} \otimes \mathrm{id}$, $\xi^2 = \mathrm{id} \otimes \xi \otimes \mathrm{id}$, $\xi^1 = \mathrm{id} \otimes \mathrm{id} \otimes \xi$ for $\xi = E, F, H$. A crucial identity satisfied by t^{ij} is the following:

$$[t^{12}, t^{13} + t^{23}] = 0. \tag{4.2.7}$$

In order to avoid questions of convergence in $\mathcal{U}_\mathfrak{g}$ the solutions will be considered to be formal series in the symbol $\tilde{h}$. Therefore in the following a solution G will be called analytic if any coefficient function of the expansion in powers of h is analytic.

2.2.4 Solutions to (4.2.5) are uniquely characterized by their asymptotic behaviour: First note that the solutions are of the general form

$$G(z_1, z_2, z_3) = (z_3 - z_1)^{\tilde{h}(t^{12}+t^{13}+t^{23})} F\left(\frac{z_2 - z_1}{z_3 - z_1}\right), \tag{4.2.8}$$

where F satisfies

$$\frac{dF}{dz} = \tilde{h}\left(\frac{t^{12}}{z} + \frac{t^{23}}{z-1}\right) F(z). \tag{4.2.9}$$

Expressions of the form $z^{\tilde{h}f}$, $f \in \mathcal{U}_\mathfrak{g} \otimes \mathcal{U}_\mathfrak{g}$ are to be understood as the following formal exponential series:

$$z^{\tilde{h}f} := exp(\tilde{h} f \ln(z)).$$

First consider the problem of finding solutions to (4.2.9) with specified asymptotic behaviour: It is a standard theorem of the theory of ordinary differential equations that there exist unique solutions of the form

$$F_1(z) = f_1(z) z^{\tilde{h}t^{12}} \qquad F_2(z) = f_2(z)(1-z)^{\tilde{h}t^{23}},$$

where $f_1(z)$ ($f_2(z)$) is analytic in a neighborhood of $z = 0$ ($z = 1$) and $f_1(0) = \mathrm{id}_{\mathcal{U}_\mathfrak{g} \otimes \mathcal{U}_\mathfrak{g} \otimes \mathcal{U}_\mathfrak{g}} = f_2(1)$.
If one now for example asks for the solution to (4.2.5) with asymptotics

$$G \approx (z_2 - z_1)^{\tilde{h}t^{12}} (z_3 - z_1)^{\tilde{h}(t^{13}+t^{23})}, \tag{4.2.10}$$

then it is given by substituting $F_1(z)$ for $F(z)$ in (4.2.8). The logarithms appearing in (4.2.10) are taken as principal values.

2.2.5 There are twelve different asymptotic zones which may be associated with expressions of the form $(ij)k$ or $i(jk)$, where (ijk) is any permutation of (123). Associated with each of these zones is an unique solution of the form

$$G_{(ij)k} = f_{(ij)k}\left(\frac{z_j - z_i}{z_k - z_i}\right)(z_j - z_i)^{\tilde{h}t^{ij}}(z_k - z_i)^{\tilde{h}(t^{ik}+t^{jk})}$$
$$G_{i(jk)} = f_{i(jk)}\left(\frac{z_k - z_j}{z_k - z_i}\right)(z_k - z_j)^{\tilde{h}t^{jk}}(z_k - z_i)^{\tilde{h}(t^{ij}+t^{ik})},$$

where the functions $f_{(ij)k}(z)$, $f_{i(jk)}(z)$ are analytic around $z=0$ and equal to the identity for $z=0$.
The solutions corresponding to different asymptotic zones are linearly related: $G_\Gamma = G_{\Gamma'}M_{\Gamma\Gamma'}$, where Γ, Γ' are expressions of the form $(ij)k$ or $i(jk)$ and $M \in \mathcal{U}_\mathfrak{g}\otimes\mathcal{U}_\mathfrak{g}\otimes\mathcal{U}_\mathfrak{g}$. The linear relation between $G_{(ij)k}$ and $G_{i(jk)}$ defines an object $\phi_{ijk}\in\mathcal{U}_\mathfrak{g}\otimes\mathcal{U}_\mathfrak{g}\otimes\mathcal{U}_\mathfrak{g}$, called reassociator:

$$G_{(ij)k} = G_{i(jk)}\phi_{ijk} \tag{4.2.11}$$

Note that there is only one such object: If $\phi := \phi_{123} = \sum_\tau \varphi_\tau^3\otimes\varphi_\tau^2\otimes\varphi_\tau^1$ then $\phi_{ijk} = \sum_\tau \varphi_\tau^k\otimes\varphi_\tau^j\otimes\varphi_\tau^i$. One has unique relations between $G_{(ij)k}$, $G_{(ji)k}$ and $G_{k(ij)}$ in the region $(z_i,z_j),(z_i,z_k)\in\mathbb{C}_+^2$ (*braiding*):

$$G_{(ij)k} = G_{(ji)k}e^{\pi i\tilde{h}t^{ij}} \qquad G_{(ij)k} = G_{k(ij)}e^{\pi i\tilde{h}(t^{ik}+t^{jk})}. \tag{4.2.12}$$

To verify these relations note that because of $\ln(z) = ln(-z)+i\pi$ in the considered region and identity (4.2.7) both sides of the eqns. (4.2.12) have the same asymptotic behaviour. Since the asymptotic behaviour specifies the solutions uniquely, the identities follow.

2.2.6 It is clear that the monodromy of solutions to the KZ equations (4.2.5) is completely determined once one knows the data t^{ij} and ϕ: The monodromy operation is nothing but braiding squared, and by using ϕ if necessary one may always transform to a solution for which the braiding operation is represented in terms of t^{ij} as in equation (4.2.12).

2.2.7 Drinfeld uses the data $(\mathfrak{g},\mathcal{R},\phi)$ $(\mathcal{R} = e^{i\pi\tilde{h}t^{12}})$ to define nontrivial examples $A_{\mathfrak{g},t}$ of an algebraic structure called quasitriangular quasi Hopf algebra. In order to define an action of such an algebra on tensor products one needs a further piece of data: A co-product $\Delta:\ \mathcal{U}_\mathfrak{g}\to\mathcal{U}_\mathfrak{g}\otimes\mathcal{U}_\mathfrak{g}$. In the present case it will be simply given by $\Delta(a) = a\otimes\mathrm{id}+\mathrm{id}\otimes a$, $a\in\mathfrak{g}$. Structures of this type are considered equivalent if related by some transformation called twist. A twist is generated by an invertible element $F\in\mathcal{U}_\mathfrak{g}\otimes\mathcal{U}_\mathfrak{g}$ that is defined to act on $(\Delta,\mathcal{R},\phi)$ as follows:

$$\begin{aligned}\tilde{\mathcal{R}} &= F^{21}\cdot\mathcal{R}\cdot F^{-1}\\ \tilde{\Delta}(a) &= F\cdot\Delta(a)\cdot F^{-1}\end{aligned} \qquad \tilde{\phi} = F^{23}\cdot(\mathrm{id}\otimes\Delta)(F)\cdot\phi\cdot(\Delta\otimes\mathrm{id})(F^{-1})\cdot(F^{12})^{-1}, \tag{4.2.13}$$

where the following notation has been used: If $F = \sum_\sigma f_\sigma^1\otimes f_\sigma^2$ then $F^{21} = \sum_\sigma f_\sigma^2\otimes f_\sigma^1$, $F^{23} = \sum_\sigma \mathrm{id}\otimes f_\sigma^1\otimes f_\sigma^2$ etc..
Drinfeld proves in [Dr3][Dr4] an uniqueness theorem for quasitriangular quasi Hopf algebras. One of its consequences for the present situation (Prop. 3.16 of [Dr3]) is the existence of an element $F\in\mathcal{U}_\mathfrak{g}\otimes\mathcal{U}_\mathfrak{g}$ such that

$$\tilde{\mathcal{R}} = R_{q^{-1}}, \qquad \tilde{\Delta}(a) = \Delta_{q^{-1}}(a), \qquad \tilde{\phi} = \mathrm{id}, \tag{4.2.14}$$

where R_q is the universal $\mathcal{U}_q(sl(2))$ R-matrix introduced in appendix A, Δ_q is the $\mathcal{U}_q(sl(2))$ co-product, and the deformation parameter q is given by $q = e^{i\pi\tilde{h}}$. All objects involved are formal series in $\tilde{h}$.
This concludes the review of Drinfeld's work.

2.2.8 The idea now is to apply a twist to solutions of the KZ-equations in order to find functions with monodromy data given by the $\mathcal{U}_q(sl(2))$ R-matrix. Let me define

$$\tilde{G}_{(ij)k} = G_{(ij)k}\cdot P^{ijk}\cdot(\Delta\otimes\mathrm{id})(F^{-1})\cdot(F^{12})^{-1}\cdot P^{ijk} \tag{4.2.15}$$
$$\tilde{G}_{i(jk)} = G_{i(jk)}\cdot P^{ijk}\cdot(\mathrm{id}\otimes\Delta)(F^{-1})\cdot(F^{23})^{-1}\cdot P^{ijk}, \tag{4.2.16}$$

where the permutation operator P^{ijk} acts as $P^{ijk}(\xi^1\otimes\xi^2\otimes\xi^3)P^{ijk} = \xi^i\otimes\xi^j\otimes\xi^k$. I will now calculate the linear relations between the different $\tilde{G}$:

$$\begin{aligned}\tilde{G}_{(ij)k} &= G_{i(jk)}\cdot\phi^{ijk}\cdot P^{ijk}\cdot(\Delta\otimes\mathrm{id})(F^{-1})\cdot(F^{12})^{-1}\cdot P^{ijk}\\ &= G_{i(jk)}\cdot P^{ijk}\cdot\phi\cdot(\Delta\otimes\mathrm{id})(F^{-1})\cdot(F^{12})^{-1}\cdot P^{ijk}\\ &= G_{i(jk)}\cdot P^{ijk}\cdot(\mathrm{id}\otimes\Delta)(F^{-1})\cdot(F^{23})^{-1}\cdot\tilde{\phi}\cdot P^{ijk}\\ &= \tilde{G}_{i(jk)}\cdot\tilde{\phi}.\\ \tilde{G}_{(ij)k} &= G_{(ji)k}\cdot\mathcal{R}^{ij}\cdot P^{ijk}\\ &= G_{(ji)k}\cdot P^{ijk}\cdot\mathcal{R}\cdot(\Delta\otimes\mathrm{id})(F^{-1})\cdot(F^{12})^{-1}\cdot P^{ijk}\\ &= G_{(ji)k}\cdot P^{ijk}\cdot(\Delta'\otimes\mathrm{id})(F^{-1})\cdot\mathcal{R}\cdot(F^{12})^{-1}\cdot P^{ijk}\\ &= G_{(ji)k}\cdot P^{ijk}\cdot(\Delta'\otimes\mathrm{id})(F^{-1})\cdot(F^{21})^{-1}\cdot\tilde{\mathcal{R}}\cdot P^{ijk}\\ &= G_{(ji)k}\cdot P^{jik}\cdot(\Delta\otimes\mathrm{id})(F^{-1})\cdot(F^{12})^{-1}\cdot P^{jik}\cdot\tilde{\mathcal{R}}^{ij}\\ &= \tilde{G}_{(ji)k}\cdot\tilde{\mathcal{R}}^{ij}.\\ \tilde{G}_{(ij)k} &= G_{k(ij)}\cdot P^{ijk}\cdot(\Delta\otimes\mathrm{id})(\mathcal{R})\cdot(\Delta\otimes\mathrm{id})(F^{-1})\cdot(F^{12})^{-1}\cdot P^{ijk}\end{aligned}$$

$$
\begin{aligned}
&= G_{k(ij)} \cdot P^{ijk} \cdot (\Delta \otimes \mathrm{id})(\mathcal{R}F^{-1}) \cdot (F^{12})^{-1} \cdot P^{ijk} \\
&= G_{k(ij)} \cdot P^{ijk} \cdot (\Delta \otimes \mathrm{id})((F^{21})^{-1}) \cdot (\Delta \otimes \mathrm{id})(\tilde{\mathcal{R}}) \cdot (F^{12})^{-1} \cdot P^{ijk} \\
&= G_{k(ij)} \cdot P^{ijk} \cdot (\Delta \otimes \mathrm{id})((F^{21})^{-1}) \cdot (F^{12})^{-1} \cdot (\tilde{\Delta} \otimes \mathrm{id})(\tilde{\mathcal{R}}) \cdot P^{ijk} \\
&= G_{k(ij)} \cdot P^{kij} \cdot (\mathrm{id} \otimes \Delta)(F^{-1}) \cdot (F^{23})^{-1} \cdot P^{kij} \cdot P^{ijk} \cdot (\tilde{\Delta} \otimes \mathrm{id})(\tilde{\mathcal{R}}) \cdot P^{ijk} \\
&= \tilde{G}_{k(ij)} \cdot P^{ijk} \cdot (\tilde{\Delta} \otimes \mathrm{id})(\tilde{\mathcal{R}}) \cdot P^{ijk}
\end{aligned}
$$

If one now chooses F such that (4.2.14) holds, one sees that indeed all linear relations between the different solutions G are given in terms of the $\mathcal{U}_q(sl(2))$ R-matrix. Especially one has $\tilde{G}_{(ij)k} = \tilde{G}_{i(jk)}$!

2.2.9 What I am really interested in is the case that G takes values in the tensor product of irreducible representations of $\mathfrak{g}$. I will therefore consider the case that

$$G_{i(jk)}, G_{(ji)k} \in \rho_k(\mathcal{U}_\mathfrak{g}) \otimes \rho_j(\mathcal{U}_\mathfrak{g}) \otimes \rho_i(\mathcal{U}_\mathfrak{g}) \subset \mathrm{End}(V_k \otimes V_j \otimes V_i), \tag{4.2.17}$$

where ρ_i, $i = 1,2,3$ are representations of $\mathcal{U}_\mathfrak{g}$ on the vector spaces V_i^T, the dual of a spin J_i representation V_i. Introduce the abbreviation $V_{ijk} := V_{J_k} \otimes V_{J_j} \otimes V_{J_i}$ Then

$$G_{i(jk)}, G_{(ij)k} \in \mathrm{End}(V_{ijk}^T) \simeq V_{ijk}^T \otimes V_{ijk}^{TT} \tag{4.2.18}$$

The representation operators of the elements t^{ij} appearing in the KZ equation will be considered to act on G from the left, the monodromies from the right. However, under the identification $V_{ijk}^{TT} \simeq V_{ijk}$ any right action on V_{ijk}^{TT} is equivalent to a left action on V_{ijk}. (This is seen as follows: Let $<\phi^T, \psi>$, $\psi \in V$, $\phi \in V^T$ be the dual pairing between a vector space V and its dual V^T. Then one has for any $\rho^T \in \mathrm{End}(V^T)$ that $<\phi^{TT} \cdot \rho^T, \psi^T> \equiv <\phi^{TT}, \rho^T \cdot \psi^T> = <\rho^T \cdot \psi^T, \phi> = <\psi^T, \rho \cdot \phi>$.)

2.2.10 Let me now introduce a Clebsch-Gordan basis $\left(e_m^{J_{ij}J}(J_k|J_jJ_i)\right)^T$ for V_{ijk}^T, a q-Clebsch-Gordan basis $\left(e_m^{J_{ij}J}(J_k|J_jJ_i)\right)_q$ for V_{ijk} and define coefficient functions $\tilde{G}_{(ij)k}^{J_{ij}J'_{ij}}\left[\begin{smallmatrix} J_k & J_j \\ J & J_i \end{smallmatrix}\right](z_1,z_2,z_3)$ by

$$\tilde{G}_{(ij)k} = \sum_{J,m,J_{ij},J'_{ij}} \left(e_m^{J_{ij}J}(J_k|J_jJ_i)\right)^T \otimes \left(e_m^{J'_{ij}J}(J_k|J_jJ_i)\right)_q \tilde{G}_{(ij)k}^{J_{ij},J'_{ij}}\left[\begin{smallmatrix} J_k & J_j \\ J & J_i \end{smallmatrix}\right] \tag{4.2.19}$$

$$\tilde{G}_{i(jk)} = \sum_{J,m,J_{jk},J'_{jk}} \left(e_m^{J_{jk}J}(J_kJ_j|J_i)\right)^T \otimes \left(e_m^{J'_{jk}J}(J_kJ_j|J_i)\right)_q \tilde{G}_{i(jk)}^{J_{jk},J'_{jk}}\left[\begin{smallmatrix} J_k & J_j \\ J & J_i \end{smallmatrix}\right] \tag{4.2.20}$$

Now it is easy to see that the functions $\tilde{G}_{(ij)k}^{J_{ij}J'_{ij}}$ do indeed have the required fusion and braiding matrices (requirements no. 4 and 5):

First, by using equation (6.1.21) of appendix A one finds the following braid relations $((z_i, z_j), (z_i, z_k) \in \mathbb{C}_+^2)$:

$$\tilde{G}_{(ij)k} = e^{i\pi\tilde{h}(J_{ij}(J_{ij}+1) - J_i(J_i+1) - J_j(J_j+1))}\tilde{G}_{(ji)k} \tag{4.2.21}$$

$$\tilde{G}_{(ij)k} = e^{i\pi\tilde{h}(J(J+1) - J_k(J_k+1) - J_{ij}(J_{ij}+1))}\tilde{G}_{k(ij)}, \tag{4.2.22}$$

One therefore has to identify $\tilde{h} = \gamma$.
Second, the relation between $\tilde{G}_{(ij)k}$ and $\tilde{G}_{i(jk)}$ is given by

$$\tilde{G}_{(ij)k}^{J_{ij}} = \sum_{J_{ij},J'_{ij}} \left\{\begin{array}{cc|c} J_k & J_j & J_{jk} \\ J_i & J & J_{ij} \end{array}\right\} \tilde{G}_{i(jk)}^{J_{jk},J'_{jk}} \left\{\begin{array}{cc|c} J_i & J_j & J'_{ij} \\ J_k & J & J'_{jk} \end{array}\right\}_q . \tag{4.2.23}$$

Equations (4.2.21),(4.2.22) hold since the action of the $\mathcal{U}_q(sl(2))$ R-matrix is represented diagonally in the q-Clebsch-Gordan basis (see appendix A), whereas equation (4.2.23) just expresses equality of $\tilde{G}_{(ij)k}$ and $\tilde{G}_{i(jk)}$ and the definition of the Racah-Wigner coefficients (ordinary and q-deformed).

2.2.11 The Riemann problem of finding functions with the required monodromies has thus been solved. It remains to satisfy requirement no. 3 on the asymptotics.

2.3 Asymptotics

In order to determine the asymptotics near the hyperplanes $z_i = z_j$ consider the case that $(ijk) = (123)$. Because of eqn. (4.2.8) it suffices to consider the case that $z_1 = 0, z_2 \equiv z, z_3 = 1$ and to find the asymptotics of solutions to (4.2.9) at $z = 0, 1, \infty$. It is convenient to rewrite (4.2.9) as an equation for the coefficient functions $F^{J_{21}J'_{21}}(z) \equiv \tilde{G}_{(12)3}^{J_{21}J'_{21}}\left[\begin{smallmatrix} J_3 & J_2 \\ J & J_1 \end{smallmatrix}\right](0,z,1)$ resp. $F^{J_{32}J'_{32}}(z) \equiv \tilde{G}_{1(23)}^{J_{32}J'_{32}}\left[\begin{smallmatrix} J_3 & J_2 \\ J & J_1 \end{smallmatrix}\right](0,z,1)$. This is done by noting that t^{21} and t^{32} are diagonal in the bases $e_m^{J_{12}J}$ and $e_m^{J_{23}J}$ respectively

$$
\begin{aligned}
t^{21}e_m^{J_{21}J}(J_3|J_2J_1) &= (J_{21}(J_{21}+1) - J_1(J_1+1) - J_2(J_2+1))e_m^{J_{21}J}(J_3|J_2J_1) \\
t^{32}e_m^{J_{32}J}(J_3J_2|J_1) &= (J_{32}(J_{32}+1) - J_2(J_2+1) - J_3(J_3+1))e_m^{J_{32}J}(J_3J_2|J_1),
\end{aligned}
$$

whereas the representation of t^{32} in the basis that diagonalizes t^{21}, $t^{32}e_m^{J_{21}J} = \sum_{J'_{21}} (t_<^{32})_{J_{21}J'_{21}} e_m^{J'_{21}J}$, may be expressed in terms of *ordinary* 6-j symbols:

$$(t_<^{32})_{J_{21}J'_{21}} = \sum_{J_{32}} \left\{\begin{array}{cc|c} J_1 & J_2 & J_{21} \\ J_3 & J & J_{32} \end{array}\right\}(J_{32}(J_{32}+1) - J_2(J_2+1) - J_3(J_3+1))\left\{\begin{array}{cc|c} J_3 & J_2 & J_{32} \\ J_1 & J & J'_{21} \end{array}\right\}. \tag{4.2.24}$$

The coefficient functions $F_<^{J_{21}}(z)$ therefore have to be solutions of

$$\frac{dF_<^{J_{21}}}{dz} = \tilde{h} \sum_{J'_{21}} \left(\frac{\tau^{21}}{z} \delta_{J'_{21}J_{21}} + \frac{(t_<^{32})_{J'_{21}J_{21}}}{z-1} \right) F_<^{J'_{21}}, \tag{4.2.25}$$

where $\tau^{21} = J_{21}(J_{21}+1) - J_1(J_1+1) - J_2(J_2+1)$.

2.3.1 An important property of the matrix $(t_<^{32})_{J'_{21}J_{21}}$ has been noted in [ZF][CF]: It is tridiagonal, i.e. the only nonvanishing matrix elements are $(t_<^{32})_{J-1,J}$, $(t_<^{32})_{J,J}$ and $(t_<^{32})_{J,J+1}$. Moreover, these matrix elements are indeed nonzero, as can be seen from eqn. (B7) of [CF]. I will use this property without proof.
Consider therefore the differential equation

$$\frac{dF}{dz} = \left(\frac{P}{z} + \frac{Q}{z-1} \right) F, \tag{4.2.26}$$

where F, P, Q are $n \times n$ matrices, $P = \mathrm{diag}(d_1, \ldots, d_n)$ and Q is tridiagonal. I want to demonstrate that in the basis where the monodromies around 0 are diagonalized, the matrix elements of F have the following asymptotics:

$$F_{ij} = z^{d_j+|i-j|}\mathcal{O}(1). \tag{4.2.27}$$

For its proof, make the ansatz $F = g \cdot z^P$ and construct an analytic matrix g with $g(0) = \mathrm{id}$ as power series solution of

$$\frac{dg}{dz} = \frac{1}{z}[P,g] + \frac{Q}{z-1} g. \tag{4.2.28}$$

By expanding $g = \sum_{r=0} z^r g_r$, $1/(1-z) = \sum_{r=0} z^r$ this is rewritten as the following recurrence relation for the coefficients g_r:

$$r g_r = [P, g_r] - Q \sum_{s=0}^{r-1} g_s. \tag{4.2.29}$$

From this recurrence relation one may read off that the matrix elements $(g_r)_{ij}$ can be nonvanishing only if $|i-j| \le r$. Moreover, $(g_r)_{ij} \neq 0$ for $|i-j| = r$. The functions F_{ij} do therefore indeed have asymptotics given in (4.2.27). The leading asymptotics of the functions F may then conveniently be expressed in terms of the conformal dimensions of the WZW-model: Let $\Delta'(J) = \tilde{h} J(J+1)$,

$$\Delta'_{ij} = \Delta'(J_{ij}) - \Delta'(J_i) - \Delta'(J_j) \qquad \Delta'_{(ij)k} = \Delta'(J) - \Delta'(J_{ij}) - \Delta'(J_k).$$

The behaviour near $0, 1, \infty$ may then be represented as

$$\begin{aligned} F_{(0)}^{J_{12}J'_{12}}(z) &= \phi_{(0)}^{J_{12}J'_{12}}(z) z^{\Delta'_{12}+|J_{12}-J'_{12}|} && \text{near} \quad z=0 \\ F_{(1)}^{J_{23}J'_{23}}(z) &= \phi_{(1)}^{J_{23}J'_{23}}(z)(z-1)^{\Delta'_{23}+|J_{23}-J'_{23}|} && \text{near} \quad z=1 \\ F_{(\infty)}^{J_{13}J'_{13}}(z) &= \phi_{(\infty)}^{J_{13}J'_{13}}(z) z^{\Delta'_{(13)2}+|J_{13}-J'_{13}|} && \text{near} \quad z=\infty, \end{aligned} \tag{4.2.30}$$

where $\phi_{(i)}^{kl}(z_i) = \mathcal{O}(1)$ for $i = 0, 1, \infty$.

2.4 Fixing the asymptotics, uniqueness

It follows from an old result of Riemann that any vector $\mathcal{G}_{(ij)k}^{J'_{ij}}$ of functions that has the same monodromies as the $\tilde{G}_{(ij)k}^{J_{ij}J'_{ij}}$ must be of the following form

$$\mathcal{G}_{(ij)k}^{J'_{ij}} = \sum_{J_{ij}} H_{(ij)k}^{J_{ij}} \tilde{G}_{(ij)k}^{J_{ij}J'_{ij}}, \tag{4.2.31}$$

where $H_{(ij)k}^{J_{ij}}(z_i, z_j, z_k)$ is a *rational* function of the differences of its arguments. A corresponding formula holds for $i(jk)$ instead of $(ij)k$. In the present situation this result may be proved as follows:
Again one may use (4.2.8) to reduce to the case $z_1 = 0, z_2 \equiv z, z_3 = 1$. Assume given a vector of functions G and a matrix of functions F with the same monodromies around their regular singular points $0, 1, \infty$. Further assume that F is solution to an ordinary differential equation of the form $\frac{dF}{dz} = A(z)F$, where $A(z)$ has only poles of first order at $0, 1, \infty$. Since $\det F(z) = exp(\int^z \mathrm{tr} \ln A) \det F(z_0)$ one finds that F will be invertible everywhere on the universal cover of $\mathbb{P}^1/\{0, 1, \infty\}$ if it is invertible somewhere. $F(z)$ is invertible for z close enough to 0. One may therefore consider $H = GF^{-1}$. This function is analytic and single valued on $\mathbb{P}^1/\{0, 1, \infty\}$ by the assumption on the monodromies of F, G. H can at most have poles at $0, 1, \infty$. It must therefore be a rational function of z.

2.4.1 In the present subsection I will show that there exist rational functions $H_\Gamma^{J_{ij}}$ with the right asymptotics for (4.2.31) to hold. Furthermore, it turns out that H and consequently $\mathcal{G}$ are unique under the given assumptions.

2.4.2 From the asymptotics of F determined in the previous subsection and the required asymptotics of $\mathcal{G}$ it follows that the functions H should have asympotics

$$H_{(ij)k}^{J_{ij}} \approx (z_j - z_i)^{J_i+J_j-J_{ij}} (z_k - z_i)^{J_k+J_{ij}-J} \quad \text{for} \quad z_j - z_i \ll z_k - z_i \tag{4.2.32}$$

$$H_{i(jk)}^{J_{jk}} \approx (z_k - z_j)^{J_k+J_j-J_{jk}} (z_k - z_i)^{J_i+J_{jk}-J} \quad \text{for} \quad z_k - z_j \ll z_k - z_i. \tag{4.2.33}$$

Because of (4.2.23) one must have the following transformation between the functions $H^{J_{ij}}_{(ij)k}$ and $H^{J_{jk}}_{i(jk)}$:

$$H^{J_{ij}}_{(ij)k}(z_i,z_j,z_k) = \sum_{J_{jk}} H^{J_{jk}}_{i(jk)}(z_i,z_j,z_k)\left\{ \begin{smallmatrix} J_i & J_j \\ J_k & J \end{smallmatrix} \middle| \begin{smallmatrix} J_{ij} \\ J_{jk} \end{smallmatrix} \right\}. \tag{4.2.34}$$

2.4.3 Such functions can be found by considering realizations of sl_2 on polynomials: On the vector space $\mathcal{P}_j$ spanned by $\{1, z, \ldots, z^{2j}\}$ one may introduce a sl_2-representation by

$$H = 2z\partial_z - 2j, \qquad F = -\partial_z, \qquad E = z^2\partial - 2jz. \tag{4.2.35}$$

A basis for $\mathcal{P}_j$ that is orthonormal w.r.t. the invariant bilinear form is given by the vectors

$$e^j_m(z) := \binom{2j}{j+m}^{\frac{1}{2}} z^{j+m}. \tag{4.2.36}$$

Tensor products of representations may be realized by taking polynomials in $z_1, z_2, \ldots$ and defining

$$H = \sum_i (z_i\partial_i - 2j_i), \quad F = -\sum_i \partial_i, \quad E = \sum_i (z_i^2\partial_i - 2j_iz_i). \tag{4.2.37}$$

2.4.4 The functions H will then be defined as

$$\begin{aligned} H^{J_{ij}}_{(ij)k}(z_i,z_j,z_k) &= \sum_{m_i+m_j+m_k=-J} \left(\begin{smallmatrix} J_i & J_j \\ m_i & m_j \end{smallmatrix} \middle| J_{ij} \right) \left(\begin{smallmatrix} J_{ij} & J_k \\ m_i+m_j & m_k \end{smallmatrix} \middle| J \right) e^{J_k}_{m_k}(z_k) e^{J_j}_{m_j}(z_j) e^{J_i}_{m_i}(z_i) \\ H^{J_{jk}}_{i(jk)}(z_i,z_j,z_k) &= \sum_{m_i+m_j+m_k=-J} \left(\begin{smallmatrix} J_j & J_k \\ m_j & m_k \end{smallmatrix} \middle| J_{jk} \right) \left(\begin{smallmatrix} J_i & J_{jk} \\ m_i & m_j+m_k \end{smallmatrix} \middle| J \right) e^{J_k}_{m_k}(z_k) e^{J_j}_{m_j}(z_j) e^{J_i}_{m_i}(z_i). \end{aligned}$$

Equation (4.2.34) holds by definition of the classical Racah-Wigner coefficients. In order to prove (4.2.32) note that it follows from the definition of the Clebsch-Gordan coefficients that the $H^{J_{ij}}_{(ij)k}$ satisfy the following differential equations

$$fH^{J_{ij}}_{(ij)k} = 0 \qquad\qquad hH^{J_{ij}}_{(ij)k} = -2JH^{J_{ij}}_{(ij)k}. \tag{4.2.38}$$

Consequently $H^{J_{ij}}_{(ij)k}$ must be of the form

$$H^{J_{ij}}_{(ij)k} = (z_k - z_i)^{J_i+J_j+J_k-J} g^{J_{ij}}_{(ij)k}\left(\frac{z_j - z_i}{z_k - z_i}\right). \tag{4.2.39}$$

In order to find the asymptotics of $g^{J_{ij}}_{(ij)k}(z)$ as $z \to 0$ note that

$$g^{J_{ij}}_{(ij)k}(z) = H^{J_{ij}}_{(ij)k}(0,z,1) = \sum_{m_j} \left(\begin{smallmatrix} J_i & J_j \\ -J_i & m_j \end{smallmatrix} \middle| J_{ij} \right) \left(\begin{smallmatrix} J_{ij} & J_k \\ m_j-J_i & J_i-J-m_j \end{smallmatrix} \middle| J \right) e^{J_k}_{J_i-J-m_j}(1) e^{J_j}_{m_j}(z),$$

where the lower bound of the summation is given by $m_j \geq J_i - J_{ij}$. One therefore has

$$g^{J_{ij}}_{(ij)k}(z) \approx z^{J_i+J_j-J_{ij}} \quad \text{for} \quad z \approx 0. \tag{4.2.40}$$

Similarly one verifies (4.2.33).

2.4.5 For the discussion of uniqueness it will again suffice to consider the case $z_1 = 0$ and $z_3 = 1$, since the general case follows from projective invariance. Assume that there is a second solution $k^{J_{ij}}_{(ij)k}(z)$. There must be a matrix M with elements $M^{J_{ij}J'_{ij}}(z)$ such that

$$k^{J_{ij}}_{(ij)k}(z) = \sum_{J'_{ij}} M^{J_{ij}J'_{ij}}(z) g^{J'_{ij}}_{(ij)k}(z). \tag{4.2.41}$$

The matrix M has to be holomorphic on the entire complex plane and bounded as $z \to \infty$. By Liouville's theorem, it is constant. By considering equation (4.2.41) i.e. in the limit $z \to 0$ one sees that it must be diagonal in the corresponding basis. The transformations to the other bases are of the form $M' = R^{-1}MR$, where the matrix elements of R are given in terms of Racah-Wigner coefficients. Requiring M' to be diagonal as well will further restrict M. Although I cannot prove $M = \mathrm{id}$ in the moment, it seems that a possible leftover freedom could at most correspond to a change of normalization.

2.5 Discussion

An interesting aspect of these results is that they provide an explicit connection between WZNW and minimal model conformal blocks. This should be interpreted by quantum Hamiltonian reduction. In fact, it seems to me that quantum Hamiltonian reduction is not yet fully understood on the operator level. In this context, the present result may be an important ingredient for an improved understanding.

In the present case it is unnecessarily heavy machinery to use Drinfeld's theorem to find correlation functions with monodromy data related to $\mathcal{U}_q(sl(2))$. However, my main motivation was to explore ways to characterize the conformal blocks that still work even if no free field representations are available. From this point of view it is desirable to have as powerful tools as possible. I will speculate on possible extensions and applications in the conclusions.

V: Concluding Remarks

SECTION V:

CONCLUDING REMARKS

I would like to conclude by pointing out some aspects of the present work that may have other applications or lead to further developments:

A new operator formalism was developed that is suitable for construction of Liouville theory on a space of states consisting of irreducible modules. It is entirely based on exploiting conformal covariance, which led to a unique construction of chiral vertex operators and fusion products.
One may view this formalism as a kind of vertex operator calculus that is well-suited for dealing with interacting conformal field theories. The formalism is also close in spirit to string field theory, since the notion of a fusion product corresponds directly to the three-string vertex. To my opinion, this kind of approach has the follwing advantages:

- It combines the advantages of the geometric approach to conformal field theory due to BPZ, namely the simple rules for moving Virasoro-generators from one insertion point to the others (conformal Ward identities) with the virtues of a true algebraic operator formalism (manifest factorization, i.e. sum over intermediate states).
- Arbitrary ways of combining chiral vertex operators (or sewing three-punctured spheres) are described on a uniform footing simply as different orders of taking the fusion product. The duality operations of fusion and braiding then get a nice interpretation as commutativity and associativity operations of the fusion product. This sheds some new light on the role of the quantum group structure. On the mathematical side, the fusion product formalism should be well suited to uncover the category-theoretic aspects of conformal field theory, as done for irrational WZNW-models in [KL].

The formalism may well be profitably extended and applied to more general conformal field theories.

In view of my original aim to construct correlation functions for 2D gravity theories it seems that I had more "spin-off" than progress. However, the approach put forward in the present work at least suggests some new possible approaches:
First, one at least knows that the chiral vertex operators one needs for the gravitationally dressing of the tachyons exist and are uniquely determined by conformal invariance. This leads to a unique definition of their conformal blocks as formal power series. I do not know whether these power series converge.
A possible approach towards a better characterization of the corresponding conformal blocks may start from an extrapolation of the quantum group structure to the case of negative spins. This will involve infinite-dimensional, nonunitary representations of $\mathcal{U}_q(sl(2))$. One would guess that fusion and braiding matrices of the relevant chiral vertex operators are given in terms of suitable generalizations of q-Racah-Wigner coefficients.[1]
However, it seems that the fusion rules one needs require consideration of lowest weight $\mathcal{U}_q(sl(2))$-representations as well as highest weight ones. Maybe there is interesting additional structure for the cases where q is a root of unity, which is the case relevant for minimal models coupled to 2D gravity. A Virasoro representation theoretic hint in this direction is the remarkable symmetry between the structure of imbeddings of a minimal model Verma module to that of its gravitational dressing (see III.1.1).
After having obtained a reasonable guess on the fusion and braiding matrices one may try to find a characterization of the conformal blocks in terms of a suitable generalization of the Knizhnik-Zamolodchikov equations along the lines of this work. However, one will probably have considerable technical problems, since the necessary systems of differential equations will presumably be infinite dimensional.

Drinfeld's theorem was proposed as kind of "universal" means to establish Knizhnik-Zamolodchikov type equations as solution to the Riemann problem of finding analytic functions with monodromy data associated with quantum group structures. This idea was used to establish the connection between minimal model conformal blocks and solutions to the Knizhnik-Zamolodchikov equations, i.e. WZNW conformal blocks. An interesting application may be the characterization of conformal blocks for W-algebra minimal models, on which not much seems to be known.
It is tempting to speculate on the generality of this approach: Given that

- the most general rational or quasirational CFT has conformal blocks, the monodromy data of which may be described in terms of fusion and braiding matrices

[1] Progress on generalization of q-Racah-Wigner coefficients has been made by Gervais and collaborators by imposing the polynomial equations. These results do not yet cover the cases relevant for 2D gravity. Moreover, it seems that the $\mathcal{U}_q(sl(2))$-representation theoretic aspects still remain to be worked out.

V: Concluding Remarks

satisfying the Moore-Seiberg polynomial equations (for evidence see [MS][FFK]),

- there is a canonical way to construct a quasitriangular quasi Hopf algebra to any set of solutions of the polynomial equations ([VS] contains arguments in this direction), and
- Drinfeld's uniqueness theorem states the twist-equivalence of any quasitriangular quasi Hopf algebra to one which is defined from the monodromy data of Knizhnik-Zamolodchikov type equations,

one may speculate that conformal blocks of the most general rational or quasirational CFT may be characterized in terms of solutions to KZ type equations. This fits to the conjecture of Moore and Seiberg that any rational or quasirational CFT can be obtained from a WZNW model by forming suitable cosets, quantum Hamiltonian reduction etc.. This is of course a rather daring conjecture in view of the following problems:

- First, I do not know how well established the first two points are.
- Drinfeld's theorem is not immediately applicable to the case of rational conformal field theories, precisely because it does not consider the truncation to weak quasi-triangular quasi Hopf algebras. One would need a general theory of the algebraic conditions that single out a subbundle of the vector bundle of solutions to KZ-eqns. such that the monodromy data of the subbundle are given by commutativity and associativity isomorphisms of a weak quasitriangular quasi Hopf algebra.
- In Drinfeld's work the deformation parameter h, $q = e^h$ is treated as a formal variable. This should be no problem as long as it is applied to finite dimensional representations for which both the solutions of the KZ equations and the fusion and braiding matrices of the quasitriangular quasi Hopf algebras are analytic in h. Perhaps there are sensible CFT for which this is not the case.
- Most importantly, it is not clear whether there really is a conformal field theory to any collection of would-be conformal blocks, the fusion and braiding matrices of which satisfy the polynomial equations. At least, one will have to consider a similiar problem on the torus, since consistency on the torus provides further restrictions. One would need a canonical way to construct a chiral algebra that produces these functions as correlators of the corresponding chiral vertex operators.

Nevertheless, even if such an approach does not work in the most general case, it seems reasonable that methods similiar to those presented here can be used to characterize the conformal blocks for a rather large class of conformal field theories.

Section VI:

Appendix A

1. Quantum group $U_q(sl(2))$

In this appendix I want to present the results on the representation theory of $\mathcal{U}_q(sl(2))$ that were used in the main text. In particular, I will prove the identities between Clebsch-Gordan coefficients, the universal R-matrix and the Racah-Wigner coefficients that are used in the main text. Among these is the orthogonality relation (6.1.39) that may be used to prove the locality of Liouville exponentials. In view of the difficulties to prove locality in some previous approaches to Liouville theory it is nice to notice that one does not need to know the explicit expressions to derive the necessary identities: These are rather simple consequences of the defining relations of the quantum group.

I shall use the following notations:

$$[x] := \frac{q^x - q^{-x}}{q - q^{-1}} \qquad [n]! := [n][n-1]\dots[1] \tag{6.1.1}$$

$$[x|x+n] := [x][x+1]\dots[x+n], \tag{6.1.2}$$

where $n \in \mathbb{N}$, $x \in \mathbb{R}$.

1.1 Algebra and representations

The algebra of $\mathcal{U}_q(sl(2))$ is a one parameter deformation of the universal enveloping algebra $\mathcal{U}(sl(2))$:

$$[H,E] = 2E, \qquad [H,F] = -2F \qquad [E,F] = \frac{q^H - q^{-H}}{q - q^{-1}}. \tag{6.1.3}$$

It is sometimes convenient to write the deformation parameter q as $q = e^{h/2}$, since the classical limit then corresponds to $h \to 0$. For conformal field theory applications I will be mainly interested in the case $|q| = 1$, i.e. h purely imaginary.

As usual one may define Verma modules V_n by acting with the generator F on a highest weight vector v_n that satisfies $Ev_n = 0$, $Hv_n = (n-1)v_n$. It has a basis consisting of the elements $v_n^r = F^r v_n$, where $Ev_n^r = [r][n-r]v_n^{r-1}$ and $Hv_n^r = (n-1-2r)v_n^r$.

The natural invariant bilinear form $(.,.)$ on V_n, called Shapovalov form, is uniquely defined by the properties $(v_n, v_n) = 1$, $(Fx, y) = (x, Ey)$. Explicitely: $(v_n^r, v_n^{r'}) = \delta^{rr'}[r]![n-r|n-1]$. In the case of irrational q and $n \in \mathbb{N}$ the bilinear form $(.,.)$ has a nontrivial kernel generated by a unique nullvector. The submodule obtained by forming $H_j := V_n/Ker(.,.)$, $j = \frac{n-1}{2}$ is irreducible. A basis for H_j is given by the vectors

$$e_m^j := ([j-m]![j+m+1|2j])^{-\frac{1}{2}} v_{2j+1}^{j-m}, \tag{6.1.4}$$

where $m = -j \dots j$ if $2j \in \mathbb{N}$ and $m = -\infty \dots j$ otherwise. The action of E, F, H on e_m^j is the following:

$$\begin{aligned} Ee_m^j &= d_m^{j+} e_{m+1}^j := ([j-m][j+m+1])^{1/2} e_{m+1}^j \\ Fe_m^j &= d_m^{j-} e_{m-1}^j := ([j+m][j-m+1])^{1/2} e_{m-1}^j, \end{aligned} \qquad He_m^j = 2me_m^j. \tag{6.1.5}$$

If $q^p = -1$ the representation theory becomes more complicated essentially due to the fact that $[p] = 0$. There is a subset of "physical" representations with basis $\{e_j^m\}$ and $\mathcal{U}_q(sl(2))$-action given by (6.1.5) for $j \leq \frac{p-2}{2}$. Other representations are in general more complicated, such as indecomposable but not irreducible representations.

1.2 Tensor product of representations

The tensor product of representations of $\mathcal{U}_q(sl(2))$ will be a representation itself provided the $\mathcal{U}_q(sl(2))$ action on tensor products is defined by means of the following coproduct:

$$\Delta(E) = E \otimes q^{-\frac{H}{2}} + q^{\frac{H}{2}} \otimes E \qquad \Delta(F) = F \otimes q^{-\frac{H}{2}} + q^{\frac{H}{2}} \otimes F \qquad \Delta(H) = H \otimes 1 + 1 \otimes H$$

In the case $q \in \mathbb{R}$ it is straightforward to apply the usual Clebsch-Gordan algorithm to decompose a tensor product into irreducible representations. One obtains a basis $e_m^j(j_1 j_2)$, orthonormal with respect to the obvious extension of the bilinear form $(.,.)$ to tensor products. The Clebsch-Gordan coefficients are defined as the coefficients of the expansion of $e_m^j(j_1 j_2)$ in the basis $e_{m_1}^{j_1} \otimes e_{m_2}^{j_2}$:

$$e_m^j(j_1 j_2) = \sum_{m_1+m_2=m} \left(\begin{smallmatrix} j_1 & j_2 \\ m_1 & m_2 \end{smallmatrix} \middle| j \right)_q e_{m_1}^{j_1} \otimes e_{m_2}^{j_2} \tag{6.1.6}$$

The properties of orthogonality and completeness of the basis $e^j_m(j_1 j_2)$ are reflected by the following identities satisfied by the Clebsch-Gordan (CG) coefficients:

$$\sum_{m_1+m_2=m} \begin{pmatrix} j_1 & j_2 \\ m_1 & m_2 \end{pmatrix}\!\Big|j\Big)_q \begin{pmatrix} j_1 & j_2 \\ m_1 & m_2 \end{pmatrix}\!\Big|j'\Big)_q = \delta_{jj'} \tag{6.1.7}$$

$$\sum_j \left(\begin{matrix} j_1 & j_2 \\ m_1 & m_2 \end{matrix}\Big|j\right)_q \left(\begin{matrix} j_1 & j_2 \\ m_1' & m_2' \end{matrix}\Big|j\right)_q \delta_{m_1+m_2, m_1'+m_2'} = \delta_{m_1 m_1'} \delta_{m_2 m_2'} \tag{6.1.8}$$

The fact that $e^j_m(j_1 j_2)$ transforms according to (6.1.5) leads to the following relations on the CG coefficients:

$$d^{j\pm}_{m_1+m_2} \left(\begin{matrix} j_1 & j_2 \\ m_1 & m_2 \end{matrix}\Big|j\right)_q = q^{m_1} d^{j_2\pm}_{m_2\mp1} \left(\begin{matrix} j_1 & j_2 \\ m_1 & m_2\mp1 \end{matrix}\Big|j\right)_q + q^{-m_2} d^{j_1\pm}_{m_1\mp1} \left(\begin{matrix} j_1 & j_2 \\ m_1\mp1 & m_2 \end{matrix}\Big|j\right)_q \tag{6.1.9}$$

A property that will be needed later is the behaviour of the CG-coefficients under exchanging $j_1 \leftrightarrow j_2$, $m_1 \leftrightarrow m_2$:

$$\left(\begin{matrix} j_1 & j_2 \\ m_1 & m_2 \end{matrix}\Big|j\right)_q = (-1)^{j_1+j_2-j} \left(\begin{matrix} j_2 & j_1 \\ m_2 & m_1 \end{matrix}\Big|j\right)_{q^{-1}} \tag{6.1.10}$$

This equation is proved by observing that the recursion relations (6.1.9) for $\left(\begin{matrix} j_1 & j_2 \\ m_1 & m_2 \end{matrix}\Big|j\right)_q$ and $\left(\begin{matrix} j_2 & j_1 \\ m_2 & m_1 \end{matrix}\Big|j\right)_{q^{-1}}$ are identical. Since (6.1.9) and normalization with respect to (.,.) determine the CG coefficients uniquely up to sign factors one must have (6.1.10) up to a sign. This sign may be fixed by taking the limit $q \to 1$.

All these properties continue to hold if one defines the q-Clebsch-Gordan coefficients for complex q by analytic continuation.

1.3 Braiding operation

The action of an ordinary Lie algebra on a tensor product of its representations $U \otimes V$ is trivially equivalent to that on $V \otimes U$: Exchanging the tensor factors after acting on $U \otimes V$ gives the same result as first exchanging and then acting on it. For co-products such as (Section VI) the equivalence of representations $V \otimes U$ and $U \otimes V$ is more complicated: One needs a nontrivial operation $B : U \otimes V \to V \otimes U$, such that $\Delta(\xi)B = B\Delta(\xi)$, $\xi \in \mathcal{U}_q(sl(2))$ which means that $B(u \otimes v)$ transforms as $u \otimes v$ for any $u \in U, v \in V$. Write B as $B = P^{12}R$, where P^{12} exchanges the two tensor factors. The condition on R is then

$$R\Delta = \Delta' R, \tag{6.1.11}$$

where $\Delta' = P^{12}\Delta P^{12}$. One needs additional properties of R in order to find that the operation B defines a representation of the braid group on multiple tensor products of representations. Most importantly

$$(\Delta \otimes \mathrm{id})(R) = R^{13}R^{23} \qquad (6.1.12) \qquad (\mathrm{id} \otimes \Delta)(R) = R^{13}R^{12}. \tag{6.1.13}$$

By combining these equations one gets the Yang-Baxter equation:

$$R^{12}R^{13}R^{23} = R^{12}(\Delta \otimes \mathrm{id})(R) = (\Delta' \otimes \mathrm{id})(R)R^{12} = R^{23}R^{13}R^{12}, \tag{6.1.14}$$

where the last equality also follows from (6.1.12) by permuting the algebra elements in the first and second position of the triple tensor product. I will also need

$$R^{12}(\Delta \otimes \mathrm{id})(R) = R^{23}(\mathrm{id} \otimes \Delta)(R), \tag{6.1.15}$$

which is a trivial consequence of (6.1.14), (6.1.12), (6.1.13). An element with these properties was constructed in [Dr1]. Explicitely it is given by

$$R_q = \sum_{k=0}^{\infty} \frac{q^{\frac{1}{2}k(k-1)}}{[k]!}(1-q^2)^k q^{-\frac{1}{2}H\otimes H} \left(q^{-\frac{kH}{2}}E^k\right) \otimes \left(q^{\frac{kH}{2}}F^k\right) \tag{6.1.16}$$

The proof that R_q satisfies the properties (6.1.14), (6.1.12), (6.1.13) is nontrivial and may most elegantly be done by the so-called double construction, see [Dr1].

One may take the matrix elements of R_q taken in the representation $\mathcal{V}_{j_1} \otimes \mathcal{V}_{j_2}$

$$\begin{aligned}(R_q^{j_1 j_2})^{n_1 n_2}_{m_1 m_2} = \delta_{m_1+m_2, n_1+n_2} q^{-2m_1 m_2}(1-q^2)^n \frac{q^{\frac{1}{2}n(n-1)}}{[n]!} q^{-n(m_1-m_2)} \times \\ \times ([j_1+m_1-n+1|j_1+m_1][j_1-m_1+1|j_1-m_1+n])^{\frac{1}{2}} \\ \times ([j_2-m_2-n+1|j_2-m_2][j_2+m_2+1|j_2+m_2+n])^{\frac{1}{2}},\end{aligned}$$

(6.1.17)

where $n = m_1 - n_1 = n_2 - m_2$. The matrix elements of R are analytic functions of q (resp. h) as long as $q^p \neq -1$. For $q^p = -1$ they are still analytic as long as one considers only physical values of j_1, j_2, since then $n < p$.

Above conditions leave some freedom in the choice of R_q. This freedom has been fixed in the explicit expressions (6.1.16) resp. (6.1.17) such that

$$\sum_{n_1 n_2} (R_q^{j_1 j_2})^{n_1 n_2}_{m_1 m_2} (R_{q^{-1}}^{j_1 j_2})^{p_1 p_2}_{n_1 n_2} = \delta_{m_1 p_1}\delta_{m_2 p_2}, \tag{6.1.18}$$

what may be checked by explicit computation using the identity

$$\sum_{n=0}^{k} (-1)^n q^{n(k-1)} \frac{[k]!}{[n]![k-n]!} = \delta_{k,0}. \tag{6.1.19}$$

Now consider the action of the braiding operation on a Clebsch-Gordan basis. Since both $B(e^j_m(j_1j_2))$ and $e^j_m(j_2j_1)$ transform as an irreducible spin j representation they must be proportional to each other:

$$B(e^j_m(j_1j_2)) = \lambda^j_{j_1j_2}(q) e^j_m(j_2j_1). \tag{6.1.20}$$

This is equivalent to the following relations between R-matrix and Clebsch-Gordan coefficients:

$$\sum_{n_1n_2} (R_q^{j_1j_2})^{m_1m_2}_{n_1n_2} \left(\begin{smallmatrix} j_1 & j_2 \\ n_1 & n_2 \end{smallmatrix}\middle| j\right)_q = \lambda^j_{j_1j_2}(q) \left(\begin{smallmatrix} j_2 & j_1 \\ m_2 & m_1 \end{smallmatrix}\middle| j\right)_q \tag{6.1.21}$$

The eigenvalues $\lambda^j_{j_1j_2}(q)$ are explicitely given as

$$\lambda^j_{j_1j_2}(q) = (-1)^{j_1+j_2-j} q^{j_1(j_1+1)+j_2(j_2+1)-j(j+1)} \tag{6.1.22}$$

Proof ([HH]): First of all consider

$$\sum_{n_1n_2} \sum_{p_1p_2} (R_q^{j_1j_2})^{n_1n_2}_{m_1m_2} (R_{q^{-1}}^{j_1j_2})^{p_1p_2}_{n_1n_2} \left(\begin{smallmatrix} j_1 & j_2 \\ p_1 & p_2 \end{smallmatrix}\middle| j\right)_{q^{-1}}. \tag{6.1.23}$$

This may on one hand be evaluated by the use of (6.1.18), on the other hand by using (6.1.10) and (6.1.21). In this way one finds a functional equation on $\lambda^j_{j_1j_2}(q)$:

$$\lambda^j_{j_1j_2}(q)\lambda^j_{j_1j_2}(q^{-1}) = 1 \tag{6.1.24}$$

From its definition, eqn.(6.1.21) it follows that $\lambda^j_{j_1j_2}(q)$ depends analytically on h: One may solve for λ, and both CG coefficients and R matrix elements are analytic in q as long as no unphysical representations are considered in the root of unity case. Analyticity in h ensures the uniqueness of the following solution to (6.1.24):

$$\lambda^j_{j_1j_2}(q) = (-1)^{n(j_1j_2j)} q^{f(j_1j_2j)}, \tag{6.1.25}$$

where $f(j_1j_2j)$ does not depend on h and $n(j_1j_2j)$ is an integer valued function of the indicated variables. By taking the classical limit $h \to 0$ one finds that $n(j_1j_2j) = j_1 + j_2 - j$. Because of the analyticity in h one may determine $f(j_1j_2j)$ from the first order term of the expansion of (6.1.21) in powers of h:
To this end expand

$$\left(\begin{smallmatrix} j_1 & j_2 \\ m_1 & m_2 \end{smallmatrix}\middle| j\right)_q = \left(\begin{smallmatrix} j_1 & j_2 \\ m_1 & m_2 \end{smallmatrix}\middle| j\right) + h\left(\begin{smallmatrix} j_1 & j_2 \\ m_1 & m_2 \end{smallmatrix}\middle| j\right)^{(1)} + \mathcal{O}(h^2) \tag{6.1.26}$$

The first order terms of (6.1.21) are

$$\begin{aligned} &\left(\begin{smallmatrix} j_1 & j_2 \\ n_1 & n_2 \end{smallmatrix}\middle| j\right)^{(1)} - 2\left(\begin{smallmatrix} j_1 & j_2 \\ n_1+1 & n_2-1 \end{smallmatrix}\middle| j\right) d^{j_1+}_{n_1} d^{j_2-}_{n_2} - 2\left(\begin{smallmatrix} j_1 & j_2 \\ n_1 & n_2 \end{smallmatrix}\middle| j\right) n_1n_2 \\ &= (-1)^{j_1+j_2-j} \left(f(j_1j_2j)\left(\begin{smallmatrix} j_2 & j_1 \\ n_2 & n_1 \end{smallmatrix}\middle| j\right) + \left(\begin{smallmatrix} j_2 & j_1 \\ n_2 & n_1 \end{smallmatrix}\middle| j\right)^{(1)} \right). \end{aligned}$$

The $\left(\begin{smallmatrix} \cdot & \cdot \\ \cdot & \cdot \end{smallmatrix}\middle| \cdot\right)^{(1)}$ terms cancel because of the symmetry (6.1.10). By using the orthogonality relation for CG coefficients (6.1.7) in the classical limit one may solve for $f(j_1j_2j)$:

$$f(j_1j_2j) = -2\sum_{n_1n_2} \left(\begin{smallmatrix} j_1 & j_2 \\ n_1 & n_2 \end{smallmatrix}\middle| j\right) \left\{ n_1n_2\left(\begin{smallmatrix} j_1 & j_2 \\ n_1 & n_2 \end{smallmatrix}\middle| j\right) + d^{j_1+}_{n_1} d^{j_2-}_{n_2} \left(\begin{smallmatrix} j_1 & j_2 \\ n_1+1 & n_2-1 \end{smallmatrix}\middle| j\right) \right\} \tag{6.1.27}$$

Consider the second term on the right hand side: By a simple shift of summation and the identities $d^{j_1+}_{n_1-1} = d^{j_1-}_{n_1}$, $d^{j_2-}_{n_2+1} = d^{j_2+}_{n_2}$ one may rewrite it as

$$\sum_{n_1n_2} \left(\begin{smallmatrix} j_1 & j_2 \\ n_1 & n_2 \end{smallmatrix}\middle| j\right) \left\{ d^{j_1+}_{n_1} d^{j_2-}_{n_2} \left(\begin{smallmatrix} j_1 & j_2 \\ n_1+1 & n_2-1 \end{smallmatrix}\middle| j\right) + d^{j_1-}_{n_1} d^{j_2+}_{n_2} \left(\begin{smallmatrix} j_1 & j_2 \\ n_1-1 & n_2+1 \end{smallmatrix}\middle| j\right) \right\}. \tag{6.1.28}$$

Now it is useful to consider the case that $n = j$, $q = 1$: The recursion relations (6.1.9) then simplify to

$$0 = \left(\begin{smallmatrix} j_1 & j_2 \\ m_1 & m_2 \end{smallmatrix}\middle| j\right) d^{j_2+}_{m_2} + \left(\begin{smallmatrix} j_1 & j_2 \\ m_1-1 & m_2+1 \end{smallmatrix}\middle| j\right) d^{j_1+}_{m_1-1}. \tag{6.1.29}$$

With their help and again $d^{j_1+}_{n_1-1} = d^{j_1-}_{n_1}$ one finds

$$f(j_1j_2j) = \sum_{n_1+n_2=j} \left(\begin{smallmatrix} j_1 & j_2 \\ n_1 & n_2 \end{smallmatrix}\middle| j\right)^2 \left(-2n_1n_2 + (d^{j_1+}_{m_1})^2 + (d^{j_2+}_{m_2})^2\right), \tag{6.1.30}$$

and finally $f(j_1j_2j) = j_1(j_1+1) + j_2(j_2+1) - j(j+1)$. q.e.d.

1.4 Racah-Wigner coefficients

There are two natural ways to find irreducible representations in the triple tensor product of $\mathcal{U}_q(sl(2))$ representations:

$$e^{j_{23}j}_m(j_1|j_2j_3) = \sum_{m_1+m_2+m_3=m} \left(\begin{smallmatrix} j_2 & j_3 \\ m_2 & m_3 \end{smallmatrix}\middle| j_{23}\right)_q \left(\begin{smallmatrix} j_1 & j_{23} \\ m_1 & m_{23} \end{smallmatrix}\middle| j\right)_q e^{j_1}_{m_1} \otimes e^{j_2}_{m_2} \otimes e^{j_3}_{m_3} \tag{6.1.31}$$

$$e^{j_{12}j}_m(j_1j_2|j_3) = \sum_{m_1+m_2+m_3=m} \left(\begin{smallmatrix} j_1 & j_2 \\ m_1 & m_2 \end{smallmatrix}\middle| j_{12}\right)_q \left(\begin{smallmatrix} j_{12} & j_3 \\ m_{12} & m_3 \end{smallmatrix}\middle| j\right)_q e^{j_1}_{m_1} \otimes e^{j_2}_{m_2} \otimes e^{j_3}_{m_3} \tag{6.1.32}$$

Since both sets of vectors $e^{j_{23}j}_m(j_1|j_2j_3)$ and $e^{j_{12}j}_m(j_1j_2|j_3)$ form complete orthogonal bases of $\mathcal{V}_{j_1} \otimes \mathcal{V}_{j_2} \otimes \mathcal{V}_{j_3}$ they must be linearly related:

$$e^{j_{12}j}_m(j_1j_2|j_3) = \sum_{j_{23}} \left\{\begin{smallmatrix} j_1 & j_2 \\ j_3 & j \end{smallmatrix}\middle| \begin{smallmatrix} j_{12} \\ j_{23} \end{smallmatrix}\right\}_q e^{j_{23}j}_m(j_1|j_2j_3). \tag{6.1.33}$$

The coefficients $\left\{\begin{smallmatrix} \cdot & \cdot \\ \cdot & \cdot \end{smallmatrix}\middle| \begin{smallmatrix} \cdot \\ \cdot \end{smallmatrix}\right\}_q$ are called the Racah-Wigner (RW) coefficients. One may rewrite their definition in terms of the CG coefficients:

$$\left(\begin{smallmatrix} j_1 & j_2 \\ m_1 & m_2 \end{smallmatrix}\middle| j_{12}\right)_q \left(\begin{smallmatrix} j_{12} & j_3 \\ m_{12} & m_3 \end{smallmatrix}\middle| j\right)_q = \sum_{j_{23}} \left\{\begin{smallmatrix} j_1 & j_2 \\ j_3 & j \end{smallmatrix}\middle| \begin{smallmatrix} j_{12} \\ j_{23} \end{smallmatrix}\right\}_q \left(\begin{smallmatrix} j_2 & j_3 \\ m_2 & m_3 \end{smallmatrix}\middle| j_{23}\right)_q \left(\begin{smallmatrix} j_1 & j_{23} \\ m_1 & m_{23} \end{smallmatrix}\middle| j\right)_q \tag{6.1.34}$$

By using the orthogonality relation for CG coefficients, eqn. (6.1.7) one may solve this for the RW coefficients:

$$\left\{\begin{matrix} j_1 & j_2 \\ j_3 & j \end{matrix}\middle|\begin{matrix} j_{12} \\ j_{23}\end{matrix}\right\}_q = \sum_{m_1+m_2+m_3=m} \left(\begin{matrix} j_1 & j_2 \\ m_1 & m_2 \end{matrix}\middle| j_{12}\right)_q \left(\begin{matrix} j_{12} & j_3 \\ m_{12} & m_3 \end{matrix}\middle| j\right)_q \left(\begin{matrix} j_2 & j_3 \\ m_2 & m_3 \end{matrix}\middle| j_{23}\right)_q \left(\begin{matrix} j_1 & j_{23} \\ m_1 & m_{23} \end{matrix}\middle| j\right)_q .$$

In this form one may immediately read off an important symmetry property of the RW coefficients (use (6.1.10)):

$$\left(\left\{\begin{matrix} j_1 & j_2 \\ j_3 & j \end{matrix}\middle|\begin{matrix} j_{12} \\ j_{23}\end{matrix}\right\}_q\right)^* = \left\{\begin{matrix} j_1 & j_2 \\ j_3 & j \end{matrix}\middle|\begin{matrix} j_{12} \\ j_{23}\end{matrix}\right\}_{q^{-1}} = \left\{\begin{matrix} j_3 & j_2 \\ j_1 & j \end{matrix}\middle|\begin{matrix} j_{23} \\ j_{12}\end{matrix}\right\}_q , \tag{6.1.35}$$

where $*$ denotes complex conjugation. Finally I want to note two important identities satisfied by the RW coefficients. They are ($c_j := j(j+1)$):

$$\sum_{j_{12}} \left\{\begin{matrix} j_1 & j_2 \\ j_3 & j \end{matrix}\middle|\begin{matrix} j_{12} \\ j_{23}\end{matrix}\right\}_q \left\{\begin{matrix} j_3 & j_2 \\ j_1 & j \end{matrix}\middle|\begin{matrix} j'_{23} \\ j_{12}\end{matrix}\right\}_q = \delta_{j_{23}j'_{23}} \tag{6.1.36}$$

$$\begin{aligned} &\sum_{j_{23}} (-1)^{j_{23}} q^{c_{j_{23}}} \left\{\begin{matrix} j_1 & j_2 \\ j_3 & j \end{matrix}\middle|\begin{matrix} j_{12} \\ j_{23}\end{matrix}\right\}_q \left\{\begin{matrix} j_2 & j_3 \\ j_1 & j \end{matrix}\middle|\begin{matrix} j_{23} \\ j_{13}\end{matrix}\right\}_q \\ &= (-1)^{j_1+j_2+j_3+j-j_{12}-j_{13}} q^{c_{j_1}+c_{j_2}+c_{j_3}+c_j-c_{j_{12}}-c_{j_{13}}} \left\{\begin{matrix} j_2 & j_1 \\ j_3 & j \end{matrix}\middle|\begin{matrix} j_{12} \\ j_{13}\end{matrix}\right\}_q \end{aligned} \tag{6.1.37}$$

As preparation for their proof introduce the following notation for braiding operations on triple tensor products: Let $P^{(12)3}$ denote the cyclic shift, i.e. $P^{(12)3}(\xi_1 \otimes \xi_2 \otimes \xi_3) = \xi_3 \otimes \xi_1 \otimes \xi_2$, and P^{ij} the permutations of the tensor factors at ith and jth position. Define braiding operations by $B^{ij} = P^{ij} R^{ij}$, $B^{(12)3} = P^{(12)3}(\Delta \otimes \mathrm{id})(R)$ and $B^{1(23)} = P^{1(23)}(\mathrm{id} \otimes \Delta)(R)$. Equations (6.1.15) and (6.1.13) are now reexpressed as the identities

$$B^{23} B^{(12)3} = B^{12} B^{1(23)} \quad \text{and} \quad B^{1(23)} = B^{23} B^{12}. \tag{6.1.38}$$

Now all one has to do is to evaluate these identities on a Clebsch-Gordan basis $e_m^{j_{12}j}(j_1 j_2 | j_3)$ by repeatedly using the definition (6.1.33) of the RW-coefficients to express in terms of a basis, where the braiding operation is diagonalized in the sense of (6.1.21).
By combining (6.1.36) and (6.1.35) one finds

$$\sum_{j_{12}} \left\{\begin{matrix} j_1 & j_2 \\ j_3 & j \end{matrix}\middle|\begin{matrix} j_{12} \\ j_{23}\end{matrix}\right\}_q \left(\left\{\begin{matrix} j_1 & j_3 \\ j_2 & j \end{matrix}\middle|\begin{matrix} j_{12} \\ j_{23}\end{matrix}\right\}_q\right)^* = \delta_{j_{23}j'_{23}}, \tag{6.1.39}$$

which is the crucial identity for the proof of locality.
There is a further identity that is needed in the chapters on free field representations and

quantum group structure. It is found by evaluating the identity $B^{23} = (B^{12})^{-1} B^{(12)3}$ on $e_m^{j_{12}j}(j_1 j_2 | j_3)$ and using (6.1.36):

$$\begin{aligned} &\sum_{m_2 m_3} \left(\begin{matrix} j_1 & j_2 \\ m_1 & m_2 \end{matrix}\middle| j_{12}\right)_q \left(\begin{matrix} j_{12} & j_3 \\ m_1+m_2 & m_3 \end{matrix}\middle| j\right)_q (R_q^{j_2 j_3})^{n_2 n_3}_{m_2 m_3} \\ &= \sum_{j_{13}} (\lambda^{j_{13}}_{j_1 j_2})^{-1} \lambda^{j}_{j_{12} j_3} \left\{\begin{matrix} j_2 & j_1 \\ j_3 & j \end{matrix}\middle|\begin{matrix} j_{12} \\ j_{13}\end{matrix}\right\}_q \left(\begin{matrix} j_1 & j_3 \\ m_1 & n_3 \end{matrix}\middle| j_{13}\right)_q \left(\begin{matrix} j_{13} & j_2 \\ m_1+n_3 & n_2 \end{matrix}\middle| j\right)_q \end{aligned} \tag{6.1.40}$$

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