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APPLYING THE ZEL'DOVICH APPROXIMATION TO GENERAL RELATIVITY

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ABSTRACT

Starting from general relativity, we give a systematic derivation of the Zel'dovich approximation describing the nonlinear evolution of collisionless dust. We begin by evolving dust along worldlines, and we demonstrate that the Szekeres line element is an exact but apparently unstable solution of the evolution equations describing pancake collapse. Next, we solve the Einstein field equations by employing Hamilton-Jacobi techniques and a spatial gradient expansion. We give a prescription for evolving a primordial or "seed" metric up to the formation of pancakes, and demonstrate its validity by rederiving the Szekeres solution approximately at third order and exactly at fifth order in spatial gradients. Finally we show that the range of validity of the expansion can be improved quite significantly if one notes that the 3-metric must have non-negative eigenvalues. With this improvement the exact Szekeres solution is obtained after only one iteration.

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1. INTRODUCTION

The Zel'dovich 1970 approximation plays a fundamental role in describing the nonlinear evolution of collisionless dust that arises, for example, in the cold-dark-matter model (Peebles 1982, Bond and Efstathiou 1984). For irrotational dust flows, Zel'dovich was able to extrapolate the results of linear perturbation theory into the nonlinear regime. His methods were initially derived using Newtonian theory. Using Einstein's theory, we now extend these results to a general relativistic setting. In fact, we are able to compute corrections to arbitrary high order.

The Zel'dovich approximation is useful in interpreting results from red-shift surveys of galaxies. For example, redshift catalogues of galaxies determined by the Infrared Astronomical Satellite (IRAS) give the first evidence of the inflationary Universe: the Universe is at critical density, $\Omega \equiv \rho/\rho_{crit} = 1$ (Kaiser et al 1992, Dekel 1991). Relativistic linear perturbation theory is insufficient in interpreting these data since sheets and clusters have formed. N-body simulations do not possess high enough resolution to describe very deep surveys. As redshift surveys look deeper, it would be useful to employ a relativistic version of the Zel'dovich approximation. Currently, the pencil beam survey of Broadhurst et al 1990 extends to a redshift of $z \sim 0.5$ which is mildly relativistic. Deeper surveys are planned for the future.

The Zel'dovich approximation is deceptively simple. It describes a cold pressure-free fluid, "dust", which flows irrotationally. The Eulerian coordinates r^i are related to the Lagrangian coordinates x^i (which may be taken to be initial coordinates) through

$$r^i = a(t)\left[x^i + \frac{3}{2}a(t)\frac{\partial \Phi_H}{\partial x^i}\right]. \tag{1.1}$$

The gravitational potential $\Phi_H(x)$, the gauge-invariant variable of Bardeen 1980, is a function only of the Lagrangian coordinates x^i. Here, $a(t) = t^{2/3}$ is the scale factor of the background metric

$$ds^2 = -dt^2 + a^2(t)\left[(dx^1)^2 + (dx^2)^2 + (dx^3)^2\right].$$

The Eulerian energy density $\rho(t,r)$ is expressed in terms of the Lagrangian energy density $\rho_0(x)$ by

$$\rho(t,r) = \frac{\rho_0(x)}{a^3(t)\det\left[\delta_{jk} + \frac{3}{2}a(t)\frac{\partial^2\Phi_H}{\partial x^j \partial x^k}\right]}. \tag{1.2}$$

For a fixed dust particle represented by x^i, the symmetric matrix $-\partial^2\Phi_H/\partial x^j\partial x^k$ may be diagonalized yielding eigenvalues, $\alpha(x)$, $\beta(x)$ and $\gamma(x)$, where $\alpha > \beta > \gamma$, so that eq.(1.2) becomes

$$\rho(t,r) = \frac{\rho_0(x)}{\left[1 - \frac{3}{2}a(t)\alpha\right]\left[1 - \frac{3}{2}a(t)\beta\right]\left[1 - \frac{3}{2}a(t)\gamma\right]}. \tag{1.3}$$

$\rho(t,r)$ first becomes infinite at the epoch $a_{sing} = 2/(3\alpha)$ and this singularity corresponds to a "pancake" collapse along the direction of the eigenvector with largest positive eigenvalue

α. (For a review, see e.g., Shandarin and Zel'dovich 1989; the Newtonian theory has been studied by Buchert 1989, 1992.)

At first sight, the relativistic version appears to be rather different since it is described in terms of a metric. For example, for scales with wavelengths exceeding the Hubble radius during the matter-dominated era, the metric can be written as

$$ds^2 = -dt^2 + a^2(t)\, k_{ij}(x) dx^i dx^j \,. \tag{1.4}$$

The "seed" 3-metric $k_{ij}(x)$ (generated during the inflationary scenario) is independent of time t, but is otherwise arbitrary. In this paper, we will give an expression for the metric after the scales have re-entered the Hubble radius and have started to form pancakes. Since the universe has undergone considerable expansion during a previous inflationary epoch, any primordial vorticity will have been diluted away. Hence it is an excellent approximation to consider irrotational flow, where the 4-velocity is the gradient of a velocity potential $\chi(t,x)$

$$U^\mu = -\chi^{,\mu} \,. \tag{1.5}$$

If the time-hypersurface in eq.(1.4) is chosen so that $\chi = t$ is uniform, then the dust particles are locally at rest. Such a time hypersurface is called comoving.

The plan of this paper is as follows. In §2.1 we use a traditional approach, integration along the fluid worldlines, to discuss the evolution of relativistic collapsing irrotational dust spacetimes. We show in §2.2 that there is an exact collapse solution, which is axisymmetric, and in §2.3 we express it in terms of the Szekeres 1975 line element. This approach illustrates the general features of gravitational collapse.

In §3.1 we describe the spatial gradient expansion utilized by Salopek and Stewart 1992. Irrotational dust is introduced using an action principle which simplifies the treatment. (More general perfect fluids can be handled as well, Salopek and Stewart 1993.) This technique allows one to expand inhomogeneous line elements in powers of spatial gradients without making any symmetry assumptions of the seed metric. Salopek and Stewart developed the theory to third order in spatial gradients, extending the first order theory (Salopek and Bond 1990, Salopek 1992a) which was initially applied to the inflationary scenario. Parry et al 1993 have extended the results to arbitrary order, and in this paper we shall use some of their results at fifth and seventh order. Using Hamilton-Jacobi methods in §3.2 we solve Einstein's equations using a generating functional which satisfies both of the constraints as well as half of the evolution equations. The remaining equations for the evolution of the spatial 3-metric may be solved by using an iteration method.

A priori one might have expected such an expansion to break down long before pancake formation. However we demonstrate in §3.3 that the third order theory correctly reproduces the third order terms in the Szekeres solution, given the appropriate seed metric. Further the fifth order terms yield exactly the Szekeres metric; there are no seventh order corrections.

Following the spirit of Zel'dovich we can improve our calculations dramatically. He noticed that a naive application of linearized theory could lead to negative energy densities. By reformulating his theory in a Lagrangian framework he overcame this. Here we require that the spatial 3-metric remain positive-definite at all times. Using this property in §4, we show how the Zel'dovich approximation may be derived from the (third order) long-wavelength theory. Thus the theory is considerably more accurate than one might have reasonably expected. We summarize our conclusions in §5.

The sign conventions of Misner et al 1973 are used throughout.

2. GENERAL RELATIVISTIC COLLAPSING DUST SPACETIMES

In this section we obtain an exact solution of the Einstein field equations representing the collapse of dust. As was explained in the introduction it is physically reasonable to assume that the flow is irrotational. We shall make a second assumption that the magnetic part of the Weyl tensor vanishes. At each point the Riemann curvature tensor has twenty independent components. Ten of these (the Ricci tensor) are determined by the local matter via the Einstein field equations. The remaining ten constitute the Weyl tensor and represent gravitational radiation. Given a 4-velocity U^α the Weyl tensor $C^{\alpha\beta\gamma\lambda}$ can be decomposed, in analogy with electrodynamics, into "electric" and "magnetic" parts. Roughly speaking a body emits electrically when it changes its shape and magnetically when it rotates. Precise definitions are given in Ellis 1971. The second assumption is certainly consistent with the first, and it leads to interesting solutions, but is made on the grounds of simplicity alone. A similar line of reasoning was employed by Barnes and Rowlingson 1989 as well as Matarrese et al 1993.

In §2.1 we review briefly a decomposition of the field equations given first by Ehlers 1961 and later exploited by Ellis 1971. This enables us to construct a system of ordinary differential equations along the world lines of the dust particles which determine some, but not all, features of the spacetime. §2.2 applies these ideas to a generic collapse situation, and we obtain an exact axisymmetric, but apparently unstable solution. In §2.3 we use this information to obtain the spacetime line element for the exact solution. It describes pancake collapse as predicted by Zel'dovich 1970.

2.1 The evolution equations along the fluid worldlines

In this section an overdot represents a directional derivative along the fluid worldlines, e.g., $\dot{U}^\alpha = U^\alpha{}_{;\beta}U^\beta$. The projection tensor into the rest space of U^α will be denoted $h^{\alpha\beta} = g^{\alpha\beta} + U^\alpha U^\beta$. Because we are considering dust with energy density ρ and energy-momentum tensor $T^{\alpha\beta} = \rho U^\alpha U^\beta$, the acceleration $\dot{U}^\alpha$ vanishes (as a consequence of $h^\alpha{}_\beta T^{\beta\gamma}{}_{;\gamma} = 0$). The velocity gradient may therefore be decomposed into

$$U_{\alpha;\beta} = \sigma_{\alpha\beta} + \tfrac{1}{3}\theta h_{\alpha\beta}. \tag{2.1}$$

Here $\theta = U^\alpha{}_{;\alpha}$ is the volume expansion rate, ($\theta = 3H$, where H is the Hubble parameter) and $\sigma_{\alpha\beta}$ is the symmetric trace-free shear tensor. ($U_{\alpha;\beta}$ has no antisymmetric part because the motion is irrotational.) It should be noted that $\sigma_{\alpha\beta}$ resides in the rest space of U^α, $\sigma_{\alpha\beta}U^\beta = 0$.

The basic idea of Ehlers was to project both the Ricci identity

$$U_{\alpha;\nu\mu} - U_{\alpha;\mu\nu} = R_{\alpha\beta\mu\nu}U^\beta, \tag{2.2}$$

and the Bianchi identity

$$C^{\alpha\beta\gamma\lambda}{}_{;\lambda} = R^{\gamma[\alpha;\beta]} - \frac{1}{6}g^{\gamma[\alpha}R^{,\beta]}\,, \tag{2.3}$$

along directions parallel and orthogonal to U^α. By doing this one can build a system of equations which describe the evolution of ρ, θ and $\sigma_{\alpha\beta}$ along the fluid world lines. We start with ρ. The conservation equation $U_\alpha T^{\alpha\beta}{}_{;\beta} = 0$ implies

$$\dot{\rho} + \rho\theta = 0, \tag{2.4}$$

which governs the evolution of ρ if we know θ. The evolution of θ is given by the *Raychaudhuri* equation

$$\dot{\theta} + \tfrac{1}{3}\theta^2 + \sigma_{\mu\nu}\sigma^{\mu\nu} + \tfrac{1}{2}\kappa\rho = 0, \tag{2.5}$$

($\kappa = 8\pi G = 8\pi/m_p^2$), provided we know $\sigma_{\alpha\beta}$. Equations (2.5) and (2.6) below are obtained by contracting eq.(2.2) with U^μ and forming the trace and trace-free parts respectively. The evolution of $\sigma_{\alpha\beta}$ is described by

$$h_\alpha{}^\mu h_\beta{}^\nu \dot{\sigma}_{\mu\nu} + \tfrac{2}{3}\theta\sigma_{\alpha\beta} + \sigma_{\alpha\gamma}\sigma^\gamma{}_\beta - \tfrac{1}{3}h_{\alpha\beta}\sigma_{\mu\nu}\sigma^{\mu\nu} + E_{\alpha\beta} = 0, \tag{2.6}$$

which requires knowledge of the electric part of the Weyl tensor, $E_{\alpha\gamma} = C_{\alpha\beta\gamma\delta}U^\beta U^\delta$. $E_{\alpha\beta}$ is symmetric and trace-free. It evolves according to

$$h_\alpha{}^\mu h_\beta{}^\nu \dot{E}_{\mu\nu} + \theta E_{\alpha\beta} + h_{\alpha\beta}\sigma_{\mu\nu}E^{\mu\nu} - \tfrac{3}{2}(E_{\alpha\gamma}\sigma^\gamma{}_\beta + \sigma_{\alpha\gamma}E^\gamma{}_\beta) + \tfrac{1}{2}\kappa\rho\sigma_{\alpha\beta} = 0, \tag{2.7}$$

which closes the system. Equation (2.7) can be obtained from equation (2.3) by projection. A detailed derivation and discussion of equations (2.1–7) can be found in e.g., Ellis 1971. It is an instructive exercise to construct their Newtonian analogues.

It is highly advantageous to introduce an orthonormal tetrad of vectors (U^α, e_i^α), $i, j, k, \ldots = 1, 2, 3$, with the e_i^α aligned along the eigenvectors of $\sigma_{\alpha\beta}$, so that

$$\sigma_{\alpha\beta} = \sigma_1 e_{1\alpha} \otimes e_{1\beta} + \sigma_2 e_{2\alpha} \otimes e_{2\beta} - (\sigma_1 + \sigma_2) e_{3\alpha} \otimes e_{3\beta}, \tag{2.8}$$

effectively diagonalizing $\sigma_{\alpha\beta}$. (Indices $i, j, \ldots$ represent tetrad labels for which there is no implicit summation convention. Further indexed quantities like $\sigma_1 = \sigma_{\alpha\beta}e_1^\alpha e_1^\beta$ are scalars.) Projecting eq.(2.6) along the eigenvectors we note that $E_{\alpha\beta}$ must take the same form

$$E_{\alpha\beta} = \epsilon_1 e_{1\alpha} \otimes e_{1\beta} + \epsilon_2 e_{2\alpha} \otimes e_{2\beta} - (\epsilon_1 + \epsilon_2) e_{3\alpha} \otimes e_{3\beta}. \tag{2.9}$$

In addition the eigenvalues satisfy

$$\dot{\sigma}_i + \tfrac{2}{3}\theta\sigma_i + \sigma_i^2 - \tfrac{2}{3}(\sigma_1^2 + \sigma_1\sigma_2 + \sigma_2^2) + \epsilon_i = 0, \tag{2.10}$$

and

$$\dot{\epsilon}_i + \theta\epsilon_i + (2\sigma_1\epsilon_1 + \sigma_1\epsilon_2 + \sigma_2\epsilon_1 + 2\sigma_2\epsilon_2) - 3\sigma_i\epsilon_i + \tfrac{1}{2}\kappa\rho\sigma_i = 0, \tag{2.11}$$

for $i = 1, 2$, with no summation convention on repeated indices. Note that the evolution equations (2.4–5, 2.10–11) involve only scalar quantities. They are therefore ordinary differential equations along the flow lines. This simplification arises because we have ignored the magnetic part of the Weyl tensor.

For a spherically symmetric Friedman-Robertson-Walker spacetime the evolution equations collapse to

$$\dot{\rho}_F + \rho_F\theta_F = 0,$$
$$\dot{\theta}_F + \tfrac{1}{3}\theta_F^2 + \tfrac{1}{2}\kappa\rho_F = 0,$$

with the well known solution

$$\theta_F = 3\frac{\dot{a}}{a}, \qquad a(t) = t^{2/3}, \qquad \rho_F = \frac{\theta_F^2}{3\kappa}. \tag{2.12}$$

Following Matarrese et al 1993 we use eq.(2.12) to partially factor out the expansion of the universe from eqs.(2.4–5, 10–11), setting

$$\rho = \rho_F(1 + a\delta), \qquad \theta = \theta_F + \dot{a}\vartheta, \qquad \sigma_i = \dot{a}s_i, \qquad \epsilon_i = \dot{a}t^{-1}e_i, \tag{2.13}$$

and treating $a(t)$ as a new independent variable along the world lines. With these changes the equations become

$$\delta' = -\vartheta\delta - (\vartheta + \delta)/a \tag{2.14a}$$
$$\vartheta' = -\tfrac{1}{3}\vartheta^2 - 2(s_1^2 + s_1s_2 + s_2^2) - \tfrac{3}{2}(\vartheta + \delta)/a, \tag{2.14b}$$
$$s_1' = -\tfrac{1}{3}s_1(s_1 + 2\vartheta) + \tfrac{2}{3}s_2(s_1 + s_2) - \tfrac{3}{2}(s_1 + e_1)/a, \tag{2.14c}$$
$$s_2' = -\tfrac{1}{3}s_2(s_2 + 2\vartheta) + \tfrac{2}{3}s_1(s_1 + s_2) - \tfrac{3}{2}(s_2 + e_2)/a, \tag{2.14d}$$
$$e_1' = e_1(s_1 - s_2 - \vartheta) - e_2(s_1 + 2s_2) - \delta s_1 - (s_1 + e_1)/a, \tag{2.14e}$$
$$e_2' = e_2(s_2 - s_1 - \vartheta) - e_1(s_2 + 2s_1) - \delta s_2 - (s_2 + e_2)/a. \tag{2.14f}$$

2.2 The Exact Solution

At first sight analysis of eq.(2.14) would seem to be a difficult task. However it is easy to see that the quadratic terms in eq.(2.14b) are manifestly negative. Thus if ϑ is negative on a world line and large enough so that the linear terms can be neglected, then $\vartheta \to -\infty$ in a finite time. This is the collapse scenario we are looking for. We therefore examine eqs.(2.14b–d) with the linear terms ignored,

$$\begin{aligned} \vartheta' &= -\tfrac{1}{3}\vartheta^2 - 2(s_1^2 + s_1s_2 + s_2^2), \\ s_1' &= -\tfrac{1}{3}s_1(s_1 + 2\vartheta) + \tfrac{2}{3}s_2(s_1 + s_2), \\ s_2' &= -\tfrac{1}{3}s_2(s_2 + 2\vartheta) + \tfrac{2}{3}s_1(s_1 + s_2). \end{aligned} \tag{2.15}$$

Equations (2.15) are approximate equations only. It proves useful to factor out the expansion setting $s_i = \vartheta\Sigma_i$, so that

$$\begin{aligned} \vartheta' &= -\vartheta^2\left[\tfrac{1}{3} + 2(\Sigma_1^2 + \Sigma_1\Sigma_2 + \Sigma_2^2)\right], \\ \Sigma_1' &= \vartheta\left(\tfrac{1}{3} + \Sigma_1\right)\left[2\left(\Sigma_1^2 + \Sigma_1\Sigma_2 + \Sigma_2^2\right) - \Sigma_1\right], \\ \Sigma_2' &= \vartheta\left(\tfrac{1}{3} + \Sigma_2\right)\left[2\left(\Sigma_1^2 + \Sigma_1\Sigma_2 + \Sigma_2^2\right) - \Sigma_2\right]. \end{aligned} \tag{2.16}$$

Clearly there is an equilibrium solution with $\Sigma_1 = \Sigma_2 = -\frac{1}{3}$. Examination of the phase portrait, Fig. 1, shows that this is an attractor for the system (2.16), assuming implicitly that $\vartheta < 0$. Indeed there are three such attractors, representing the same solution under a relabeling of the spacelike members of the orthonormal tetrad. There are also three repellers (more precisely saddles) given by, e.g., $\Sigma_1 = \Sigma_2 = \frac{1}{6}$. Note that the attractors and repellers are all axisymmetric, whereas the general flow is not.

We now return to the more general equations (2.14), and look for the exact solution corresponding to the attractor, setting $s_1 = s_2 = -\frac{1}{3}\vartheta$. The solution has already been given by Matarrese et al 1993 as

$$\delta = \frac{f}{1-af}, \qquad \vartheta = -\frac{f}{1-af}, \qquad s_1 = s_2 = -e_1 = -e_2 = \frac{1}{3}\,\frac{f}{1-af}, \tag{2.17}$$

where f is a constant on each world line, but will be different on different world lines. Matarrese et al specialized to the Newtonian case and identified solution (2.17) with the Zel'dovich pancake collapse. It is easy to see that the solution (2.17) satisfies also the approximate equations (2.15); the linear terms vanish in this case. Of course, we wish to check whether it is a stable solution for the more general equations (2.14). For this we used a numerical integration with initial data which deviated randomly by 1% from that of the exact solution. The behaviour is shown in Fig. 2 where we have used $\log(-\vartheta)$ as a time coordinate so as to expand the collapse epoch. s_1, s_2 and δ behave as expected, and as shown in Fig. 1. However e_1/ϑ and e_2/ϑ diverge as the singularity is approached. This can be verified by a linear stability analysis or by the following simple consideration. Near the singularity it is permissible to set $s_1 = s_2 = -\frac{1}{3}\vartheta$, $\delta = -\vartheta$. Then (2.14b, e, f) become

$$\vartheta' = -\vartheta^2, \qquad e_1' = -\vartheta(e_1 - e_2) - \tfrac{1}{3}\vartheta^2, \qquad e_2' = -\vartheta(e_2 - e_1) - \tfrac{1}{3}\vartheta^2, \tag{2.18}$$

showing

$$e_1 + e_2 \sim \tfrac{2}{3}\vartheta, \qquad e_1 - e_2 \propto \vartheta^2. \tag{2.19}$$

Clearly this instability requires further examination; we suspect that it is caused by neglect of the magnetic part of the Weyl tensor. (Note that our initial data was chosen to be consistent with Matarrese et al.) The exact solution, see §2.3, has a Weyl tensor of Petrov type D with zero magnetic part. However a linearized perturbation of it might be expected to be of Petrov type I with a nonzero magnetic part.

2.3 The Szekeres solution

Although the exact result (2.17) looks extremely simple it gives only partial information about the spacetime — we have still to determine the line element. In principle this has already been done by Barnes and Rowlingson 1989 but it is instructive to construct it from eq.(2.17). In terms of our original variables eq.(2.17) implies

$$\theta = 3\frac{\dot a}{a} - \frac{\dot a f}{1-af}, \qquad \sigma_1 = \sigma_2 = \frac{1}{3}\,\frac{\dot a f}{1-af}. \tag{2.20}$$

Barnes and Rowlingson have shown that in the case of axisymmetry, as here, one can rotate the spacelike members of the orthonormal tetrad so as to make them hypersurface

orthogonal; in other words spatial coordinates can be chosen so that the metric tensor is diagonal. Thus in comoving synchronous gauge

$$ds^2 = -dt^2 + \gamma_{ik}(t, x^j)\,dx^i\,dx^k, \tag{2.21}$$

with γ_{ik} diagonal. In this coordinate system $U^\alpha = \delta^\alpha{}_0$, and

$$U^i{}_{;k} = \sigma^i{}_k + \tfrac{1}{3}\theta\delta^i{}_k = \tfrac{1}{2}\gamma^{ij}\frac{\partial\gamma_{jk}}{\partial t}.$$

Using eq.(2.20) we have

$$\dot\gamma_{11} = 2\frac{\dot a}{a}\gamma_{11}, \qquad \dot\gamma_{22} = 2\frac{\dot a}{a}\gamma_{22}, \qquad \dot\gamma_{33} = 2\left(\frac{\dot a}{a} - \frac{\dot a f(x^i)}{1-af(x^i)}\right)\gamma_{33}.$$

Using the axisymmetry, and some relabeling we obtain

$$\gamma_{11} = \gamma_{22} = a^2(t)b^2(x^i), \qquad \gamma_{33} = a^2(t)\left[d(x^i) - c(x^i)a(t)\right]^2, \tag{2.22}$$

where b, c and d are functions of position that remain to be determined.

Using computer algebra, we have computed the Einstein tensor for the line element (2.20–22) (see appendix). We require that all terms vanish except G_{00}, which can be taken as the definition of the density ρ. Setting $G_{01} = G_{02} = 0$ we find $c_{,1} = c_{,2} = 0$, i.e., $c = c(x^3)$. $G_{03} = 0$ implies $b_{,3} = 0$, i.e., $b = b(x^1, x^2)$. It is convenient to introduce cylindrical coordinates $x^i = (r, \theta, z)$ and to require $b = b(r)$, $d = d(r, z)$. Setting $a = t^{2/3}$ in $G_{33} = 0$ implies

$$b\left(\frac{d^2b}{dr^2} + \frac{1}{r}\frac{db}{dr}\right) = \left(\frac{db}{dr}\right)^2,$$

with solution $b = \alpha r^\beta$, α, β constants. Reasonable behavior at both large and small r requires $\beta = 0$, and by rescaling $b = 1$. The remaining equations imply

$$d(x^i) = d_0(z) - \frac{5}{9}c(z)r^2, \quad \text{where} \quad r^2 = x^2 + y^2. \tag{2.23}$$

Thus there are two arbitrary functions of z, $d_0(z)$ and $c(z)$, in the line element

$$ds^2 = -dt^2 + a^2(t)\left[dr^2 + r^2\,d\theta^2 + \left(d_0(z) - \tfrac{5}{9}c(z)r^2 - c(z)a(t)\right)^2\right]. \tag{2.24}$$

This metric was derived first, in a totally different way, by Szekeres 1975. One can see immediately that all the features of Zel'dovich's Newtonian treatment can be read off the line element (2.24). Proper distances between fluid particles in the (r, θ) plane expand as in a FRW universe. Proper distances in the z-direction while initially expanding, eventually collapse to zero and then re-expand. However unlike earlier treatments, the line element (2.24) is an exact solution of Einstein's field equations.

3. IRROTATIONAL DUST FLOWS VIA AN ACTION PRINCIPLE

While supplying useful insights the results of §2 do not solve the main problem: given a line element (1.4) describing the initial conditions of an inhomogeneous, matter-dominated cosmological system, what is its subsequent evolution? How do pancake structures arise under these more general circumstances where the seed metric $k_{ij}(x)$ is arbitrary?

In order to obtain an accurate approximate solution to this problem we now switch approaches and describe an irrotational dust flow interacting with Einstein gravity via an action principle:

$$\mathcal{I} = \int d^4x\sqrt{-g}\left\{ \frac{1}{2\kappa}{}^{(4)}R - \frac{1}{2}\rho(g^{\mu\nu}\chi_{,\mu}\chi_{,\nu} + 1) \right\}. \tag{3.1}$$

${}^{(4)}R$ is the Ricci scalar of the 4-metric. Here we have assumed that the rest mass of the dust particles is unity, so that the number density is identical to the energy density ρ. The metric can be decomposed using the Arnowitt-Deser-Misner (ADM) formalism (see, e.g., Misner et al 1973) as

$$g_{00} = -N^2 + \gamma^{ij}N_iN_j, \quad g_{0i} = g_{i0} = N_i, \quad g_{ij} = \gamma_{ij}. \tag{3.2}$$

N, N_i and γ_{ij} are the lapse function, the shift function, and the 3-metric, respectively. The 4-velocity is given by the gradient of the velocity potential χ

$$U^\mu = -g^{\mu\nu}\chi_{,\nu}. \tag{3.3}$$

The energy density $\rho(t,x)$ serves a Lagrange multiplier in eq.(3.1) which ensures that the square of the 4-velocity is -1

$$U^\mu U_\mu = -1. \tag{3.4}$$

The action is more transparent if one employs a Hamiltonian formalism. It then becomes

$$\mathcal{I} = \int d^4x\left(\pi^\chi\frac{\partial\chi}{\partial t} + \pi^{ij}\frac{\partial\gamma_{ij}}{\partial t} - N\mathcal{H} - N^i\mathcal{H}_i \right), \tag{3.5}$$

where the energy and momentum densities, $\mathcal{H}$ and $\mathcal{H}_i$ are given by

$$\mathcal{H} = \sqrt{1+\gamma^{ik}\chi_{,i}\chi_{,k}}\,\pi^\chi + \kappa\gamma^{-1/2}\pi^{ij}\pi^{kl}\left(2\gamma_{jk}\gamma_{li} - \gamma_{ij}\gamma_{kl}\right) - \frac{1}{2\kappa}\gamma^{1/2}R, \tag{3.6a}$$

$$\mathcal{H}_i = -2\left(\gamma_{il}\pi^{lk}\right)_{,k} + \pi^{lk}\gamma_{lk,i} + \pi^\chi\chi_{,i}. \tag{3.6b}$$

R is the Ricci scalar of the spatial 3-metric γ_{ij} and the π^{ij} are the momenta conjugate to γ_{ij}. The momentum conjugate to the dust field χ is related to the density ρ through

$$\pi^\chi = \rho\gamma^{1/2}\sqrt{1+\gamma^{ik}\chi_{,i}\chi_{,k}}. \tag{3.7}$$

The lapse and shift functions are Lagrange multipliers that demand that the energy and momentum densities vanish

$$\mathcal{H} = 0 \quad \text{and} \quad \mathcal{H}_i = 0. \tag{3.8a, b}$$

In this section, we shall need to solve only the two evolution equations,

$$\left(\frac{\partial\chi}{\partial t} - N^i\chi_{,i}\right)/N = \sqrt{1+\gamma^{ik}\chi_{,i}\chi_{,k}}, \tag{3.9a}$$

$$\left(\frac{\partial\gamma_{ij}}{\partial t} - N_{i|j} - N_{j|i}\right)/N = 2\kappa\gamma^{-1/2}\pi^{kl}\left(2\gamma_{ik}\gamma_{jl} - \gamma_{ij}\gamma_{kl}\right), \tag{3.9b}$$

found by varying eq.(3.5) with respect to π^χ and π^{ij}. Here $|$ will denote a covariant derivative with respect to the spatial 3-metric γ_{ij}.

3.1 Hamilton-Jacobi Theory

The basic idea of Hamilton-Jacobi theory is to construct a canonical transformation in which the new Hamiltonian vanishes identically, so that the new coordinates and momenta are constants of the motion. See e.g., Salopek and Stewart 1992 for a detailed description and motivation. The generating functional $\mathcal{S} \equiv \mathcal{S}[\gamma_{ij}(x), \chi(x)]$ for the canonical transformation is the object of primary interest. Through functional differentiation, one obtains the canonical momenta:

$$\pi^{ij}(x) = \frac{\delta\mathcal{S}}{\delta\gamma_{ij}(x)}, \quad \pi^\chi(x) = \frac{\delta\mathcal{S}}{\delta\chi(x)}. \tag{3.10}$$

When these expressions are substituted into eqs.(3.8a,b), one obtains the Hamilton-Jacobi equation

$$\begin{aligned}\mathcal{H}(x) =& \sqrt{1+\gamma^{ik}\chi_{,i}\chi_{,k}}\;\frac{\delta\mathcal{S}}{\delta\chi(x)} + \\ & \kappa\gamma^{-1/2}\frac{\delta\mathcal{S}}{\delta\gamma_{ij}(x)}\frac{\delta\mathcal{S}}{\delta\gamma_{kl}(x)}\left[2\gamma_{jk}(x)\gamma_{il}(x) - \gamma_{ij}(x)\gamma_{kl}(x)\right] - \frac{1}{2\kappa}\gamma^{1/2}R = 0,\end{aligned} \tag{3.11a}$$

and the momentum constraint

$$\mathcal{H}_i(x) = -2\left(\gamma_{ik}\frac{\delta\mathcal{S}}{\delta\gamma_{kj}(x)}\right)_{,j} + \frac{\delta\mathcal{S}}{\delta\gamma_{lk}(x)}\gamma_{lk,i} + \frac{\delta\mathcal{S}}{\delta\chi(x)}\chi_{,i} = 0. \tag{3.11b}$$

(From now on, we have set $\kappa = 1$.) The remainder of Einstein's equations are automatically satisfied provided that eqs.(3.9) and (3.11) hold.

3.2 The gradient expansion

Salopek and Stewart 1992 have shown how to solve the Hamilton-Jacobi equation using a gradient expansion. One of the advantages of solving the Hamilton-Jacobi equation rather than the usual Einstein equations is that it is not necessary to make a choice of time and spatial coordinates until the end of the calculation. Eq.(3.11a) is hypersurface

and gauge independent. However, for simplicity, we will consider a restricted generating functional $\mathcal{S}[\gamma_{ij}(x)|\chi]$, where the dust field $\chi(x)$ is assumed to be homogeneous:

$$\mathcal{S}[\gamma_{ij}(x)|\chi] \equiv \mathcal{S}[\gamma_{ij}(x), \chi(x) = \chi]. \tag{3.12}$$

This corresponds to choosing a comoving and synchronous time hypersurface since $U^\mu = \delta^\mu_0$ and $N = 1$ (see eqs.(3.3) and (3.9a)). The rate of change of the restricted generating functional $\mathcal{S}[\gamma_{ij}(x)|\chi]$ is

$$\frac{\partial \mathcal{S}[\gamma_{ij}(x)|\chi]}{\partial \chi} = \int d^3x \left[\frac{\delta \mathcal{S}[\gamma_{ij}(x), \chi(x)]}{\delta \chi(x)}\right]_{\chi(x)=\chi}, \tag{3.13}$$

in which case the Hamilton-Jacobi equation simplifies to a single integral equation for $\mathcal{S}[\gamma_{ij}(x)|\chi]$

$$0 = \frac{\partial \mathcal{S}}{\partial \chi} + \int d^3x \left\{\kappa \gamma^{-1/2} \frac{\delta \mathcal{S}}{\delta \gamma_{ij}(x)} \frac{\delta \mathcal{S}}{\delta \gamma_{kl}(x)} \left[2\gamma_{jk}(x)\gamma_{il}(x) - \gamma_{ij}(x)\gamma_{kl}(x)\right] - \frac{1}{2\kappa}\gamma^{1/2} R\right\}, \tag{3.14}$$

and the momentum constraint reduces to

$$0 = -2\left(\gamma_{ik} \frac{\delta \mathcal{S}}{\delta \gamma_{kj}(x)}\right)_{,j} + \frac{\delta \mathcal{S}}{\delta \gamma_{lk}(x)} \gamma_{lk,i}. \tag{3.15}$$

The last equation states that the restricted generating functional $\mathcal{S}[\gamma_{ij}(x)|\chi]$ is invariant under reparametrizations of the spatial coordinates. We therefore look for a solution in the form of a spatial gradient expansion

$$\mathcal{S} = \mathcal{S}^{(0)} + \mathcal{S}^{(2)} + \mathcal{S}^{(4)} + \ldots, \tag{3.16}$$

where

$$\mathcal{S}^{(0)} = -2 \int d^3x \gamma^{1/2} H(\chi), \tag{3.17a}$$

$$\mathcal{S}^{(2)} = \int d^3x \gamma^{1/2} J(\chi) R, \tag{3.17b}$$

$$\mathcal{S}^{(4)} = \int d^3x \gamma^{1/2} \left[L(\chi) R^2 + M(\chi) R^{ij} R_{ij}\right]. \tag{3.17c}$$

$\mathcal{S}^{(0)}$ contains no spatial gradients whereas $\mathcal{S}^{(2)}$ contains two spatial gradients, and so on. H, J, L, M are arbitrary functions of the velocity potential χ which are chosen so as to satisfy the Hamilton-Jacobi equation order by order:

$$H^2 + \frac{2}{3}\frac{dH}{d\chi} = 0, \qquad \text{(zeroth order)} \tag{3.18a}$$

$$\frac{dJ}{d\chi} + JH - \frac{1}{2} = 0, \qquad \text{(second order)} \tag{3.18b}$$

$$\frac{dL}{d\chi} - LH - \frac{3}{4}J^2 = 0, \qquad \text{(fourth order)} \tag{3.18c}$$

$$\frac{dM}{d\chi} - MH + 2J^2 = 0. \qquad \text{(fourth order)} \tag{3.18d}$$

In 3 spatial dimensions, the Riemann tensor R_{ijkl} may be expressed in terms of of the Ricci tensor R_{ij} and the 3-metric, (the Weyl tensor vanishes), and hence it does not appear in the generating functional. Moreover, the term $\int d^3x \gamma^{1/2} P(\chi) R^{|i}{}_{|i}$ is absent because it is a total derivative (since χ is homogeneous) and hence contributes nothing to the canonical momenta (3.10). The nonlinear equation (3.18a) is referred to as the separated Hamilton-Jacobi equation (SHJE) of order zero. The remaining equations are linear and they are referred to as the separated Hamilton-Jacobi equations of order two ((3.18b)) and of order four ((3.18c,d)).

The functional derivatives appearing in eq.(3.14) are readily computed if one notes that for a small deviation $\delta\gamma_{ij}$ of the 3-metric, the corresponding change in the Ricci tensor is (e.g., Misner et al 1973, p.500)

$$\delta R_{ij} = \frac{1}{2}\gamma^{kl}\left[\delta\gamma_{li|jk} + \delta\gamma_{lj|ik} - \delta\gamma_{ij|lk} - \delta\gamma_{lk|ij}\right].$$

For example,

$$\frac{\delta \mathcal{S}^{(0)}}{\delta \gamma_{ij}(x)} = -\gamma^{1/2}\gamma^{ij} H,$$

$$\frac{\delta \mathcal{S}^{(2)}}{\delta \gamma_{ij}(x)} = \gamma^{1/2} J \left[\frac{1}{2} R \gamma^{ij} - R^{ij}\right],$$

$$\frac{\delta \mathcal{S}^{(4)}}{\delta \gamma_{ij}(x)} = \gamma^{1/2} L \left[\frac{1}{2} R^2 \gamma^{ij} - 2RR^{ij} + 2R^{|ij} - 2\gamma^{ij} R^{|k}{}_{|k}\right] + \gamma^{1/2} M \left[\frac{1}{2}\gamma^{ij} R^{kl} R_{kl} - 2R^i{}_l R^{jl} + R^{ik|j}{}_{|k} + R^{jk|i}{}_{|k} - R^{ij|k}{}_{|k} - \frac{1}{2}\gamma^{ij} R^{|k}{}_{|k}\right].$$

A solution to eq.(3.18a) is immediate:

$$H = \frac{2}{3\chi}. \tag{3.19a}$$

This is the well-known expression for the Hubble parameter of a homogeneous Friedman model where χ is comoving and synchronous time. This result is actually valid for inhomogeneous fields provided the wavelength of the fluctuations exceeds the Hubble radius. The growing mode solutions to eqs.(3.18b–d) are

$$J = \frac{3}{10}\chi, \quad \text{(second order)}, \tag{3.19b}$$

$$L = \frac{81}{2800}\chi^3, \quad M = -\frac{27}{350}\chi^3, \quad \text{(fourth order)}. \tag{3.19c, d}$$

The gradient expansion is similar to solving Einstein's equations using a power-series expansion (e.g. Müller et al 1990). Hamilton-Jacobi methods are more powerful because they yield a systematic way of finding explicit solutions to the momentum constraint which was not treated adequately before. In addition, they may be useful in illuminating the quantum theory (Salopek 1992b).

The solution (3.16) of the Hamilton-Jacobi equation automatically ensures that the constraint equations (3.8a,b) are satisfied. The only equation that remains to be solved is (3.9b). For simplicity, we will assume that the shift vanishes $N^i = 0$ and that the lapse N is 1 (consistent with our choice of χ as our time hypersurface):

$$\frac{\partial \gamma_{ij}}{\partial \chi} = 2\gamma^{-1/2} \frac{\delta \mathcal{S}}{\delta \gamma_{kl}(x)} \left(2\gamma_{ik}\gamma_{jl} - \gamma_{ij}\gamma_{kl} \right), \tag{3.20}$$

This is a first-order nonlinear differential equation for γ_{ij}, and we will solve it in terms of a gradient expansion. To first order in spatial gradients, eq.(3.9b) becomes

$$\frac{\partial \gamma_{ij}}{\partial \chi} = 2H(\chi)\gamma_{ij}, \tag{3.21}$$

whose solution is immediate,

$$\gamma^{(1)}_{ij}(\chi, x) = \chi^{4/3} k_{ij}(x), \tag{3.22}$$

which is expressed in terms of a conformal metric $k_{ij}(x)$ which is independent of χ, but is otherwise arbitrary; it is determined by the generation of quantum noise in an inflationary epoch. The solution accurate to third order is found by solving eq.(3.20) using an iteration method. First one neglects spatial derivatives fourth order and higher, and then substitutes the first order result into those terms which have two spatial derivatives:

$$\frac{\partial \gamma_{ij}}{\partial \chi} = 2H\gamma_{ij} + J \left(\hat{R} k_{ij} - 4\hat{R}_{ij} \right). \tag{3.23}$$

where $\hat{R}_{ij} \equiv R_{ij}(k_{lm})$ and $\hat{R} \equiv R(k_{lm})$ are the Ricci tensor and Ricci scalar, respectively, of the conformal 3-metric k_{lm}; they are independent of χ. The third-order expression for the 3-metric is:

$$\gamma^{(3)}_{ij}(\chi, x) = \chi^{4/3}\, k_{ij}(x) + \frac{9}{20}\chi^2 \left[\hat{R}(x)\, k_{ij}(x) - 4\hat{R}_{ij}(x) \right]. \tag{3.24}$$

(This is the same as the result quoted by Salopek and Stewart 1992 if one identifies $k_{ij}(x) = (2m/3)^{-2/3} h_{ij}(x)$.)

It should now be clear how to solve eq.(3.20) to higher order. For example, the fifth-order result

$$\begin{aligned} \gamma^{(5)}_{ij}(\chi, x) = {} & \chi^{4/3}\, k_{ij} + \frac{9}{20}\chi^2 \left[\hat{R}\, k_{ij} - 4\hat{R}_{ij} \right] + \\ & \frac{81}{350}\chi^{8/3} \Bigg[k_{ij} \left(-4\hat{R}^{lm}\hat{R}_{lm} + \frac{89}{32}\hat{R}^2 + \frac{5}{8}\hat{R}_{;k}{}^{;k} \right) \\ & + \frac{5}{8}\hat{R}_{;ij} - 10\hat{R}\hat{R}_{ij} + 17\hat{R}^l{}_i\hat{R}_{lj} - \frac{5}{2}\hat{R}_{ij;k}{}^{k} \Bigg], \end{aligned} \tag{3.25}$$

has been derived by Parry et al 1993. They give a systematic method of solving the higher order terms. (A semi-colon ; in eq.(3.25) denotes a covariant derivative with respect to k_{ij}.)

3.3 Comparison with the Szekeres Solution

Embedded in the previous section is a prescription for studying the non-linear formation of structure in cosmology. At early times (when the inhomogeneities are actually quite small), one writes the spatial 3-metric in the form (3.22) (synchronous gauge is assumed). We assert that successively more accurate approximations for the evolution of the spatial 3-metric up to the formation of pancakes, the growth of the seed, are given by eqs.(3.24) and (3.25). Further they can be improved using the techniques of §4. Although these expressions look complicated, they involve only algebraic and derivative operations applied to the seed. Thus they can be generated straightforwardly using computer algebra programs.

This may seem a radical proposal. Eqs.(3.24) and (3.25) were derived under the assumption that spatial gradients were small. Will they be valid even as pancakes form? Fortunately we have an exact solution, the Szekeres line element derived in §2 which can be used to test these expressions. In eq.(3.22), the seed metric $k_{ij}(x)$ was arbitrary, but we will now consider the special case

$$k_{ij}(x^k) = \mathrm{diag}\left[1, 1, d^2(x^k)\right] \tag{3.27}$$

where $d(x^i)$ was defined in eq.(2.23). For convenience, we have chosen Cartesian coordinates rather than cylindrical coordinates. The non-vanishing components of the Ricci curvature tensor for k_{ij} are

$$\hat{R}_{11} = \hat{R}_{22} = \frac{1}{2}\hat{R}_{33}/d^2(x^i) = \frac{10}{9}\frac{c(z)}{d(x^i)}. \tag{3.28}$$

If we identify $a(t) = t^{2/3} = \chi^{2/3}$, one may readily evaluate

$$\gamma^{(3)}_{ij} = \chi^{4/3}\mathrm{diag}\left[1, 1, d^2(x^i) - 2c(z)\, d(x^i)\, \chi^{2/3}\right], \tag{3.29a}$$

$$\gamma^{(5)}_{ij} = \chi^{4/3}\mathrm{diag}\left[1, 1, \left(d(x^i) - c(z)\, \chi^{2/3}\right)^2\right]. \tag{3.29b}$$

Eq.(3.29a) is not exactly the Szekeres solution, but in §4 we demonstrate how an improved expansion yields the Szekeres line element. Eq.(3.29b) is precisely the Szekeres solution (2.24). Using the techniques of Parry et al 1993 we have computed the seventh order corrections to eq.(3.25) which are quite complicated. With the Ansatz (3.27), we have verified that they do indeed vanish.

In Fig. 3, we compare the exact solution with the various approximations. The third order expression gives a good qualitative account of the evolution but it gives a poor estimate for the value of the expansion parameter $a^{(3)}_{sing}$ when the metric becomes singular:

$$a^{(3)}_{sing} = \frac{1}{2}d(x^i)/c(z), \quad \text{(third order)}.$$

The fifth order result gives the exact result

$$a_{sing} = a^{(5)}_{sing} = d(x^i)/c(z)\,, \quad \text{(fifth order)}.$$

In §4, we show that the Zel'dovich approximation (1.1) yields the same result.

The excellent agreement of the spatial gradient expansion with the exact Szekeres solution is reassuring. One surmises that the expansion will quite useful in cosmological systems. It appears that that the domain of validity of the spatial gradient expansion is greater than might have been expected; indeed it seems that it gives accurate results for the 3-metric until the first caustics form.

4. IMPROVING THE SPATIAL GRADIENT APPROXIMATION

The spatial gradient expansion given in §3 has neglected a very useful piece of information. The three eigenvalues of the 3-metric are all non-negative (otherwise the signature of space-time would change). Hence, one can improve the range of validity by expressing the expansion in terms of a 'perfect square.' We illustrate the basic point using a Taylor series for a non-negative function of a single variable.

4.1 Improved Taylor series for a non-negative function

If a non-negative function

$$g(x) = \sum_{n=0}^{\infty} a_n x^n, \tag{4.1}$$

is expanded in a Taylor series, then it is a trivial observation to note that the truncated series

$$g_m(x) = \sum_{n=0}^{m} a_n x^n, \tag{4.2}$$

may not be positive. As a result, the truncated series can be a poor approximation to $g(x)$ in certain regimes. In fact, if the physics of the situation demands that $g(x)$ be non-negative, then the truncated series is nonsense whenever it drops below zero. However, this disastrous situation may be avoided if one expands the square root of $g(x)$ in a Taylor series. If one writes

$$g(x) = [h(x)]^2\,, \tag{4.3}$$

and then expands $h(x)$ in a series,

$$h(x) = \sum_{n=0}^{\infty} b_n x^n\,, \tag{4.4}$$

one obtains the "improved" (guaranteed positive) truncated series

$$\tilde{g}_m(x) = [h_m(x)]^2\,, \tag{4.5a}$$

where

$$h_m(x) = \sum_{n=0}^{m} b_n x^n\,. \tag{4.5b}$$

Note that both $g_m(x)$ and $\tilde{g}_m(x)$ both agree to order x^m but that $\tilde{g}_m(x)$ has higher order terms which guarantee that it is positive. Moreover, if one has an explicit expression for $g_m(x)$, one may uniquely determine $\tilde{g}_m(x)$ (up to a sign) since

$$b_0 = a_0^{1/2}, \tag{4.6a}$$

$$b_1 = \frac{1}{2} a_0^{-1/2} a_1, \tag{4.6b}$$

$$b_2 = \frac{1}{2} a_0^{-1/2} \left(a_2 - \frac{a_1^2}{4a_0} \right) \quad \text{etc.} \tag{4.6c}$$

As a concrete example of these ideas, let us consider the non-negative function $g(x) = \cos^2 x$. The first few terms (truncated Taylor series for $g(x)$) about $x = 0$ are

$$g_2(x) = 1 - x^2, \tag{4.7a}$$

$$g_4(x) = 1 - x^2 + \frac{x^4}{3}, \tag{4.7b}$$

whereas those for $h(x)$ are

$$h_2(x) = 1 - \frac{x^2}{2}, \tag{4.8a}$$

$$h_4(x) = 1 - \frac{x^2}{2} + \frac{x^4}{24}\,. \tag{4.8b}$$

In Figs. 4a,b we compare $g_2(x)$ with $\tilde{g}_2(x) = [h_2(x)]^2$ and $g_4(x)$ with $\tilde{g}_4(x) = [h_4(x)]^2$. In both cases, the improved expansion is much better because it extends the range of validity by a factor of about two into a regime where one would not expect the expansion to apply. Note the $g_2(x)$ becomes negative for $x > 1$ yielding nonsensical results; however, $\tilde{g}_2(x)$ is surprisingly accurate. Even though $g_4(x)$ is everywhere positive, it is a mediocre approximation of $\cos^2 x$, whereas the agreement of $\tilde{g}_4(x)$ for $x < \frac{1}{2}\pi$ is quite remarkable.

4.2 Improved Taylor series for a positive-definite matrix

The above results may be easily extended to describe a positive definite matrix A which has a series expansion

$$A = I + \sum_{n=1}^{\infty} A_n = \left(I + \sum_{n=1}^{\infty} B_n \right)^2, \tag{4.9}$$

where I denotes the identity matrix. Given the matrices A_n, one finds that the first few B_n are

$$B_1 = \frac{1}{2}A_1\,, \tag{4.10a}$$

$$B_2 = \frac{1}{2}\left(A_2 - \frac{1}{4}A_1^2\right)\,, \tag{4.10b}$$

$$B_3 = \frac{1}{2}\left[A_3 - \frac{1}{4}\left(A_1A_2 + A_2A_1\right) + \frac{1}{8}A_1^3\right]\,, \quad \text{etc.} \tag{4.10c}$$

These are basically the same as eqs.(4.6) except that the A_n's do not necessarily commute.

4.3 Application of Zel'dovich approximation to general relativity

Using the series expansion of §3, and the improved expansion of §4.2, one can apply the Zel'dovich approximation to general relativity.

In the first instance, we will work with the expansion (3.24) of the 3-metric accurate to third order in spatial gradients. To that order, we may write it as a perfect square

$$\tilde{\gamma}_{ij}^{(3)} = \chi^{4/3}\left\{k_{il} + \frac{9}{40}\chi^{2/3}\left[\hat{R}k_{il} - 4\hat{R}_{il}\right]\right\} k^{lm}\left\{k_{jm} + \frac{9}{40}\chi^{2/3}\left[\hat{R}k_{jm} - 4\hat{R}_{jm}\right]\right\} \tag{4.11}$$

Using the seed (3.27), we verify that this improved approximation exactly reproduces the exact Szekeres solution. Eq.(4.11) is one the major results of this paper. It demonstrates that a straightforward improvement of the gradient expansion can lead to reliable results. Because of its (relative) simplicity and wide application we call it the generalization of the Zel'dovich approximation to a general relativistic context.

In order to justify our claims further, we will write the seed metric in the form (Salopek and Stewart 1992)

$$k_{ij}(x) = \delta_{ij}\left(1 + \frac{10}{3}\Phi_H(x)\right)\,. \tag{4.12}$$

In problems of physical interest $\Phi_H(x) \sim 10^{-5}$ is very small. Hence we are justified in using a linear approximation to the combination

$$D_{jm} = k_{jm} + \frac{9}{40}\chi^{2/3}\left[\hat{R}k_{jm} - 4\hat{R}_{jm}\right] \tag{4.13}$$

appearing in eq.(4.11):

$$D_{jm} \approx \delta_{jm}\left(1 + \frac{10}{3}\Phi_H(x)\right) + \frac{3}{2}\chi^{2/3}\Phi_{H,jm}\,. \tag{4.14}$$

Consequently, eq.(4.11) certainly reproduces the standard results of linear perturbations in comoving synchronous gauge (e.g. Salopek and Stewart 1992)

$$\gamma_{ij}(\chi, x) = \chi^{4/3}\left(1 + \frac{10}{3}\Phi_H(x)\right) + 3\chi^2\Phi_{H,jm}\,. \tag{4.15}$$

However, eq.(4.11) is valid far beyond linear theory. If in addition, we assume that the wavelength of the fluctuations are much smaller than the Hubble radius, then eq.(4.14) simplifies to

$$D_{jm} \approx \delta_{jm} + \frac{3}{2}\chi^{2/3}\Phi_{H,jm}\,. \tag{4.16}$$

The 3-metric (4.11) becomes singular when $a(t) = 2/(3\alpha)$ where α is the largest positive eigenvalue of $-\partial^2\Phi_H/\partial x^j\partial x^k$. This is precisely the same epoch of collapse discussed in the introduction §1. We conclude that the improved third order gradient expansion reproduces the Zel'dovich approximation.

4.4 Higher Order Zel'dovich Approximation

Using eqs.(3.25) and (4.10b), it is straightforward algebra to improve the fifth order solution:

$$\tilde{\gamma}_{ij}^{(5)} = \chi^{4/3}\left\{k_{il} + \frac{9}{40}\chi^{2/3}\left[\hat{R}k_{il} - 4\hat{R}_{il}\right] + \chi^{4/3}F_{il}\right\} k^{lm} \left\{k_{jm} + \frac{9}{40}\chi^{2/3}\left[\hat{R}k_{jm} - 4\hat{R}_{jm}\right] + \chi^{4/3}F_{jm}\right\}\,, \tag{4.17}$$

where

$$F_{ij} = \frac{81}{5600}\left\{k_{ij}\left[-32\hat{R}^{lm}\hat{R}_{lm} + 5\hat{R}_{;k}{}^{;k} + \frac{41}{2}\hat{R}^2\right] - 66\hat{R}\hat{R}_{ij} + 108\hat{R}^l{}_i\hat{R}_{lj} - 20\hat{R}_{ij;k}{}^{;k} + 5\hat{R}_{;ij}\right\}\,. \tag{4.13}$$

For the seed metric (3.27), there is no difference between the improved third order result (4.12) and the improved fifth order one (4.13).

(In a Newtonian cosmological setting, numerous authors have investigated improvements to the Zel'dovich approximation including inter alia Bond and Couchman 1987, Moutarde et al 1991 and Nusser et al 1991. Boucher et al 1992 have employed a small gradient expansion which resembles some of the techniques of §3.)

It is tempting to extrapolate our solutions beyond the time that the first caustics form. Because additional physical assumptions are required, we will postpone these considerations for future work.

5. CONCLUSIONS

Using traditional techniques, we have reduced Einstein's equations for irrotational dust flows with zero magnetic part of the Weyl tensor to a set of ordinary differential equations. We showed that the Szekeres line element is an exact solution of the resulting evolution equations. Unfortunately, numerical integrations demonstrate that this solution is apparently unstable. The interpretation of this instability is unclear. It is probably a result of assuming zero magnetic part of the Weyl tensor in which case this approach will have a narrow range of applications.

In an alternative line of research, we have solved Einstein's equations analytically using a spatial gradient expansion. Our methods are nonlinear and they do not assume any special symmetry. The generality of our results is a promising feature. Our key tool was the application of Hamilton-Jacobi methods which can be applied to cosmological systems with dust, general perfect fluids, scalar fields, etc.. In fact, the Hamilton-Jacobi equation is the semi-classical approximation to the Wheeler-DeWitt equation (Salopek, Stewart and Parry 1993) — our techniques also shed some light on the quantum theory of the gravitational field.

The main advantage of Hamilton-Jacobi theory is a systematic and straightforward solution of the constraint equations of general relativity. The momentum constraint implies that the generating functional is invariant under reparametrizations of the spatial coordinates. It motivates a series expansion in terms of the Ricci curvature. We have basically assumed that all peculiar velocities are small compared to the speed of light. The Hamilton-Jacobi equation is a natural starting point for a gauge and hypersurface invariant analysis because it neither refers to the lapse nor to the shift functions. However, for simplicity, we have restricted ourselves to comoving hypersurfaces.

The spatial gradient expansion does not immediately reproduce the Zel'dovich approximation. It is useful to note that the 3-metric should remain positive-definite at all times. Employing this property, we extended the effectiveness of the series. The improved third order expression eq.(4.11) essentially yields the Zel'dovich approximation. If we consider a special "seed" metric, we obtain the exact Szekeres solution.

This work can be used to interpret redshift surveys of galaxies. In very deep surveys, a relativistic treatment of the Zel'dovich approximation should be a distinct advantage, although it currently is not essential. Our formalism can also be used to compute the skewness of the density distribution at the present epoch given various initial conditions. Our derivation is elegant and powerful and it is straightforward to obtain higher order corrections. It removes much of the mystery behind the Zel'dovich approximation.

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APPENDIX: The Einstein Tensor for the Exact Solution

Consider a 4-metric of the form

$$ds^2 = -dt^2 + a^2(t)\left[b^2(x,y,z)(dx^2+dy^2) + w^2(t,x,y,z)\,dz^2\right],$$

where

$$w(t,x,y,z) = d(x,y,z) - a(t)c(x,y,z),$$

and $a(t) = t^{2/3}$. It is a routine exercise in computer algebra to compute the Einstein tensor components

$$^{(4)}G_{01} = \frac{\dot{a}c_{,1}}{w}, \qquad ^{(4)}G_{02} = \frac{\dot{a}c_{,2}}{w}, \qquad ^{(4)}G_{03} = \frac{-2\dot{a}cb_{,3}}{bw},$$

$$^{(4)}G_{11} = \frac{w^2(b_{,1}w_{,1} - b_{,2}w_{,2}) + b^2(wb_{,33} - b_{,3}w_{,3}) + bw^2w_{,22} + (10/9)b^3cw^2}{bw^3},$$

$$^{(4)}G_{22} = \frac{w^2(b_{,2}w_{,2} - b_{,1}w_{,1}) + b^2(wb_{,33} - b_{,3}w_{,3}) + bw^2w_{,11} + (10/9)b^3cw^2}{bw^3},$$

$$^{(4)}G_{12} = \frac{b_{,1}w_{,2} + b_{,2}w_{,1} - bw_{,12}}{w},$$

$$^{(4)}G_{13} = \frac{wb_{,1}b_{,3} + bb_{,3}w_{,1} - bwb_{,13}}{b^2w},$$

$$^{(4)}G_{23} = \frac{wb_{,2}b_{,3} + bb_{,3}w_{,2} - bwb_{,23}}{b^2w},$$

$$^{(4)}G_{33} = \frac{bw^2(b_{,11} + b_{,22}) - w^2(b_{,1}^2 + b_{,2}^2) + b^2b_{,3}^2}{b^4}.$$

References

Bardeen, J.M. 1980, Phys. Rev. D, 22, 1882

Barnes, A. & Rowlingson, R.R. 1989, Class. Quantum Grav., 6, 949

Bond, J.R. & Couchman, H. 1987, in *Proceedings of the Second Canadian Conference on General Relativity and Relativistic Astrophysics*, A. Coley, C. Dyer, eds., World Scientific

Bond, J.R. & Efstathiou, G. 1984, ApJ, 285, L45

Bouchet, F.R., Juszkiewicz, R., Colombi, S. & Pellat, R. 1992, ApJ, 394, L5

Broadhurst, T.J., Ellis, R.S., Koo, D.C. & Szalay, A.S. 1990, Nature, 343, 726

Buchert, T. 1989, A & A, 223, 9

———. 1992, MNRAS, 254, 729

Dekel, A. 1991, Proc. of *NATO Advanced Research Workshop*, Durham, England, edited by T. Shanks et al, 365 (Kluwer Academic, Norwell, MA)

Ehlers, J. 1961, Abh. Math-Naturwiss. Klasse der Akad. Wiss. und Lit., Mainz, nr. 11

Ellis, G.F.R. 1971, in *General Relativity and Cosmology*, ed. R.K. Sachs, Academic Press (New York)

Kaiser, N. , Efstathiou, G., Ellis, R., Frenk, C., Lawrence, A., Rowan-Robinson, M., Saunders, W. 1992, MNRAS, 252, 1

Matarrese, S., Pantano, O. & Saez, D. 1993, Phys. Rev. D, 47, 1311

Misner, C.W., Thorne, K.S. & Wheeler, J.A. 1973, *Gravitation*, Freeman (New York)

Moutarde, F., Alimi, J.-M., Bouchet, F.R., Pellat, R. & Ramani, A. 1991, ApJ, 382, 377

Müller V., Schmidt H. J. & Starobinsky A. A. 1990, Class. Quan. Grav., 7, 1163
Nusser, A., Dekel, A., Bertschinger, E. & Blumenthal, G.R. 1991, ApJ, 379, 6
Parry, J., Salopek, D.S. & Stewart, J.M. 1993, in preparation
Peebles, P.J.E. 1982, ApJ, 263, L1
Salopek, D.S. 1992a Phys. Rev. D, 45, 1139
——. 1992b Phys. Rev. D, 46, 4373
Salopek, D.S. & Bond, J.R. 1990, Phys. Rev. D, 42, 3936
Salopek, D.S. & Stewart, J.M. 1992, Class. Quantum Grav., 9, 1943
————. 1993, Phys. Rev. D (in press)
Salopek, D.S., Stewart, J.M. & Parry, J. 1993, submitted to Phys. Rev. D
Shandarin, S.F. & Zel'dovich 1989, Rev. Mod. Phys., 61, 185
Szekeres, P. 1975, Commun. Math. Phys., 41, 55
Zel'dovich, Ya.B. 1970, A & A, 5, 84

FIGURE CAPTIONS

Fig. 1 For a dust-dominated Universe where rotational effects are unimportant, Einstein's evolution equations reduce to a set of ordinary differential equations at each spatial point. The phase portrait for this reduced system is shown assuming the approximation eq.(2.16). The normalized eigenvalues Σ_1, Σ_2 of the shear tensor are plotted horizontally and vertically. The third eigenvalue is $\Sigma_3 = -(\Sigma_1 + \Sigma_2)$. There are three attractors (given by any two of Σ_1, Σ_2 and Σ_3 equal to $-\frac{1}{3}$). They correspond to axisymmetric pancake collapse. The three crosses represent axisymmetric cigar collapse situations (given by any two of Σ_1,Σ_2 and Σ_3 equal to $\frac{1}{6}$). They are repellers.

Fig. 2 The worldline equations (2.14) demonstrate an instability which is shown numerically. The horizontal coordinate is $\log(-\vartheta)$ which approaches infinity at the time of collapse. The initial data differs from that of the exact solution (2.17) by 1%. Initially the solution stays close to the exact one (which are horizontal lines). s_1/ϑ, s_2/ϑ and δ/ϑ always stay close to their initial values, whereas e_1/ϑ and e_2/ϑ diverge as demonstrated in eq.(2.19).

Fig. 3 The various orders of the gradient expansion are compared with the exact Szekeres solution (bold curve) for the evolution of the 3-metric component γ_{33} in terms of the scale factor $a(t) = t^{2/3}$. Pancake formation occurs when $\gamma_{33} = 0$. The thin curve is the first order term (long-wavelength), whereas the dotted graph is the third order result which underestimates the epoch of collapse by a factor of two. The improved third order expansion yields the exact result. (The fifth order result also reproduces the exact result.)

Fig. 4a,b For a non-negative function, *i.e.* $g(x) = \cos^2 x$, one can improve the range of validity of a Taylor series by expanding the square root $\sqrt{g(x)} = \cos(x)$ in a power series. In (a), $g_2(x)$ is the usual Taylor series approximation valid to second order, whereas $\tilde{g}_2(x)$ is the improved approximation which is much superior. In (b), $g_4(x)$ is the fourth order Taylor series and $\tilde{g}_4(x)$ is the improved fourth order. For $x < \frac{1}{2}\pi$, $\tilde{g}_4(x)$ is indistinguishable from $g(x)$. In the same way, one may improve the spatial gradient expansion for general relativity leading to the Zel'dovich approximation.

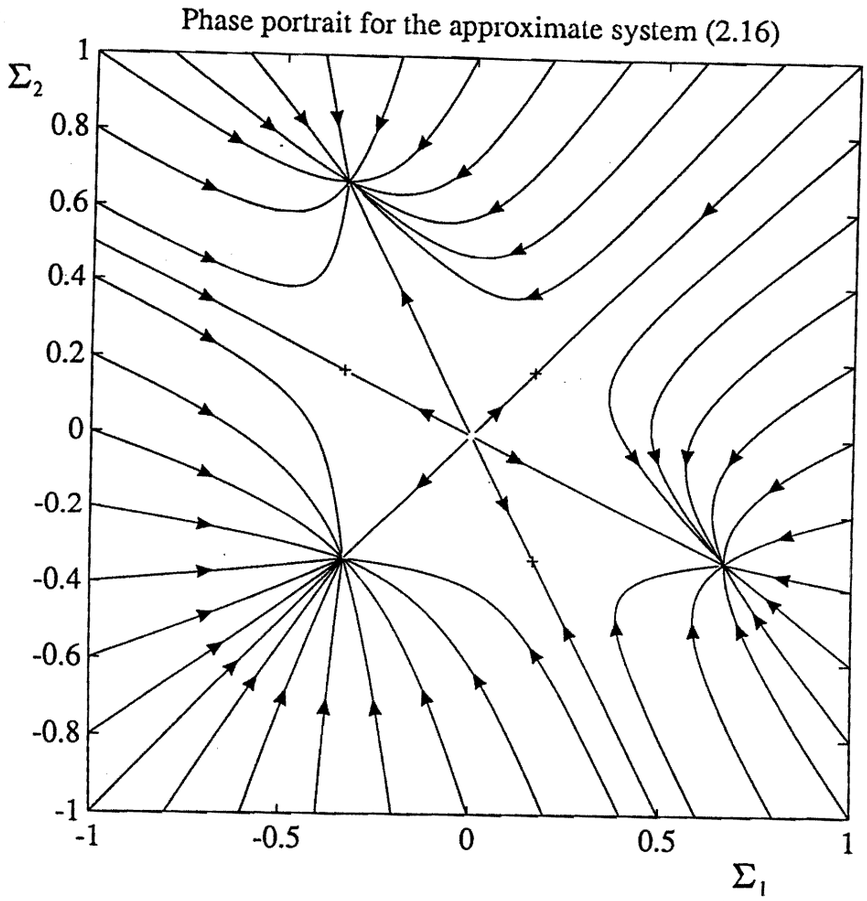
Phase portrait for the approximate system (2.16)
Σ_2
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
-1
-0.5
0
0.5
1
Σ_1

Fig. 1

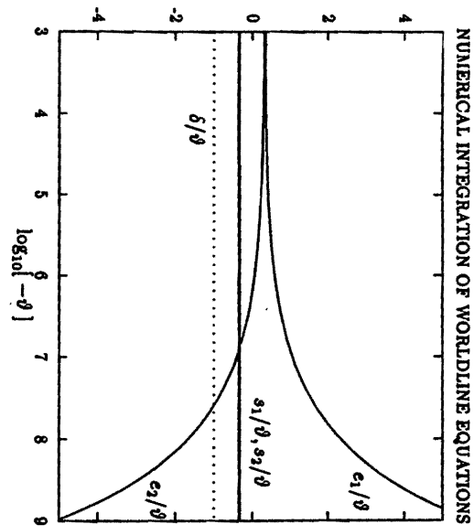
NUMERICAL INTEGRATION OF WORLDLINE EQUATIONS
e_1/ϑ
$s_1/\vartheta, s_2/\vartheta$
δ/ϑ
e_2/ϑ
$\log_{10}[-\vartheta]$
4
2
0
-2
-4
3
4
5
6
7
8
9

Fig. 2

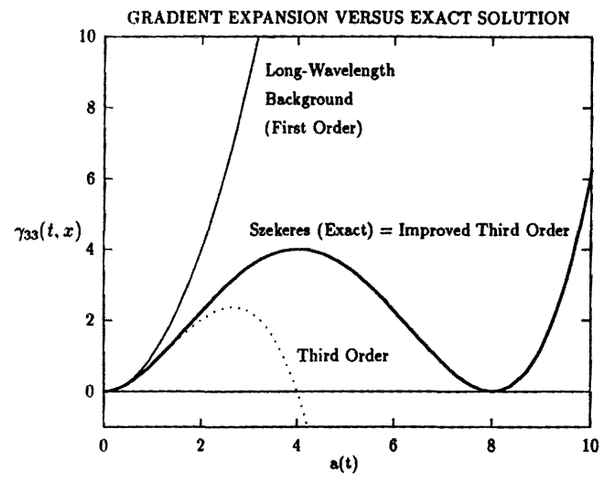
GRADIENT EXPANSION VERSUS EXACT SOLUTION
Long-Wavelength
Background
(First Order)
Szekeres (Exact) = Improved Third Order
Third Order
$\gamma_{33}(t,x)$
a(t)
0
2
4
6
8
10

Fig. 3

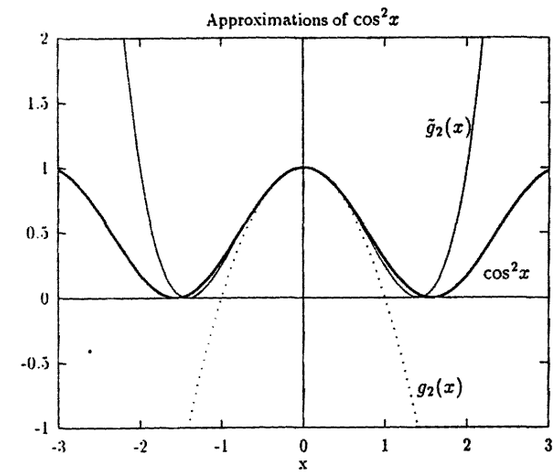
Approximations of $\cos^2 x$
$\tilde{g}_2(x)$
$\cos^2 x$
$g_2(x)$
x
2
1.5
1
0.5
0
-0.5
-1
-3
-2
-1
0
1
2
3

Fig. 4a

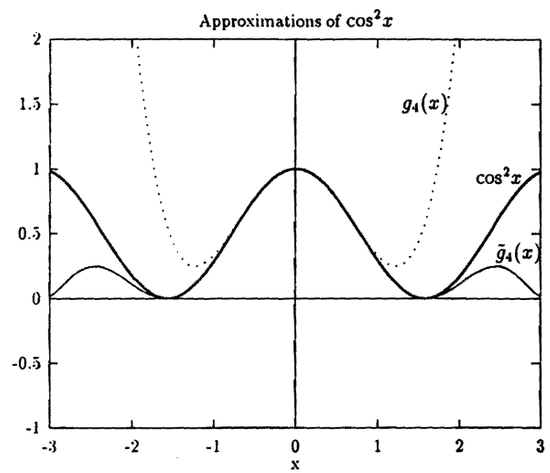
Approximations of $\cos^2 x$
$g_4(x)$
$\cos^2 x$
$\tilde{g}_4(x)$
2
1.5
1
0.5
0
-0.5
-1
-3
-2
-1
0
1
2
3
x

Fig. 4b