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Supersymmetric Bianchi models and the square root of the Wheeler–DeWitt equation

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Abstract

By solving the full set of quantum constraint equations for the Bianchi I mini-superspace model in supergravity, it is found that the wave function depends only on two arbitrary constants. General solutions to the Wheeler–DeWitt equation or to a corresponding Dirac square root equation are not permitted. Possible extensions of this work to Bianchi IX and other Bianchi types are discussed.

1. Introduction

There is a long-standing conjecture that, if one quantizes in supergravity the Bianchi I mini-superspace model, one will obtain wave-functions obeying a Dirac square root of the Wheeler–DeWitt equation. Macías, Obregón and Ryan [1] studied this problem, taking a diagonal Bianchi-I 3-metric:

$$ds^2 = (N^j N_j - N^2)\, dt^2 + 2N_i dt\, dx^i + e^{-2\Omega(t)} \left(e^{2\beta(t)}\right)_{ij} dx^i dx^j \,, \tag{1.1}$$

where $N(t)$ is the lapse function and $N^i(t)$ the shift vector; β_{ij} is taken of the form $\beta_{ij} = \mathrm{diag}\,(\beta_+ + \sqrt{3}\beta_-,\ \beta_+ - \sqrt{3}\beta_-,\ -2\beta_+)$. These authors postulated the existence of

a representation in which the supersymmetry constraints take the form

$$\left(i\frac{\partial}{\partial\Omega} - i\sigma_1\frac{\partial}{\partial\beta_+} + i\sigma_2\frac{\partial}{\partial\beta_-}\right)\Psi = 0$$

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acting on a two-component wave-function Ψ; here σ_1 and σ_2 are Pauli matrices. The 'Dirac equation' (1.2) then implies the familiar Wheeler–DeWitt equation [2]

$$\left(-\frac{\partial^2}{\partial\Omega^2} + \frac{\partial^2}{\partial\beta_+{}^2} + \frac{\partial^2}{\partial\beta_-{}^2}\right)\Psi = 0 \,, \tag{1.3}$$

for each component of the wave function.

Unfortunately, in [1] the quantum constraints requiring the wave function to be invariant under Lorentz transformations were not solved. We shall see that, when the Lorentz invariance is taken into account, the general solution for the supergravity wave function in the Bianchi I case has a very restricted form.

In supergravity, the spatial metric $h_{ij}(i,j = 1,2,3)$ is described by tetrad components $e^a{}_i$, with $h_{ij} = \eta^{ab} e_{ai} e_{bj}$, η^{ab} being the Minkowski metric. Equivalently, one may use the spinor version $e^{AA'}{}_i$. (For spinor conventions, see [3].) The 4-metric

$$ds^2 = (N^j N_j - N^2)\, dt^2 + 2N_i dt\, dx^i + h_{ij} dx^i dx^j \tag{1.4}$$

describes a general non-diagonal Bianchi-I model, when all the metric components are taken to be functions of t. The gravitino field is described by the spinor-valued forms $\psi^A{}_i, \tilde{\psi}^{A'}{}_i$, again taken to be functions of t. The $e^{AA'}{}_i$ are even Grassmann quantities, while $\psi^A{}_i$ and $\tilde{\psi}^{A'}{}_i$ are odd. A quantum state may be described by a wave function $\Psi\left(e^{AA'}{}_i, \psi^A{}_i\right)$, or equivalently by a wave function $\tilde{\Psi}\left(e^{AA'}{}_i, \tilde{\psi}^{A'}{}_i\right)$ [3]. The wave function $\Psi\left(e^{AA'}{}_i, \psi^A{}_i\right)$ may be expanded in even powers of $\psi^A{}_i$, in pieces symbolically of the form ψ^0, ψ^2, ψ^4 and ψ^6. Applying the full set of quantum constraints to Ψ, we find below that

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the general solution has the form

$$\Psi\left(e^{AA'}{}_i, \psi^A{}_i\right) = \text{const.}_1\ h^{-\frac{1}{2}} + \text{const.}_2\ h^{-\frac{1}{2}}(\psi)^6\ , \tag{1.5}$$

where $h = \det(h_{ij})$. Thus no ψ^2 and ψ^4 pieces are permitted at all, and the ψ^0 and ψ^6 pieces are unique up to constant factors. One does not even see the general solution to a Wheeler–DeWitt equation analogous to (1.3), let alone the square-root structure of (1.2).

2. Solution of the quantum constraint equations

The classical Hamiltonian for supergravity, subject to the spatially homogeneous Bianchi-I Ansatz (1.4) with homogeneous spinor fields $\left(\psi^A{}_i(t),\ \tilde{\psi}^{A'}{}_i(t)\right)$, takes the standard form for a theory with constraints arising from gauge invariances [3]:

$$H = \tilde{N}\mathcal{H} + \rho_A S^A + \tilde{S}^{A'}\tilde{\rho}_{A'} + M_{AB}J^{AB} + \tilde{M}_{A'B'}\tilde{J}^{A'B'} \tag{2.1}$$

Here $\mathcal{H}$ is the generator of local time translations, S^A and $\tilde{S}^{A'}$ are the generators of local supersymmetry transformations, and J^{AB} and $\tilde{J}^{A'B'}$ are the generators of local Lorentz transformations; they are all formed from the basic dynamical variables $\left(e^{AA'}{}_i, \psi^A{}_i, \tilde{\psi}^{A'}{}_i\right)$. The quantities $\tilde{N}, \rho_A, \tilde{\rho}_{A'}$, M_{AB} and $\tilde{M}_{A'B'}$ are Lagrange multipliers. Classically, all the constraint generators vanish, so that $\mathcal{H} = 0$ etc. Further, the constraint generators form a first-class 'algebra' (the constraint algebra), so that the Dirac bracket of any pair of generators gives an expression linear in the generators, which vanishes when the classical constraints $\mathcal{H} = 0$ etc. are imposed.

Quantum-mechanically, the constraint generators become operators which annihilate physical wave functions Ψ:

$$\begin{aligned} &\mathcal{H}\Psi = 0\ , \\ &S^A\Psi = 0\ , \quad \overline{S}^{A'}\Psi = 0\ , \\ &J^{AB}\Psi = 0\ , \quad \overline{J}^{A'B'}\Psi = 0\ . \end{aligned} \tag{2.2}$$

The classical Dirac brackets become (anti-) commutators, and the quantum version of the constraint algebra is the set of consistency conditions for the quantum constraints (2.2). In particular, the quantum constraint $\mathcal{H}$ is related to the anti-commutator of S^A and $\overline{S}^{A'}$, and the constraint $\mathcal{H}\Psi = 0$ (with suitable factor ordering for $\mathcal{H}$) follows from the other constraints in (2.2) [4]. (The Bianchi-I constraints neglecting J^{AB} and $\overline{J}^{A'B'}$ are described in [1].) Thus it is only necessary to solve the $S^A, \overline{S}^{A'}, J^{AB}$ and $\overline{J}^{A'B'}$ quantum constraints.

As described in the Introduction, the wave function can be taken as $\Psi\left(e^{AA'}{}_i, \psi^A{}_i\right)$ or $\tilde{\Psi}\left(e^{AA'}{}_i, \tilde{\psi}^{A'}{}_i\right)$. These two representations are related by a fermionic Fourier transform [3]:

$$\tilde{\Psi}\left(e^{AA'}{}_i, \tilde{\psi}^{A'}{}_i\right) = D^{-1}(e)\int \Psi\left(e^{AA'}{}_i, \psi^A{}_i\right)\exp\left(i\epsilon^{ijk}e_{AA'k}\psi^A{}_i\tilde{\psi}^{A'}{}_j\right)d\psi^A{}_i\ , \tag{2.3}$$

using Berezin integration. Here

$$D(e) = \det\left(i\epsilon^{ijk}e_{AA'k}\right)\ . \tag{2.4}$$

Under the transformation (2.3), the $\psi^0, \psi^2, \psi^4, \psi^6$ parts of the wave-function become $\tilde{\psi}^6, \tilde{\psi}^4, \tilde{\psi}^2, \tilde{\psi}^0$ parts, which helps considerably in solving the quantum constraints below.

The constraints $J^{AB}\Psi = 0$, $\overline{J}^{A'B'}\Psi = 0$ simply imply that $\Psi\left(e^{AA'}{}_i, \psi^A{}_i\right)$ is a Lorentz invariant function, formed from expressions in which all spinor indices have been contracted together. In forming such expressions, one makes use additionally of the normal spinor $n^{AA'}$, the spinor version of the unit future-pointing normal vector n^μ to the surface $t =$ const. This is a function of the $e^{AA'}{}_i$, defined by

$$n^{AA'}e_{AA'i} = 0\ , \quad n^{AA'}n_{AA'} = 1\ . \tag{2.5}$$

The $e^{AA'}{}_i$ and $n^{AA'}$ together form a basis for spinors $v^{AA'}$.

Turning to the supersymmetry constraints, classically for Bianchi I these are given by $S_A = -\frac{1}{2}i\kappa^2 p_{AA'}{}^i \tilde{\psi}^{A'}{}_i$, $\tilde{S}_{A'} = \frac{1}{2}i\kappa^2 \psi^A{}_i p_{AA'}{}^i$ [3]. Here $\kappa^2 = 8\pi$, and $p_{AA'}{}^i$ denotes the momentum conjugate to $e_{AA'}{}^i$; note that $p_{AA'}{}^i$ has non-zero Dirac brackets with $\psi^A{}_i$ and $\tilde{\psi}^{A'}{}_i$ [3]. Quantum-mechanically, we choose the 'averaged' ordering

$$\begin{aligned} S_A &= -\tfrac{1}{4}i\kappa^2 \left(\overline{\psi}^{A'}{}_i p_{AA'}{}^i + p_{AA'}{}^i \overline{\psi}^{A'}{}_i\right) , \\ \overline{S}_{A'} &= \tfrac{1}{4}i\kappa^2 \left(\psi^A{}_i p_{AA'}{}^i + p_{AA'}{}^i \psi^A{}_i\right) . \end{aligned} \tag{2.6}$$

This has the property that S_A and $\overline{S}_{A'}$ are hermitian adjoints, using the standard inner product [3] appropriate to the holomorphic representation for fermions being used here. Further, the $\overline{S}_{A'}\Psi = 0$ constraint implies a simple transformation property of the wave function $\Psi\left(e^{AA'}{}_i, \psi^A{}_i\right)$ under a supersymmetry transformation $\delta e^{AA'}{}_i = -i\kappa\tilde{\epsilon}^{A'}\psi^A{}_i$, $\delta\psi^A{}_i = 0$ [see (2.9) below]. And the $S_A\Psi = 0$ constraint implies an analogous transformation property for a wave function in the representation $\tilde{\Psi}\left(e^{AA'}{}_i, \tilde{\psi}^{A'}{}_i\right)$. There is a similar symmetry between the constraints $S_A\Psi = 0$ and $\overline{S}_{A'}\tilde{\Psi} = 0$. The two representations $\Psi\left(e^{AA'}{}_i, \psi^A{}_i\right)$ and $\tilde{\Psi}\left(e^{AA'}{}_i, \tilde{\psi}^{A'}\right)$ are thus treated symmetrically. As a result, we need only solve the supersymmetry constraints for $\Psi\left(e^{AA'}{}_i, \psi^A{}_i\right)$ at levels ψ^0 and ψ^2; the levels ψ^4 and ψ^6 can be dealt with by transforming to the representation $\tilde{\Psi}\left(e^{AA'}{}_i, \tilde{\psi}^{A'}{}_i\right)$ and studying the (formally identical) constraints for levels $\tilde{\psi}^2$ and $\tilde{\psi}^0$.

Explicitly, the constraint $S_A\Psi = 0$ in the representation $\Psi\left(e^{AA'}{}_i, \psi^A{}_i\right)$ reads

$$\frac{\partial}{\partial e^{AA'}{}_i}\left(D^{BA'}{}_{ji}\frac{\partial\Psi}{\partial\psi^B{}_j}\right) - \tfrac{1}{2}e_{AA'}{}^i D^{BA'}{}_{ji}\frac{\partial\Psi}{\partial\psi^B{}_j} = 0 , \tag{2.7}$$

where $\partial/\partial\psi^B{}_j$ denotes left differentiation [5], and

$$D^{BA'}{}_{ji} = -2ih^{-\frac{1}{2}}e^{BB'}{}_i e_{CB'}{}_j n^{CA'} , \tag{2.8}$$

with $h = \det(h_{ij})$. The constraint $\overline{S}_{A'}\Psi = 0$ is

$$\psi^A{}_i\frac{\partial\Psi}{\partial e^{AA'}{}_i} + \tfrac{1}{2}\psi^A{}_i e_{AA'}{}^i\Psi = 0 . \tag{2.9}$$

Thus from (2.9), under the supersymmetry transformation

$$\delta e^{AA'}{}_i = -i\kappa\tilde{\epsilon}^{A'}\psi^A{}_i, \quad \delta\psi^A{}_i = 0 , \tag{2.10}$$

with $\tilde{\epsilon}^{A'}$ an infinitesimal odd Grassmann parameter, one has the simple transformation property

$$\delta\Psi = \tfrac{1}{2}i\kappa\psi^A{}_i\tilde{\epsilon}^{A'}e_{AA'}{}^i\Psi . \tag{2.11}$$

Because there are no spatial derivatives present in the Bianchi I model, there is no constraint $\mathcal{H}^i$, generating spatial coordinate transformations, present among the constraints of (2.1). However, one may regard the Bianchi I model as a special limiting case of a more general class of inhomogeneous cosmological models, given by perturbations of the Bianchi I model. For such models there will be a quantum constraint $\mathcal{H}^i\Psi = 0$ implying invariance of the wave function under spatial coordinate transformations. Accordingly, we shall insist that all wave functions Ψ are spatial scalars, and carry no spatial indices $i, j, \ldots$.

The purely bosonic ψ^0 part $\Psi_0\left(e^{AA'}{}_i\right)$ of the wave function $\Psi\left(e^{AA'}{}_i, \psi^A{}_i\right)$, being a Lorentz invariant, has the form

$$\Psi_0\left(e^{AA'}{}_i\right) = F_0(h_{ij}) . \tag{2.12}$$

The only constraint to be satisfied, $\overline{S}_{A'}\Psi_0 = 0$, gives

$$\psi^A{}_i e_{AA'}{}_j\frac{\partial F_0}{\partial h_{ij}} + \tfrac{1}{2}\psi^A{}_i e_{AA'}{}^i F_0 = 0 , \tag{2.13}$$

for all $\psi^A{}_i$. The $\psi^A{}_i$ factors can be cancelled, and after contraction with $e^{AA'k}$ one obtains

$$\frac{\partial F_0}{\partial h_{ik}} + \tfrac{1}{2}h^{ik}F_0 = 0 . \tag{2.14}$$

with the solution

$$F_0(h_{ij}) = \text{const. } h^{-\frac{1}{2}} , \tag{2.15}$$

as in (1.5). By symmetry, the ψ^6 part of $\Psi\left(e^{AA'}{}_i, \psi^A{}_i\right)$ has the corresponding form.

The quadratic ψ^2 part of the wave function must be constructed in a Lorentz-invariant way from $e^{AA'}{}_i$ and $\psi^A{}_i$. Writing $\psi^A{}_{BB'} = e_{BB'}{}^i \psi^A{}_i$, one can split $\psi^A{}_{BB'}$ into a spin-$\frac{3}{2}$ and a spin-$\frac{1}{2}$ part, as [3]

$$\psi_{ABB'} = -2n^C{}_{B'}\gamma_{ABC} + \tfrac{2}{3}(\beta_A n_{BB'} + \beta_B n_{AB'}) - 2\epsilon_{AB} n^C{}_{B'}\beta_C , \tag{2.16}$$

where $\gamma_{ABC} = \gamma_{(ABC)}$ is totally symmetric and ϵ_{AB} is the alternating spinor. The general Lorentz-invariant wave function at order ψ^2 then has the form

$$\begin{aligned}\Psi_2\left(e^{AA'}{}_i, \psi^A{}_i\right) =& F_1(h_{ij})\beta_A\beta^A \\ &+ F_2(h_{ij})\gamma_{ABC}\gamma^{ABC} .\end{aligned} \tag{2.17}$$

Applying first the constraint $\overline{S}_{B'}\Psi_2 = 0$, one finds

$$\begin{aligned}&-4n^E{}_{C'}e^{FC'}{}_k e_{BB'j}\frac{\partial F_1}{\partial h_{jk}}\beta_A\beta^A\gamma^3{}_{FE} \\ &-4n^E{}_{C'}e^{FC'}{}_k e_{BB'j}\frac{\partial F_2}{\partial h_{jk}}\gamma^B{}_{FE}\,\gamma_{ABD}\gamma^{ACD} \\ &+\tfrac{16}{3}n^B{}_{C'}e^{EC'}{}_k e_{BB'j}\frac{\partial F_2}{\partial h_{jk}}\beta_E\,\gamma_{ACD}\,\gamma^{ACD} \\ &+F_2\begin{pmatrix}\tfrac{8}{3}n_{CB'}\beta_B\,\gamma^B{}_{DA}\,\gamma^{ACD} \\ -2n^E{}_{B'}\beta_E\,\gamma_{BDA}\,\gamma^{ABD} \\ +4n^E{}_{B'}\,\gamma^B{}_{DA}\,\gamma^{ACD}\,\gamma_{CBE}\end{pmatrix} = 0 .\end{aligned} \tag{2.18}$$

Now β_A and γ_{ABC} are independent odd Grassmann quantities. In particular, the symmetrized coefficient of $\beta_A\beta^A\gamma^B{}_{FE}$ in (2.18) must vanish. Hence

$$n_{(E|C'|}e_F{}^{C'}{}_k e_{B)B'j}\frac{\partial F_1}{\partial h_{jk}} = 0 . \tag{2.19}$$

Contracting this equation with $e^{BB'}{}_\ell n^E{}_{D'}e^{FD'}{}_m$ yields

$$\left(\tfrac{7}{4}h_{km}h_{j\ell} - \tfrac{3}{4}h_{\ell m}h_{jk}\right)\frac{\partial F_1}{\partial h_{jk}} = 0 , \tag{2.20}$$

which implies

$$F_1 = \text{constant} . \tag{2.21}$$

The constraint $S_A\Psi_2 = 0$ yields

$$\begin{aligned}&\tfrac{1}{2}i\frac{\partial F_1}{\partial h_{ij}}h_{ij}h^{-\frac{1}{2}}\beta_A \\ &+\tfrac{11}{8}iF_1h^{-\frac{1}{2}}\beta_A \\ &-4e_{AA'k}D^{BA'}{}_{ji}n_E{}^{C'}e_{DC'}{}^j\epsilon_{BC}\frac{\partial F_2}{\partial h_{ik}}\gamma^{CDE} = 0 .\end{aligned} \tag{2.22}$$

Since $F_1 = \text{constant}$, the coefficient of β_A in (2.22) shows further that

$$F_1 = 0 . \tag{2.23}$$

The coefficient of γ^{CDE} in (2.22) gives

$$e_{AA'k}D^{BA'}{}_{ji}n_{(E}{}^{C'}e_{D|C'}{}^j\epsilon_{B|C)}\frac{\partial F_2}{\partial h_{ik}} = 0 . \tag{2.24}$$

Contracting with $n^{AC'}e^C{}_{C'\ell}n^{DD'}e^E{}_{D'm}$ shows

$$(3h_{im}h_{k\ell} - h_{ik}h_{\ell m})\frac{\partial F_2}{\partial h_{ik}} = 0 , \tag{2.25}$$

which implies

$$F_2 = \text{constant} \ . \tag{2.26}$$

Finally, substituting (2.23), (2.26) back into (2.18) shows that F_2 multiplies a non-zero cubic in β_A and γ_{ABC} there, so that

$$F_2 = 0 \ . \tag{2.27}$$

Hence there is no ψ^2 part to the wave function $\Psi\left(e^{AA'}{}_i, \psi^A{}_i\right)$. By symmetry between the $\psi^A{}_i$ and $\tilde{\psi}^{A'}{}_i$ representations, there is also no ψ^4 part. Hence the general solution to the quantum constraints is given by (1.5).

3. Further developments

It will be of interest to study other Bianchi types in this framework, in particular the Bianchi IX model studied by Graham [6, 7]. For example, in [6] the spatial metric is taken as

$$ds^2_{\text{spatial}} = (6\pi)^{-\frac{1}{2}} e^{2\alpha} \left(e^{2\beta}\right)_{ij} \sigma^i \sigma^j \ , \tag{3.1}$$

where, as in the Introduction, $\beta_{ij} = \text{diag}\left(\beta_+ + \sqrt{3}\beta_-,\ \beta_+ - \sqrt{3}\beta_-,\ -2\beta_+\right)$, and σ^i are the left-invariant one-forms on the three-sphere. The model is quantised in [6], but without solving the full set of quantum constraints of supergravity. In particular, translating to the language of the present paper, the ψ^0 and ψ^6 parts of the wave function are found in [6] to have the simple forms

$$\text{const.}_1\ e^{-I} \ \text{ and } \ \text{const.}_2\ \psi^6 e^I \ , \tag{3.2}$$

where

$$I = \tfrac{1}{6} e^{2\alpha} Tr\left(e^{2\beta}\right) \ . \tag{3.3}$$

The bosonic solution e^{-I} has also been found as a solution to the Wheeler–DeWitt equation in [8].

The results (3.2) can be derived from the constraint equations $\overline{S}_{A'}\Psi = 0$ and $S_A\Psi = 0$ given in [3]. Indeed, analogous exponential solutions exist for all Bianchi types in class A [9]. This work will be described in a subsequent paper.

Further, based on the results of the present paper, one is led to conjecture that the general solution of the quantum constraints in the Bianchi IX case is similarly the sum of the two solutions in (3.2), without the order ψ^2 and ψ^4 solutions of the Wheeler–DeWitt equations found in [6]. It was already remarked in [6] that the state $\psi^6 e^I$ should be the Hartle–Hawking state [10]. One further expects that the $\psi^0 e^{-I}$ solution found in [6] is the wormhole ground quantum state [11] for the Bianchi IX model. If these conjectures are correct, then the supergravity constraint equations for Bianchi IX are so restrictive that they select out only the most symmetrical quantum states, usually defined via path integrals.

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