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Three Lectures on Supersymmetry and Extended Objects*

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1. Extended objects revisited
2. Solitons and supersymmetry
3. String/Fivebrane duality

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1: Extended objects revisited

A few years ago I gave some lectures at the Trieste spring school [1] on supermembranes and, to a lesser extent, supersymmetric p-dimensional objects, or p-branes for short. The emphasis there was on the 11-dimensional supermembrane [2], and the main issue was the nature of its quantum spectrum. As explained in [1], there was good evidence that the supermembrane, unlike the bosonic membrane, contains zero energy states, which it was hoped would correspond to the massless particles of $d = 11$ supergravity, but the non-uniqueness of the configuration of a collapsed object for $p > 1$ suggested that the spectrum might be continuous, which would preclude a particle-like interpretation. Shortly afterwards, de Wit, Lüscher and Nicolai [3] showed, for the regularized version of the $d = 11$ supermembrane reviewed in [1], that the spectrum *is* continuous, from zero. Although it has been questioned whether this result will continue to hold for the full, unregularized, supermembrane, or for extended objects of higher dimension, where the mathematics of [3] may fail, convincing physical arguments in favour of this scepticism have not been found.

However, as was also explained in [1], the theory of super p-branes is also of interest in the context of topological defects in supersymmetric field theories, some examples of which are provided by instantons of Euclidean d-dimensional field theories interpreted as p-dimensional topological defects of the $(d+p+1)$-dimensional Lorentzian field theory (provided that $(d+p+1)$ does not exceed the maximum dimension for which this field theory has a supersymmetric extension). One such example is the familiar Yang-Mills (YM) instanton, which becomes a fivebrane of supersymmetric $d = 10$ YM theory [4]. An attempt to extend this to a solution of the $d = 10$ supergravity/YM theory that serves as a low-energy approximation to the heterotic string was made by Strominger [5], but what he found was really a solution of the Chapline-Manton theory, as I shall explain. A number of related solutions have been found, at least one of which is known to be an *exact* solution of *classical* heterotic string theory [6,7], but it is unclear whether any of them can survive as approximate solutions of the quantum theory.

As an introduction I shall begin with a discussion of bosonic extended objects, as in [1] (and using the same notation as far as possible) but where there is an overlap I shall make use of alternative methods, which will hopefully clarify points skipped over in [1]; I shall

also take the opportunity to correct a few mis-statements. In lecture 2 I shall review the connection between supersymmetry and solitons, generalizing the original observations of Witten and Olive [8]. The most intriguing aspect of fivebranes, taken up in lecture 3, is the electric-magnetic-type duality between strings and fivebranes in $d = 10$, as first pointed out by Nepomechie [9] in the context of 'elementary' magnetic sources. The implications for the effective low-energy supergravity theory of string and/or fivebrane theory have been pursued by Duff and Lu [10,11,12] to the extent that the issue of quantum consistency deserves to be reassessed. One barrier to progress is the fact that the heterotic fivebrane, *as a six-dimensional worldvolume field theory*, has yet to be constructed; its field content must differ from the standard super-fivebrane action [13] by the inclusion of fields required for coupling to a YM background (as for the heterotic string) but it is not known how this can be done in a way that is consistent with all required symmetries.

A convenient starting point is the action for a p-dimensional extended object in d-dimensional spacetime $\mathcal{M}$ with Minkowski metric η. Let W be the $(p+1)$-dimensional worldvolume, with coordinates $\{\xi^i; i = 0, 1, \ldots, p\}$, and let $\{X^\mu; \mu = 0, 1, \ldots, d-1\}$ be coordinates for $\mathcal{M}$. The embedding of W in $\mathcal{M}$ is specified by a map $f : W \to \mathcal{M}$ such that $\xi^i \mapsto X^\mu(\xi)$, by virtue of which η induces a metric M on W with components

$$M_{ij} = \partial_i X^\mu \partial_j X^\nu \, \eta_{\mu\nu} \; . \tag{1.1}$$

We take as our action

$$S = -T \int d^{p+1}\xi \, \sqrt{-\det M} \; , \tag{1.2}$$

where T is the p-volume tension. As in [1], we shall suppose that the object is closed, i.e. has no boundary, so the only boundary conditions on the integral that need to be specified are at the initial and final times. For $p = 0$ this action is that of a relativistic point particle of mass $m = T$, while for $p = 1$ it is the Nambu-Goto action for a (closed, bosonic) relativistic string. We are using units in which c, the speed of light, is unity. An interesting point [14] here is that if the factors of c are reinstated, the non-relativistic limit $c \to \infty$ is straightforward only for $p = 0$. In that case the Lagrangian in the gauge $X^0 = ct$ is $L = mc^2 + (1/2)mv^2 + O(c^{-2})$ so the $c \to \infty$ limit can be taken after discarding the constant rest-mass energy. For the string, the leading term in the expansion of the Lagrangian in inverse powers of c is Tlc^2, where $l = \oint d\sigma \sqrt{|\mathbf{X}'|^2}$ is the length of the string (σ being the string's coordinate and the prime indicating a derivative with respect to it); this is again the rest-mass energy but now it cannot simply be discarded because it is *not* constant. The relativistic string is intrinsically relativistic in a way that the relativistic particle is not because the dynamics of even a free string does not separately conserve rest-mass energy. Similar remarks of course apply for $p > 1$.

As explained in [1], an action equivalent to (1.2) (in the sense that the classical Euler-Lagrange equations are equivalent) is

$$S = -\frac{1}{2}T \int d^{p+1}\xi \, \sqrt{-\det\gamma}\left[\gamma^{ij} M_{ij} - (p-1)\right] \; , \tag{1.3}$$

where γ^{ij} are the components of the inverse of an *independent* worldvolume metric γ. This action can be interpreted as a set of scalar fields $X^\mu(\xi)$ of a $(p+1)$-dimensional field theory

coupled to 'gravity', and with a cosmological constant proportional to $(p-1)$. For $p = 0$ we may set $T = m$ and $\gamma_{00} = -m^2V^2$ to arrive at the action

$$S = \int dt \left[\frac{1}{2V}\dot{X}^\mu \dot{X}^\nu \, \eta_{\mu\nu} + m^2 V\right] \tag{1.4}$$

for a relativistic particle of mass m. Taking $m \to 0$ we get the action of a massless particle for which the 'cosmological' term vanishes. It might at first appear that there is no analogous limit for $p > 0$ because for a string (1.3) reduces to the well-known action

$$S = -\frac{1}{2}T \int d^2\xi \, \sqrt{-\det\gamma}\, \gamma^{ij} \partial_i X \cdot \partial_j X \; , \tag{1.5}$$

for which there is already no 'cosmological term', while for $p > 1$ this term is present but cannot be consistently omitted. However, an alternative form of the action, equivalent (classically) to (1.2), but more closely analogous to the particle action (1.4), is

$$S = \int d^{p+1}\xi \, \frac{1}{2}\left[\frac{1}{V}\det M - T^2 V\right] \; , \tag{1.6}$$

where V is an independent worldvolume *scalar density*. We may now take the $T \to 0$ limit for any p to arrive at the action of the *null* p-brane [15]

$$S = \int d^{p+1}\xi \, \frac{1}{2V}\det M \; . \tag{1.7}$$

This action has the feature that it is *spacetime* conformal invariant. Specifically, if k^μ is a conformal Killing vector $(k_{(\mu;\nu)} = (1/d)g_{\mu\nu}k^\lambda{}_{;\lambda})$ then S is invariant under the infinitesimal transformations

$$\delta X^\mu = k^\mu \qquad \delta V = -2\frac{(p+1)}{d} V k^\mu{}_{;\mu} \; . \tag{1.8}$$

The physics of null objects is clearer from the phase-space form of the action. Make the worldvolume space/time split $\xi^i \to (t, \boldsymbol{\sigma})$, where $\boldsymbol{\sigma} = (\sigma^I, I = 1, \ldots, p)$ are the p-brane's coordinates. Then an action equivalent to (1.7) is

$$S = \int dt \oint d^p\sigma \, [\dot{X}^\mu p_\mu - \ell p^2 - s^I p \cdot \partial_I X] \; , \tag{1.9}$$

where $\ell(t, \boldsymbol{\sigma})$ and $s^a(t, \boldsymbol{\sigma})$ are simultaneously gauge fields for, respectively, the time and space reparametrization invariances and Lagrange multipliers for the constraints (cf. general relativity). The equivalence can be shown by eliminating p_μ and s^a by their equations of motion and making the identification $V = \ell \det M_{ab}$. Note in particular the constraint $p^2(\boldsymbol{\sigma}) = 0$ which shows that *all points of a null extended object move at the speed of light.* This is not very sensible physically but the null p-brane is a useful starting point for the discussion of some mathematical issues.

The spacetime conformal invariance of the null p-brane action should not be confused with the well-known *worldsheet* Weyl invariance of the string action (1.5). Weyl invariance

is often supposed to be an exclusive property of string theory but consider the following action for a p-dimensional extended object [16]:

$$S = -T \int d^{p+1}\xi \sqrt{-\det\gamma}\left[\frac{\gamma^{ij}M_{ij}}{(p+1)}\right]^{\frac{p+1}{2}} . \tag{1.10}$$

This action is Weyl invariant for *any* value of p. It is (classically) equivalent to our original action (1.2) because this is what it reduces to on elimination of the metric γ by means of its (algebraic) field equation (for $p = 0$ this step is unnecessary) and reduces to (1.5) for $p = 1$. The action (1.10) is *not* spacetime conformal invariant since, as a consequence of Weyl invariance, a rescaling of the metric now has no effect.

One interpretation of the action (1.2) is as an effective action at low energy, i.e. long wavelength, for a p-dimensional topological defect of a d-dimensional field theory (note that d is the dimension of spacetime whereas p is number of *spatial* dimensions of the defect). To fix ideas I shall concentrate in this lecture on domain walls in d=4 scalar field theories. This has the virtue of potential physical relevance, as well as simplicity. As our first example we shall consider a complex scalar field A with the action

$$S = \int d^4x \left[\partial\bar{A}\cdot\partial A - |U'(A)|^2\right] , \tag{1.11}$$

where U is a holomorphic function of A and U' is its derivative. This choice of potential is motivated by supersymmetry, in which context U is called the superpotential. We shall suppose that U has at least one critical point so that the minimum of the potential is at zero. If there is at least one other critical point then there will be at least two possible vacua. If A has one vacuum value as, say, $x \to -\infty$ and another one as $x \to +\infty$ then there must be a domain wall between. Let us look for a domain wall solution to the A equations of motion for A independent of y and z, assuming periodicity in the y and z directions in order to achieve finite total energy. The energy per unit area $\mathcal{E}$ can then be written as [17]

$$\begin{aligned}\mathcal{E} &= \int dx \left[|\dot{A}|^2 + |\partial_x A|^2 + |U'|^2\right] \\ &= \int dx \left[|\dot{A}|^2 + |\partial_x A - e^{i\alpha}\overline{U'(A)}|^2\right] \;+\; \mathcal{R}e(e^{-i\alpha}\mathcal{T}) ,\end{aligned} \tag{1.12}$$

where

$$\mathcal{T} = 2\big|U(A)\big|_{x=-\infty}^{x=\infty} \tag{1.13}$$

is a complex *topological* charge (depending only on the boundary conditions at $x = \pm\infty$). By choosing $\alpha = \arg\mathcal{T}$ we deduce the energy density bound

$$\mathcal{E} \geq |\mathcal{T}| , \tag{1.14}$$

with equality for a static solution satisfying the *first-order* equation

$$\partial_x A = e^{i\alpha}\overline{U'(A)} . \tag{1.15}$$

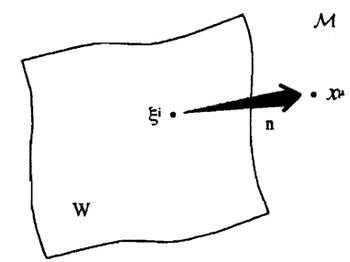
Multiplying by U' and integrating over x we find that $\mathcal{T} = -e^{i\alpha}2\int_{-\infty}^{\infty}dx\mathcal{L}(x)$, where $\mathcal{L}(x) = \left|U'(A(x))\right|^2$ is the Lagrangian density for the solution $A(x)$ of (1.15), so that α is indeed the argument of $\mathcal{T}$.

We now wish to consider [18] what might be the effective action governing the dynamics of fluctuations of this domain wall away from the static configuration assumed above. Let $X^\mu(\xi)$ be the locus in spacetime of the wall's worldvolume, W. The cartesian coordinates of a point p near W can be written as

$$x^\mu = X^\mu(\xi) + xn^\mu(\xi) , \tag{1.16}$$

where x is the perpendicular distance of p from W and n^μ are the components of an outward normal, n, to W passing through p and the point on W with coordinates ξ^i, as shown in Fig. 1.1.

Fig. 1.1: Coordinates of a point in $\mathcal{M}$ near W.



We must suppose that x is less than the radii of curvature of W in order to ensure the uniqueness of n. The metric of $\mathcal{M}$ at p is

$$ds^2 = \eta_{\mu\nu}\, dx^\mu dx^\nu . \tag{1.17}$$

Now (1.16) provides a change of coordinates from x^μ to (ξ^i, x). In terms of these new coordinates the metric is

$$ds^2 = \left[M_{ij} - 2x\Omega_{ij} + O(x^2)\right]d\xi^i d\xi^j + (dx)^2 , \tag{1.18}$$

where

$$\Omega_{ij} = n\cdot X_{,ij} \tag{1.19}$$

are the components of the extrinsic curvature Ω, or second fundamental form, of W as a hypersurface in $\mathcal{M}$. Thus, if R^{-1} is the lowest eigenvalue of Ω then

$$ds^2 = M_{ij}d\xi^i d\xi^j + (dx)^2 + O(x/R) , \tag{1.20}$$

and the volume element of $\mathcal{M}$ is therefore

$$\sqrt{-\det M}\big(1 + O(x/R)\big)d^3\xi dx . \tag{1.21}$$

In these new coordinates the locus of W is $x = 0$. Neglecting the domain wall's curvature we may assume that it is produced by the static scalar field configuration $A(x)$. Then, to order (x/R), $\mathcal{L}(A)$ is a function only of x and so

$$\begin{aligned} S &= \left[\int dx \mathcal{L}(x)\right] \int d^3\xi \sqrt{-\det M}(1 + O(x/R)) \\ &= -|T| \int d^3\xi \sqrt{-\det M}(1 + O(x/R)) \ . \end{aligned} \tag{1.22}$$

Neglecting the O(x/R) terms we arrive at (1.2) with $T = |\mathcal{T}|$. Note that the terms neglected can be expected to include not only corrections proportional to the extrinsic curvature but also interaction terms between the domain wall and the scalar field A. It is a curious feature of extended object solutions of *relativistic* field theories that, to leading order, the dynamics of the object decouples from that of the fields from which it is composed. The difficulty with the non-relativistic limit mentioned previously is presumably a reflection of the fact that such a decoupling does not happen for extended objects in non-relativistic field theories (e.g. vortices in superfluids). This is reminiscent of the fact that the discovery of the relativistic invariance of Maxwell's equations made the ether hypothesis unnecessary.

Some interesting new issues are raised by another type of scalar field theory with domain walls [19]. The Lagrangian density has the form

$$\mathcal{L} = \frac{1}{2}(\partial\phi^I \cdot \partial\phi^J - m^2 k^I(\phi)k^J(\phi))g_{IJ}(\phi) \ . \tag{1.23}$$

The scalar fields constitute a map $\phi : \mathcal{M} \to M$, from spacetime to an n-dimensional target space M with coordinates $\{\phi^I,\ I = 1, 2 \ldots, n\}$ and metric g_{IJ}. M is assumed to have a Killing vector k, and k^I are its components. A particularly interesting case, again motivated by supersymmetry, is (M, g) *hyper-Kähler* and k *triholomorphic*. These terms require some explanation. A hyper-Kähler manifold has the following properties:

(i) A quaternionic structure. That is, a triplet of complex structures $\mathbf{J}(\phi)$ obeying the algebra of the quaternions,

$$(J^a)_I{}^K (J^b)_K{}^J = -\delta^{ab}\delta_I{}^J + \varepsilon^{abc}(J^c)_I{}^J \tag{1.24}$$

(ii) A metric g that is Hermitian with respect to all three complex structures. That is, the second rank tensors

$$\mathbf{\Omega}_{IJ} = (\mathbf{J})_I{}^K g_{KJ} \tag{1.25}$$

are *antisymmetric*. They can therefore be taken to be the components of a triplet of two-forms $\mathbf{\Omega}$, called the *Kähler two-forms*.

(iii) The Kähler two forms are closed. That is, $d\mathbf{\Omega} = 0$.

A triholomorphic Killing vector k is one for which $\mathcal{L}_k\mathbf{\Omega} = 0$, where $\mathcal{L}_k$ is the Lie derivative with respect to k. Since $\mathcal{L}_k = di_k + i_k d$, where i_k indicates the interior product with k (i.e. contraction) we have $d(i_k\mathbf{\Omega}) = 0$, i.e. $i_k\mathbf{\Omega}$ is a triplet of closed one-forms on M. Its pullback $f^*(i_k\mathbf{\Omega})$ is a triplet of closed one-forms on $\mathcal{M}$. Again assuming independence of the fields on y and z, we identify

$$\mathbf{Q} = \int_{-\infty}^{\infty} dx\, \phi'^I k^J \mathbf{\Omega}_{IJ} \tag{1.26}$$

as a *topological* 3-vector charge, where the prime now indicates differentiation with respect to x. There is also, of course, the Noether charge

$$Q_0 = \int_{-\infty}^{\infty} dx\, \dot{\phi}^I k^J g_{IJ} \ , \tag{1.27}$$

associated with invariance of (1.23) under $\delta\phi^I \propto k^I(\phi)$.

As in the previous example we now write the energy density as

$$\begin{aligned} E &= \int_{-\infty}^{\infty} dx\, \frac{1}{2} g_{IJ}(\dot{\phi}^I\dot{\phi}^J + \phi'^I\phi'^J + m^2 k^I k^J) \\ &= \int_{-\infty}^{\infty} dx \left\{\frac{1}{2} g_{IJ}(\phi'^I - m(\mathbf{n}\cdot\mathbf{J})^I{}_K k^K)(\phi'^J - m(\mathbf{n}\cdot\mathbf{J})^J{}_L k^L)\right. \\ &\qquad \left. + \frac{1}{2} g_{IJ}(\dot{\phi}^I - mn_0 k^I)(\dot{\phi}^J - mn_0 k^J)\right\} \\ &\quad + m(n_0 Q_0 + \mathbf{n}\cdot\mathbf{Q}) \ , \end{aligned} \tag{1.28}$$

where $(n_0, \mathbf{n})$ is a *unit* Euclidean 4-vector, i.e. $n_0^2 + \mathbf{n}\cdot\mathbf{n} = 1$. By choosing $n = (n_0, \mathbf{n})$ parallel to $Q = (Q_0, \mathbf{Q})$ we obtain from (1.28) the energy bound

$$E \geq m|Q| = m\sqrt{Q_0^2 + \mathbf{Q}\cdot\mathbf{Q}} \ , \tag{1.29}$$

which is saturated by solutions of the first-order equations

$$\dot{\phi}^I = mn_0 k^I \qquad \phi'^I = m(\mathbf{n}\cdot\mathbf{J})^I{}_J k^J \ . \tag{1.30}$$

Conversely, any solution of eqs. (1.30) has the property that the 4-vector Q is *parallel* to the 4-vector n. This can be shown by substitution of eqs. (1.30) into (1.26) and (1.27) and use of the quaternion algebra satisfied by the three complex structures.

Consider now the special case for which (M, g) is the four-dimensional Eguchi-Hanson manifold. There exist coordinates $(\phi^0, \boldsymbol{\phi})$ for which the metric takes the form [20]

$$ds^2 = \Phi^{-1}(d\phi^0 + \boldsymbol{\omega}\cdot d\boldsymbol{\phi})^2 + \Phi d\boldsymbol{\phi}\cdot d\boldsymbol{\phi} \ , \tag{1.31}$$

where

$$\Phi = \frac{1}{2}\left[\frac{1}{|\boldsymbol{\phi} - \boldsymbol{\phi}_0|} + \frac{1}{|\boldsymbol{\phi} + \boldsymbol{\phi}_0|}\right] \tag{1.32}$$

and $\boldsymbol{\omega}$ is a solution to $\nabla \times \boldsymbol{\omega} = \pm\nabla\Phi$ (in the usual notation of Euclidean 3-vector calculus). The three closed Kahler 2-forms are

$$\mathbf{\Omega} = (d\phi^0 + \boldsymbol{\omega}\cdot d\boldsymbol{\phi})d\boldsymbol{\phi} - \Phi d\boldsymbol{\phi} \times d\boldsymbol{\phi} \ , \tag{1.33}$$

where the wedge product of forms is understood. The metric (1.31) is singular at $\boldsymbol{\phi} = \pm\boldsymbol{\phi}_0$ but this is only a coordinate singularity if we make the identification $\phi^0 \sim \phi^0 + 2\pi$. The triholomorphic Killing vector is $\partial/\partial\phi^0$, except at the points $\boldsymbol{\phi} = \pm\boldsymbol{\phi}_0$ where ϕ^0 is not defined.

Clearly, k vanishes at these points. The scalar field potential therefore has two isolated minima at $\boldsymbol{\phi} = \pm\boldsymbol{\phi}_0$ and we expect there to exist kink-type solutions interpolating between them with topological charge $\mathbf{Q} = 2\boldsymbol{\phi}_0$. To find them we look for solutions to eqs. (1.30), which now read

$$\dot{\phi}^0 = mn_0 \qquad \dot{\boldsymbol{\phi}} = 0 \qquad \phi'^0 = 0 \qquad \boldsymbol{\phi}' = m\Phi^{-1}\mathbf{n} \ . \tag{1.34}$$

For $\mathbf{n}$ parallel to $\boldsymbol{\phi}_0$ they have the finite energy solutions

$$\phi^0 = \varphi + mn_0 t \qquad \boldsymbol{\phi} = \boldsymbol{\phi}_0 \tanh\left(m|\mathbf{n}|(x - x_0)\right) \ , \tag{1.35}$$

where x_0 and φ are constants. The novel features here are:

(i) In addition to the static solutions there are also stable *time-dependent* solutions with Noether charge $Q_0 = (n_0/|\mathbf{n}|)|\mathbf{Q}|$, and hence energy

$$E = \frac{m}{\sqrt{1 - n_0^2}}|\mathbf{Q}| \ . \tag{1.36}$$

(ii) In addition to depending on a position x_0 in space, the kink solution (1.35) also depends on the additional parameter φ.

The latter point has implications for the effective action because in addition to its motion in space the domain wall's motion in the 'internal' space parametrized by φ must be taken into account. It is not difficult to see how this should be done. Although we started with a $d = 4$ field theory, it can be viewed as a $d = 5$ theory with a spacetime of the form $\mathcal{M} \times S^1$ and a prescribed dependence of the fields on the extra coordinate. Specifically, starting from the Lagrangian (1.23) but *without a potential*, and taking x^5 to be the S^1 coordinate, the Lagrangian with the potential is found on imposing

$$\frac{\partial}{\partial x^5}\phi^I = mk^I \ , \tag{1.37}$$

which is consistent with (1.35) if $\varphi = m^{-1}x^5$. Motion in the internal S^1 is therefore the same as motion in the extra dimension. A time-dependent kink solution can be viewed as static solution that has been boosted in the extra dimension; the square root factor in (1.36) is just the usual relativistic factor in the expression for the energy of a moving particle. The effective action of the domain walls in this model should therefore be

$$S = -T\int d^3\xi\,\sqrt{-\det(M_{ij} + m^2\partial_i\varphi\partial_j\varphi)} \ . \tag{1.38}$$

2. Solitons and Supersymmetry

A soliton is a solution of a classical field theory with a finite localized energy density, so that at scales large compared to its size it appears to be a point particle. In the original meaning of the word 'soliton' there was also a condition that the particles scatter elastically with an S-matrix determined entirely by phase shifts. This is possible only in dimension 2 but may be good approximation for low-energy scattering in higher dimensions. In any case we shall here use the term in its looser sense of 'particle-like object'. Note that here we avoid the terminology 'extended' particle (which indicates that it is not actually point-like but only apparently so at large wavelengths) because of possible confusion with extended objects.

In the previous lecture we saw how, for a particular field theory, a lower bound on the energy could be deduced by expressing it as a sum of squares. Not all field theories with solitons have this property, but those that do have the simplifying feature that the soliton solutions can be found by solving associated *first-order* equations. The possibility of expressing the energy as a sum of squares is also a feature of supersymmetric field theories. In fact, any purely bosonic field theories with soliton solutions saturating a Bogomolnyi-type bound is *supersymmetrizable* in the sense that there exists a supersymmetric theory of which it is the purely bosonic sector. In this supersymmetric theory the first-order equations that are solved by the soliton field configuration can alternatively be deduced from the requirement that some of the supersymmetry of the vacuum solution be preserved. This connection is well-illustrated by the supersymmetric extension of the first model considered in the previous lecture, i.e. the Wess-Zumino model. Since we are at present interested in solitons rather than domain walls, we immediately dimensionally reduce to $d = 2$ by requiring that no field depend on y or z. It is then convenient to introduce the notation

$$\partial_{\ddagger} = \partial_t + \partial_x \qquad \partial_{=} = \partial_t - \partial_x \ . \tag{2.1}$$

These derivatives are Lorentz covariant; they scale with weight $+2$ or -2, respectively, under Lorentz transformations. The scale weight, or Lorentz 'charge', is indicated by a suffix with that number of plus or minus signs. The fermion partners of the complex scalar field A, which are for $d = 4$ the four real components of a Majorana spinor, may here be taken to be two *complex* anticommuting chiral spinor fields ψ_+ and ψ_- (the chirality being given by the sign of the Lorentz charge). The $d = 2$ action is

$$\begin{aligned} S = \int dtdx\Big[&\partial_{=}A\partial_{\ddagger}\bar{A} - |U'(A)|^2 \\ &+ i\bar{\psi}_+\partial_{=}\psi_+ + i\bar{\psi}_-\partial_{\ddagger}\psi_- - iU''(A)\psi_+\psi_- - i\overline{U''(A)}\bar{\psi}_+\bar{\psi}_-\Big] \ . \end{aligned} \tag{2.2}$$

The supersymmetry transformations are

$$\begin{aligned} \delta A &= i\epsilon_+\psi_- + i\epsilon_-\psi_+ \\ \delta\psi_+ &= -\partial_{\ddagger}A\bar{\epsilon}_- - \overline{U'(A)}\epsilon_+ \\ \delta\psi_- &= -\partial_{=}A\bar{\epsilon}_+ + \overline{U'(A)}\epsilon_- \ . \end{aligned} \tag{2.3}$$

Note that we have one complex spinor parameter of one chirality and another of the other chirality, which means that we have an action with (2.2) supersymmetry, as expected from its $d = 4$ origin. The corresponding (real) Noether charges may be parametrized by two (real) phases β and γ as follows:

$$\begin{aligned} Q_+(\beta) &= \frac{1}{\sqrt{2}}\Big\{e^{-\frac{i}{2}\beta}\int_{-\infty}^{\infty} dx\big[\partial_{\ddagger} A\bar{\psi}_+ - U'(A)\psi_-\big] \;+\; c.c.\Big\} \\ Q_-(\gamma) &= \frac{1}{\sqrt{2}}\Big\{e^{-\frac{i}{2}\gamma}\int_{-\infty}^{\infty} dx\big[\partial_{=} A\bar{\psi}_- + U'(A)\psi_+\big] \;+\; c.c.\Big\}\,. \end{aligned} \tag{2.4}$$

From (2.2) we see that $\dot{A} = \partial L/\partial\dot{\bar{A}} := \overline{\Pi_A}$, where Π_A is the variable conjugate to A. Making the replacement $\dot{A} \to \overline{\Pi_A}$ and using the (anti)commutation relations

$$[\Pi_A\,, A] = -i \qquad \{\psi_+, \bar{\psi}_+\} = \{\psi_-, \bar{\psi}_-\} = 1\,, \tag{2.5}$$

from canonical quantization of (2.2) (setting $\hbar = 1$), we find that

$$\begin{aligned} \big(Q_+(\beta)\big)^2 = H + P \qquad\qquad \big(Q_-(\gamma)\big)^2 = H - P \\ \{Q_+(\beta), Q_-(\gamma)\} = 2\mathcal{R}e\big(e^{-\frac{i}{2}(\beta+\gamma)}\mathcal{T}\big)\,, \end{aligned} \tag{2.6}$$

where H is the Hamiltonian, P the total momentum and $\mathcal{T}$ the topological charge introduced in the previous lecture, which now appears in the (2,2) supersymmetry algebra as a *central charge*. Now consider the particular (hermitian) supersymmetry charge

$$S = \frac{1}{\sqrt{2}}\big(Q_+(\alpha) + Q_-(\alpha)\big)\,, \tag{2.7}$$

where $\alpha = \arg \mathcal{T}$. This charge satisfies

$$S^2 = H - T\,, \tag{2.8}$$

where $T = |\mathcal{T}|$, as before. As the left hand side is a positive definite operator we deduce that

$$H \geq T\,. \tag{2.9}$$

If $|Sol\rangle$ is the eigenstate of H representing the static soliton, and if we identify $\mathcal{E}$ as its eigenvalue, then we recover from (2.9) the energy bound of (1.6). Moreover, as we saw previously, this bound is *saturated* by the static soliton solution so

$$S^2|Sol\rangle = 0 \;\Rightarrow\; \langle Sol|S^2|Sol\rangle = 0 \;\Rightarrow\; \big\|S|Sol\rangle\big\|^2 = 0\,, \tag{2.10}$$

the last step following from the Hermiticity of S. Assuming a positive definite Hilbert space we conclude that

$$S|Sol\rangle = 0\,. \tag{2.11}$$

It is not difficult to show that there are two such hermitian charges annihilating $|Sol\rangle$ so precisely half of the supersymmetry is preserved by the soliton. This can also deduced by

an analysis of the transformation laws (2.3) in the soliton background. In this background, $\psi_+ = \psi_- = 0$ so δA vanishes. Since $\partial_{\ddagger} A = -\partial_{=} A = \partial_x A = e^{i\alpha}\overline{U'(A)}$, for the soliton background, we see from (2.3) that $\delta\psi_+ = \delta\psi_- = 0$ provided that $\epsilon_+ + e^{i\alpha}\bar{\epsilon}_- = 0$. This is one complex condition on two complex parameters, so precisely *half the supersymmetry is broken by the soliton, and half preserved.* This of course remains true if we now re-interpret the $d = 2$ soliton as a string in $d = 3$ or a membrane in $d = 4$.

The above analysis was introduced in [7] in the context of a (1,1) supersymmetric model for which the topological charge is real. The possibility of a complex central charge in the (2.2) model introduces some interesting new features [21,16,22] when we consider whether the collision of two solitons of masses $M_1 = |\mathcal{T}_1|$ and $M_2 = |\mathcal{T}_2|$ can produce a third soliton of mass $M_3 = |\mathcal{T}_3|$. By topological charge conservation, $\mathcal{T}_3 = \mathcal{T}_1 + \mathcal{T}_2$, so

$$M_3 = |\mathcal{T}_1 + \mathcal{T}_2| \leq M_1 + M_2\,, \tag{2.12}$$

with equality only if the phases of $\mathcal{T}_1$ and $\mathcal{T}_2$ are equal. If the phases differ then $M_3 < M_1 + M_2$ and the third soliton is stable in the sense that energy is required to disassociate it. Given the existence of this third soliton the stability of the first two requires that $M_1 < M_2 + M_3$ and $M_2 < M_3 + M_1$, i.e that *the phases of all three solitons differ.* This is not possible for real charges, for which at least two have the same sign and hence the same phase, but for complex charges we may have a situation in which any two of three types of soliton can fuse to form the third. The simplest model for which this occurs has the superpotential

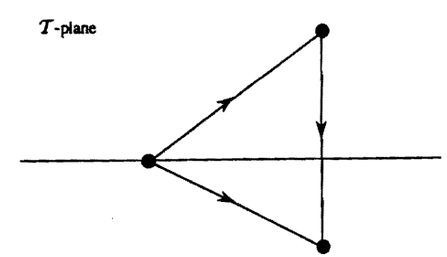
$$U(A) = A^4 - 4A\,. \tag{2.13}$$

This has three critical points; at the cube roots of unity. To each pair of critical points at A_a and A_b is associated the topological charge

$$\mathcal{T}_{ab} = 2\big[U(A_b) - U(A_a)\big] = -6(A_b - A_a)\,. \tag{2.14}$$

There are therefore three *possible* topological charges, which form an equilateral triangle, as shown in Fig. 2.1.

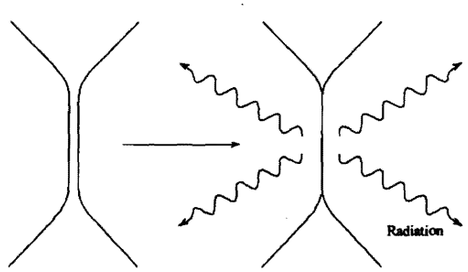
Fig. 2.1: Topological charges for the superpotential $U = A^4 - 4A$.



The three vertices of this triangle represent the three different vacua. For this example symmetry considerations ensure that each vertex is indeed connected to each of the other two

by a soliton solution. Hence any two solitons could *in principle* fuse to form a third of lower energy, radiating quanta of the scalar field A in the process. However, it happens that the model with superpotential (2.13) belongs to a class of quantum field theory for which there is no radiation produced by the collision of solitons and for which the S-matrix for soliton scattering can be found *exactly* [21]. It is crucial for this integrability that the topological charge triangle be equilateral. For a generic fourth-order superpotential the triangle formed by the three possible topological charges will not be equilateral, but in this case there is no guarantee that there is a soliton solution corresponding to every leg of the triangle. Further analysis [17] shows that for some ranges of the parameters defining the superpotential there are only two distinct solitons, so that two of the three vacua are not directly connected to each other, but for other ranges of these parameters, which of course include the special case of (2.13), all three solitons exist. In the latter case the process of soliton fusion described above is possible and since the model is integrable only for the special case of (2.13) there will generically be a non-zero probability for soliton fusion to occur. In the context of the original $d = 4$ Wess-Zumino model this process of soliton fusion corrsponds to the fusion of two domain walls to form a third one of lower tension. This produces a system of *intersecting* walls [17] (although it seems that when gravity is included this is no longer possible [23]). This is more easily visualized for strings in $d = 3$ as shown in Fig. 2.2:

Fig. 2.2: Fusing of two strings (in $d = 3$) to produce intersections.



Similar remarks apply to the second model considered in the previous lecture, i.e the hyper-Kähler sigma model with a potential given by the length squared of a triholomorphic Killing vector, although there are several differences. Firstly, the maximal supersymmetry is now (4,4). Secondly, the Noether charge Q_0 *also* appears as a central charge in the supersymmetry algebra, which is not surprising given that this charge can be considered, as we saw, to be the momentum in an extra dimension.

An example of a soliton in a $d > 2$ field theory is provided by the vortex of the $d = 3$ Abelian-Higgs model. The maximally supersymmetric extension has an $N = 4$ supersymmetry. It can be obtained by dimensional reduction from $d = 6$, where the vortex solution has the interpretation of a threebrane [24], but we shall remain here in $d = 3$. The bosonic Lagrangian density of the maximally supersymmetric Abelian-Higgs model is

$$\mathcal{L} = (D_\mu\phi_1)\overline{(D^\mu\phi_1)} + (D_\mu\phi_2)\overline{(D^\mu\phi_2)} - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - V(\phi_1,\phi_2) \ , \tag{2.15}$$

where ϕ_1 and ϕ_2 are two complex scalar fields of charge e, i.e. $D_\mu\phi = \partial_\mu + ieA_\mu\phi$, and A_μ is an Abelian gauge potential with field strength $F_{\mu\nu}$. The potential V is given by

$$V = \frac{1}{2}e^2\Big[(|\phi_1|^2 - \lambda^2)^2 + |\phi_2|^2(|\phi_2|^2 + \lambda^2)\Big] \ , \tag{2.16}$$

in terms of another (real) parameter λ, which we may assume to be positive. The potential is minimized when

$$\phi_2 = 0 \qquad \phi_1 = \lambda e^{i\theta} \ , \tag{2.17}$$

for arbitrary phase $\theta(x)$.

The Hamiltonian associated with L is

$$\begin{aligned} H = \int d^2x\Big\{ \sum_r |\pi^r|^2 + \frac{1}{2}\mathbf{E}\cdot\mathbf{E} + \sum_r |(\nabla + ie\mathbf{A})\phi_r|^2 + \frac{1}{2}B^2 + V \\ + A_0\Big[\nabla\cdot\mathbf{E} - ie\sum_r(\pi^r\phi_r - \bar{\pi}^r\bar{\phi}_r)\Big]\Big\} \ , \end{aligned} \tag{2.18}$$

where $B = F_{12}$ is the magnetic field, $\mathbf{E}$ (the electric field) is the variable conjugate to $\mathbf{A}$, and π^r are the variables conjugate to ϕ_r, $r = 1,2$. For configurations with $\mathbf{E} = 0$, $\pi^r = 0$ and $\phi_2 = 0$ the Gauss law constraint is satisfied and the energy is

$$\begin{aligned} \mathcal{E} &= \int d^2x\{\mathbf{D}\phi_1\cdot\overline{\mathbf{D}\phi_1} + \frac{1}{2}B^2 + \frac{1}{2}e^2(|\phi_1|^2 - \lambda^2)^2\} \\ &= \int d^2x\Big\{|(D_1 \pm iD_2)\phi_1|^2 + \frac{1}{2}\big[B \mp e(|\phi_1|^2 - \lambda^2)\big]^2\Big\} \ + \ T \ , \end{aligned} \tag{2.19}$$

where (D_1, D_2) are the components of $\mathbf{D}\phi = (\nabla + ie\mathbf{A})\phi$, and

$$T = \pm e\oint d\mathbf{l}\cdot\big[(|\phi_1|^2 - \lambda^2)\mathbf{A} + i\phi_1\nabla\bar{\phi}_1\big] \tag{2.20}$$

is a topological charge since the line integral is taken over the circle at spatial infinity. Assuming that ϕ_1 tends to one of its vacuum values parametrized by the angular variable θ as $|x| \to \infty$, we find that $T = \pm e\lambda^2\oint d\mathbf{l}\cdot\nabla\theta$. We therefore deduce [25] the energy bound

$$\mathcal{E} \geq (2\pi)e\lambda^2|\nu| \ , \tag{2.21}$$

where ν is an integer. This bound is saturated by solutions of the first-order equations

$$(D_1 + \mathrm{sgn}(\nu)D_2)\phi_1 = 0 \qquad B = \mathrm{sgn}(\nu)e(|\phi_1|^2 - \lambda^2) \ . \tag{2.22}$$

Vortex solutions of (2.22) exist for any integer ν. It can be shown that the topological charge $T = (e\lambda^2)\nu$ appears in the $d = 3$, $N = 4$, supersymmetry algebra as a central charge; if Q^a_α, $a = 1,\ldots,4$ are the four real two-component spinor charges, then

$$\{Q^a_\alpha, Q^b_\beta\} = \delta^{ab}(\Gamma^\mu C)_{\alpha\beta}P_\mu + T\,\Omega^{ab}C_{\alpha\beta} \tag{2.23}$$

where C is the (antisymmetric) charge-conjugation matrix, and Ω^{ab} are the components of a real antisymmetric matrix (the details for a different model, but one with the same algebra, may be found in [26]). For this model there is a single real topological charge but the consequences are otherwise the same as before, i.e. the algebra (2.23) implies the bound (2.21) and the vortex solutions, which saturate the bound, break half the supersymmetry.

We now turn to the question of the effective action for solitons in supersymmetric field theories. We shall concentrate here on the $d = 3$ vortices since the $d = 2$ case has some non-generic features. One approach, reviewed in [1], starts from the observation that since a soliton breaks translation invariance, its low-energy dynamics should be governed by an associated Goldstone variable $\mathbf{X}(t)$. This can be promoted to the Lorentz-vector $X^\mu(t)$ by requiring worldline parametrization invariance of the action because the unphysical X^0 variable can then be 'gauged away'. The usual particle action is then found to be the lowest dimension one with the required symmetry properties. These ideas can be put on a firmer foundation by means of the theory of non-linear realizations of spacetime symmetries. We shall not pursue this direction here because the details are rather involved, especially in the supersymmetric case [27,28,29]; instead we shall rely on educated guesswork. A first guess at the effective action in the supersymmetric case might be

$$S = -m \int dt \sqrt{\omega \cdot \omega} \,, \tag{2.24}$$

where

$$\omega^\mu = \dot{X}^\mu - i\bar{\theta}_a \Gamma^\mu \dot{\theta}_a \tag{2.25}$$

since we expect fermionic Goldstone variables θ_a, and ω^μ is invariant under the $N = 4$ supersymmetry transformations

$$\delta X^\mu = i\bar{\epsilon}_a \Gamma^\mu \theta_a \qquad \delta\theta_a = \epsilon_a \,. \tag{2.26}$$

This cannot be right because the supersymmetry Noether charges of the action (2.24) obey the usual supersymmetry algebra, without a central charge. In fact, *any* Lagrangian invariant under (2.26) will have this property. In order to circumvent it we need a Lagrangian that is *not* invariant, but since the *action* must remain invariant the Lagrangian is limited to change by a total derivative. A term of this type which also has the desired effect of modifying the algebra of charges is called a Wess-Zumino (WZ) term (no connection with the Wess-Zumino model, other than the authors). It happens that there is a unique WZ term of the same dimension as (2.24). By its inclusion we arrive at the massive superparticle action

$$S = -m \int dt \sqrt{-\omega \cdot \omega} \; + \; iT\Omega^{ab} \int dt \bar{\theta}_a \dot{\theta}_b \,. \tag{2.27}$$

It can be shown that the supersymmetry charges of this action satisfy the algebra (2.23), as required. There remains one point to check. We know that when the bound $m \geq T$ is saturated only half of the supersymmetry is broken so only half the components of θ_a are needed as Goldstone variables. Remarkably, this is taken into account by the superparticle action as a result of a fermionic gauge invariance, or 'kappa symmetry', which appears precisely when $m = T$ (as explained in [1] for general p).

Once we have understood the principles that underlie the construction of the effective action we can run the previous arguments in reverse. That is, we can restrict the possibilities for the existence of solitons in supersymmetric field theories by the requirement that there must exist an effective action for them. This restriction is disappointingly weak; for example, the massive superparticle action exists in *any* spacetime dimension (but possibly requiring extended supersymmetry). However, once we take into account the fact that a soliton solution of a d-dimensional field theory can generally be used to represent a p-dimensional object in a higher-dimensional field theory for some $p \geq 1$ we may also require the existence of an effective action for these objects. This turns out to be very restrictive. By arguments similar to those used above, and as explained in [1], the effective action analogous to (2.27) for p-dimensional extended objects in a d-dimensional spacetime is the super p-brane action

$$S = -T \int d^{p+1}\xi \sqrt{-\det(\omega_i^\mu \omega_j^\nu \eta_{\mu\nu})} + TS_{WZ} \,, \tag{2.28}$$

where

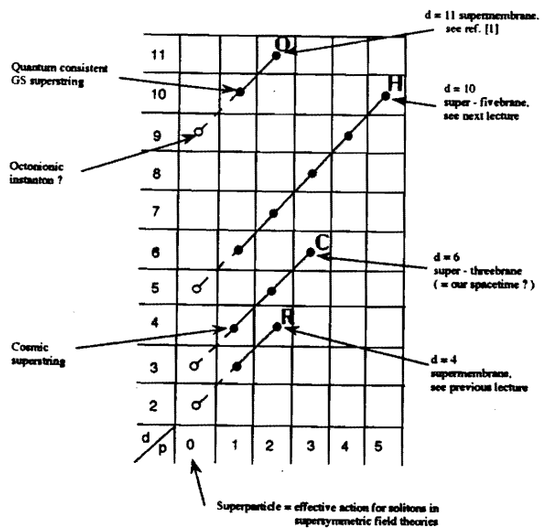
$$\omega_i^\mu := \partial_i X^\mu - i\bar{\theta}\Gamma^\mu \partial_i \theta \tag{2.29}$$

generalizes (2.25). I refer to [1] for more details. Here it will suffice to recall that the WZ term can be constructed only for specific values of (p, d) [13], as summarized in the 'Brane-Scan' of Fig.2.3. Note the four sequences labelled **R**,**C**,**H**,**O** with common co-dimension $d - p - 1 = 1, 2, 4, 8$, respectively. A feature of the Brane-Scan not shown on the figure is that the number N of supersymmetries (counting each "minimal" spinor as one) is restricted to $N > 1$ for $p > 1$. The maximal number of supersymmetries within each of the **R**,**C**,**H**, and **O** sequences, *counting separately each real spinor component*, is therefore $4, 8, 16$ and 32 respectively.

The particular field theories that we have been studying provide explicit examples of the **R** and **C** sequences. A candidate for the **H** sequence is the four-dimensional Euclidean Yang-Mills (YM) instanton, viewed as a soliton in $d = 5$. In agreement with the brane-scan, its maximally supersymmetric extension is obtainable by dimensional reduction from $d = 10$, where it can be interpreted as a fivebrane with a YM core [4]. This example has an unsatisfactory feature however in that the $d = 5$ instantonic soliton has an arbitrary size and perturbations away from an exact static solution will cause it to spread out indefinitely or shrink to a singularity. This can be overcome by considering instead the $d = 4$ YM/Higgs monopole, which has a definite size determined by the vacuum value of the Higgs field. The existence of the $d = 4$ monopole (in the BPS limit, where the YM/Higgs system is supersymmetrizable) does not contradict the Brane-Scan because, like the Q-kink solution considered in lecture 1, its effective action depends on an additional S^1 variable which can be interpreted as an extra dimension. The $d = 4$ monopole may therefore be regarded as a soliton in a $d = 5$ spacetime of the form $\mathcal{M}_4 \times S^1$.

A candidate for a $d = 9$ soliton of the **O** sequence is the 'octonionic instanton' of eight dimensional Euclidean SO(8) YM theory [30], which can be viewed as a string in $d = 10$ [31], but this has rather different properties. For instance, it breaks 15/16 of the supersymmetry, rather than half, and it cannot account for the eleven-dimensional supermembrane.

Fig.2.3: The Brane Scan



An alternative way to understand the restrictions on (p, d) imposed by the Brane-Scan is as a consequence of (i) *worldvolume supersymmetry*, and (ii) *the absence of worldvolume fields of spin* $> 1/2$. The worldvolume supersymmetry is at first surprising because only *spacetime* supersymmetry is built into the construction described above; the worldvolume supersymmetry emerges only after fixing the gauge invariances. I refer to [18] for details of how this happens for the $d = 4$ supermembrane, for which the worldvolume superspace form of the action has recently been found [28]. On second thoughts the worldvolume supersymmetry should not be too surprising because the fact that a soliton breaks only half the supersymmetry implies that the gauge-fixed action must be invariant under some 'linearly-realized' fermionic symmetry. By 'linear' I mean that the transformations do not contain field-independent terms so that, in particular, the fermion field variations vanish when the fermion fields do. This is sufficient for us to be able to invoke the Haag-Lopusanski-Sohnius theorem to the effect that any fermionic symmetry must be supersymmetry. The requirement that worldvolume fields have spins $\leq 1/2$ follows directly from the assumption that all of them are Goldstone fields resulting from the spontaneous breakdown, *at the locus of the object's worldvolume*, of (super)symmetries, but this assumption is not always justifiable since worldvolume fields may also appear for topological reasons unrelated to symmetry, as we shall see in the following lecture. If worldvolume vectors or antisymmetric tensors are permitted then various new possibilities arise for $d = 10$. In fact, fivebrane [6] and threebrane [32,33] solutions of type II $d = 10$ supergravity theories have been found that would not be allowed by the brane-scan. Although their full effective actions have not

yet been constructed the field content is known and does include a worldvolume vector or antisymmetric tensor. There are no known examples for which there are higher worldvolume spins, but neither do I know of a theorem forbidding them; the absence of any example with worldvolume spin two is the major obstacle to an interpretation of our (3+1)-dimensional spacetime as the worldvolume of a threebrane embedded in a higher-dimensional spacetime

A feature of the vortex and other solitons that is absent in the simpler scalar field models is the existence of *multi-vortex* solutions, corresponding to $\nu > 1$. The phase of ϕ_1 increases by $2\pi\nu$ as the circle at spatial infinity is circumscribed once, which means that ϕ_1 must have ν zeros in the interior. The space of vortex solutions of charge ν is therefore the space of polynomials of order ν in one complex variable [34]. The coefficients of this polynomial are the moduli, i.e. coordinates, of the solution space. The Gauss-law constraint in (2.18) can in principle be solved for an infinite set of unconstrained momenta P_A conjugate to a set of gauge-invariant variables Q^A, the coordinates of the quotient space $\mathcal{Q}$ of all field configurations $(\phi_r, \mathbf{A})$ modulo gauge transformations. The kinetic term in the Hamiltonian will then take the form $T = g^{AB} P_A P_B$, which defines a metric on the space $\mathcal{Q}$. This induces a metric on the ν complex dimensional subspace of ν-vortex solutions. The effective action for (non-relativistic) multi-vortex solutions is therefore a sigma-model with the moduli space as the target space and the low-energy scattering of vortices is given by geodesic motion on this space [35]. This constitutes an interesting generalization of the ideas discussed above for one-soliton solutions (although it seems unlikely that it can be made relativistic).

The importance of supersymmetry in this context is that the effective action is then a *supersymmetric* sigma model with half the supersymmetry of the original field theory, the half that is preserved when the Bogomoln'yi bound is saturated (this has recently been demonstrated directly [36] for multi-solitons in the model of [26], following methods used in [37] for instantons). This fact has interesting consequences because extended supersymmetry imposes strong constraints on the target space metric. For the vortex solutions, for example, we started with a maximum of 8 supersymmetries (counting each component as one) so we expect an effective sigma model with 4 supersymmetries. But this number (corresponding to $(2,2)$ supersymmetry in $d = 2$) requires that the moduli space metric be Kähler, as indeed it is [34]. Similarly the maximally supersymmetric model for monopoles is the $N = 4$ YM theory which have a total of 16 supersymmetries. The effective sigma-model action will therefore have 8 supersymmetries (corresponding to $(4,4)$ supersymmetry in $d = 2$) and this implies [38] that the metric on the multi-monopole space must be hyper-Kähler, which is again known to be true [39].

3. String/fivebrane duality

There is a natural generalization of the super p-brane action that describes the dynamics of supersymmetric extended objects in the presence of background supergravity fields, as briefly explained in [1]. In particular, if there is a $(p+1)$-form gauge potential in the supergravity multiplet then it couples to the p-brane via a generalization of the Lorentz coupling of an electrically charged particle to the electromagnetic gauge potential, and this is all that we shall need here. It is customary in string theory to denote by B the (two-form) potential in the $d = 10$, $N = 1$, supergravity multiplet but for generality, and because the heterotic string theory case requires further modifications, let us here denote the $(p+1)$-form gauge potential by A. Thus we consider

$$A = \frac{1}{(p+1)!} dX^{\mu_1} \cdots dX^{\mu_{p+1}} A_{\mu_1 \cdots \mu_{p+1}} \tag{3.1}$$

coupled to a p-brane via the worldvolume integral $q_e \int_W f^*(A)$, where q_e is an 'electric' charge density (i.e. charge per unit p-volume). This can be rewritten as the *spacetime* integral

$$\frac{1}{(p+1)!} \int d^d x \sqrt{-g}\, J_e^{\mu_1 \cdots \mu_{p+1}}(x) A_{\mu_1 \cdots \mu_{p+1}}(x) \ , \tag{3.2}$$

where the $(p+1)$th rank antisymmetric 'electric' current tensor J_e is given by

$$\sqrt{-g}(J_e)^{\mu_1 \cdots \mu_{p+1}}(x) = q_e \int_{t_i}^{t_f} dt \int d^p\sigma\, \delta^d\big(x - X(t,\sigma)\big) \varepsilon^{i_1 \cdots i_{p+1}} \partial_{i_1} X^{\mu_1} \cdots \partial_{i_{p+1}} X^{\mu_{p+1}} \ , \tag{3.3}$$

and g is the determinant of the spacetime metric; this current satisfies the conservation condition $\partial_\nu\big(\sqrt{-g} J_e^{\nu\mu_1 \cdots \mu_p}\big) \equiv 0$ except at the initial and final times t_i and t_f.

If we take the spacetime action for A to be

$$S = -\frac{1}{2(p+2)!} \int d^d x \sqrt{-g} F_{\mu_1 \cdots \mu_{p+2}} F^{\mu_1 \cdots \mu_{p+2}} \ , \tag{3.4}$$

where $F_{\mu_1 \cdots \mu_{p+2}} = (p+2)\partial_{[\mu_1} A_{\mu_2 \cdots \mu_{p+2}]}$ are the components of $F = dA$, then variation of the combined action with respect to A yields

$$\frac{1}{\sqrt{-g}} \partial_\mu\big(\sqrt{-g}\, F^{\mu\nu_1 \cdots \nu_{p+1}}\big) = (J_e)^{\nu_1 \cdots \nu_{p+1}} \ , \tag{3.5}$$

which we can rewrite as

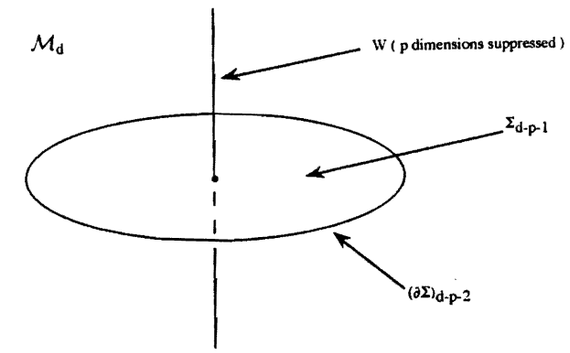
$$\star d \star F = J_e \ , \tag{3.6}$$

where J_e is the differential form with components $(J_e)_{\mu_1 \cdots \mu_{p+1}}$, and the star indicates the Hodge dual. This equation and the Bianchi identity $dF \equiv 0$ generalize to p-branes the electrodynamics of point particles [40]. The total charge density q_e is given by the integral

$$q_e = \int_{\Sigma_{d-p-1}} \star J_e \ , \tag{3.7}$$

where Σ_{d-p-1} is a $(d-p-1)$-dimensional spacelike subspace of $\mathcal{M}$ which intersects W once, as shown in Fig. 1.2 (but with p dimensions supressed):

Fig 3.1: Spacelike subspace intersects W on p-brane.



The formula (3.7) is an identity for the singular source assumed above (generalizing a point particle) but is valid generally. Using (3.6), q_e can be expressed as the surface integral

$$q_e = \int_{(\partial\Sigma)_{d-p-2}} \star F \ , \tag{3.8}$$

in direct analogy with Gauss' law in electrodynamics. As in electrodynamics, there is a generalization that allows for magnetic sources as well as electric ones [9,40]. The equations for F in the presence of both types of source are

$$\star d \star F = J_e \qquad \star\, dF = J_m \ , \tag{3.9}$$

where J_m is a $(d-p-3)$-form 'magnetic' current. In the absence of the electric source we could solve the first of equations (3.9) for a $(d-p-3)$-form, *magnetic dual*, potential $\tilde{A}$ by setting

$$\star F := G = d\tilde{A} \ . \tag{3.10}$$

This equation then becomes the Bianchi identity $dG \equiv 0$ while the second equation of (3.9) now reads $\star d \star G = J_m$. By comparison with (3.6) it can be seen that this is an equation for the potential $\tilde{A}$ in the presence of a magnetic source of dimension

$$\tilde{p} = d - p - 4 \ . \tag{3.11}$$

The total magnetic charge density (i.e. charge per unit $\tilde{p}$-volume) is given by the formula analogous to (3.8),

$$q_m = \int_{\Sigma_{p+3}} \star J_m = \int_{(\partial\Sigma)_{p+2}} \star G \ . \tag{3.12}$$

Equivalently,

$$q_m = \int_{(\partial\Sigma)_{p+2}} F \,. \tag{3.13}$$

As for monopoles in electrodynamics, quantum consistency requires the Dirac-like quantization condition [9,40,41]

$$\frac{q_e q_m}{2\pi\hbar} = \text{integer} \,. \tag{3.14}$$

The formula (3.11) confirms that the magnetic objects dual to electric particles in four-dimensional electrodynamics are particle-like, but we also learn that *the magnetic dual of a string in ten dimensions is a fivebrane.* The usual conformal gauge worldsheet action for the bosonic string, with tension $T = 1/2\pi\alpha'$, in a metric and antisymmetric tensor background is the sigma-model action

$$S = \frac{1}{2\pi\alpha'}\int d^2\xi\,\frac{1}{2}\Big[\delta^{ij}\partial_i X^\mu \partial_j X^\nu G_{\mu\nu} + \epsilon^{ij}\partial_i X^\mu \partial_j X^\nu B_{\mu\nu}\Big] \,. \tag{3.15}$$

This shows that in order to apply the quantization condition (3.19) to string theory we should set (e.g.) $q_e = 1$ and $\hbar = 2\pi\alpha'$. The magnetic charge of a fivebrane is therefore quantized in integer units of

$$Q \equiv \frac{1}{4\pi^2\alpha'}\int_{(\partial\Sigma)_3} H \,, \tag{3.16}$$

which need not vanish because H need not be *globally* exact.

In all the discussion so far we have supposed the magnetic source to be a higher-dimensional analogue of the *singular* Dirac magnetic monopole. Is there a string theory with a *non-singular magnetic fivebrane*, analogous to the 'tHooft-Polyakov monopole? As a first approach [5] to this question, we might take as our starting point the leading order effective ($d = 10$) supergravity/YM theory for the heterotic string, which can be found to by a one-loop sigma-model calculation [42] followed by a 'by hand' supersymmetrization. Instead, I shall begin with what might be described as a zeroth order approach in which the starting point is 'ordinary' $d = 10$ supergravity/YM theory, i.e. the Chapline-Manton theory [43], with gauge group $SO(32)$ (for simplicity I ignore the $E_8 \times E_8$ possibility). I shall make use of the formulation of this theory (but not all of the conventions) given in [44] (which also contains results on the supersymmetrization of the string-induced corrections, which we shall subsequently have to confront). After rescaling some of the fields, the bosonic part of the action in units for which the gravitational coupling constant is unity is

$$S = \int d^{10}x\,\sqrt{-g}e^{-2\phi}\left[R + 4(\partial\phi)^2 - \frac{1}{2.3!}H^{MNP}H_{MNP} - \frac{1}{8}\alpha'\mathrm{tr}(F^{MN}F_{MN})\right] \,, \tag{3.17}$$

where R is the scalar curvature of the $d = 10$ spacetime metric g_{MN} in spacetime coordinates $\{x^M,\ M = 0, 1 \ldots 9\}$, ϕ is the dilaton, F_{MN} is the field strength tensor of an $SO(32)$ YM potential and *tr* indicates a trace in the *vector* representation. The third rank antisymmetric tensor H_{MNP} defines the 'modified' field strength three-form

$$H = dB + \frac{1}{4}\alpha' K_3 \,, \tag{3.18}$$

where K_3 is the (Chern-Simons) three-form potential for the four-form

$$dK_3 = \mathrm{tr}(F \wedge F) \,. \tag{3.19}$$

The constant α' will later be identified as the inverse of 2π times the string tension, but in the present context it is just the inverse square root of the YM coupling constant. Note that H now satisfies the 'anomalous' Bianchi identity

$$dH = \frac{1}{4}\alpha'\mathrm{tr}(F \wedge F) \,. \tag{3.20}$$

The fermion fields of the theory are the gravitino ψ_M and gaugino χ, which are chiral, and the dilatino ϕ, which is antichiral. We shall need their supersymmetry transformation laws, which are

$$\begin{aligned} \delta\psi_M &= (\partial_M - \frac{1}{4}(\omega_+)_{MAB}\Gamma^{AB})\epsilon \\ \delta\lambda &= -\frac{1}{4}(\Gamma^M\partial_M\phi - \frac{\sqrt{2}}{12}\Gamma^{MNP}H_{MNP})\epsilon \\ \delta\chi &= \frac{1}{4}(\Gamma^{MN}F_{MN})\epsilon \,. \end{aligned} \tag{3.21}$$

where $\epsilon(x)$ is a $d = 10$ chiral spinor parameter, $\{\Gamma^A\}$ are the $d = 10$ Dirac matrices, and

$$(\omega_\pm)_{MAB} = \omega_{MAB} \pm \frac{1}{2}E_M{}^C H_{CAB} \tag{3.22}$$

is a Lorentz connection with torsion ($E_M{}^A$ is the zehnbein and ω_{MAB} the standard torsion-free connection).

Rather than attempt to solve the second order equations that follow from the action (3.17) we reduce the problem to solving a set of first-order equations, as in previous examples, by seeking field configurations that partially preserve the supersymmetry of the 'vacuum', by which we mean here the trivial configuration for which ϕ is constant, g_{MN} flat and all other fields vanish. In can be shown, as before, that such configurations automatically solve the full field equations. For a purely bosonic configuration to partially preserve supersymmetry there must exist at least one non-vanishing solution for the spinor parameter ϵ of the equations obtained by requiring the supersymmetry variations of the fermion fields to vanish. We shall seek solutions of these conditions for which the ten-dimensional spacetime is a direct product $M_6 \times T_4$ of six-dimensional Minkowski spacetime, M_6, to be identified as the worldsheet of an infinite static fivebrane, and a four-dimensional 'transverse' space, T_4. Let $\{x^\mu;\ \mu = 0, 1, \ldots, 5\}$ be the worldvolume coordinates and $\{y^m;\ m = 1, 2, 3, 4\}$ the coordinates of the transverse space, which we shall assume to be conformally flat. Thus, by assumption,

$$g_{\mu\nu} = \eta_{\mu\nu} \qquad g_{mn} = e^{2f(y)}\delta_{mn} \,, \tag{3.23}$$

where e^{2f} is the conformal factor. We further assume the vanishing of all other fields with worldvolume indices, and that the remaining fields depend only on the transverse coordinates. The problem is thereby reduced to solving four-dimensional Euclidean equations for the function $f(y)$, the dilaton $\phi(y)$, the antisymmetric tensor $B_{mn}(y)$ and the YM potential

$A_m(y)$. Moreover, we shall restrict A_m to take values in an SO(3) subgroup of SO(32) such that the 496-dimensional adjoint of $SO(32)$ has the $SO(3) \times SO(29)$ decomposition

$$(\mathbf{3},\mathbf{1}) \oplus (\mathbf{1},\mathbf{406}) \oplus (\mathbf{3},\mathbf{29}) \ . \tag{3.24}$$

With these restrictions the fermion variations of (3.22) reduce to

$$\begin{aligned}
\delta\psi_\mu &= \partial_\mu \epsilon \\
\delta\psi_m &= \partial_m \epsilon - \frac{1}{8} e^{-2f} \gamma_{pq} \big(H_{mpq} + 2e^{2f} \varepsilon_{mpqn} \partial_n f \, \gamma_5 \big) \epsilon \\
\delta\lambda &= \frac{e^{-3f}}{24\sqrt{2}} \gamma_{mpq} \big(H_{mpq} + \sqrt{2} e^{2f} \varepsilon_{mpqn} \partial_n \phi \, \gamma_5 \big) \epsilon \\
\delta\chi &= -\frac{1}{8} e^{-2f} \gamma_{mn} \big(F_{mn} - \frac{1}{2} \varepsilon_{mnpq} F_{pq} \, \gamma_5 \big) \epsilon \ .
\end{aligned} \tag{3.25}$$

where γ_5 is the product of the four (flat space) transverse $d = 10$ Dirac matrices, and satisfies $\gamma_5^2 = 1$.

Let $\epsilon_\pm$ be an eigenspinor of γ_5 with eigenvalue ± 1. Then, by choosing $\epsilon = \epsilon_\pm$ we find that $\delta\psi_M = 0$ has a solution for constant ϵ provided that $H_{mpq} = \mp 2e^{2f} \varepsilon_{mnpq} \partial_n f$. Given this, $\delta\lambda = 0$ requires

$$f = \frac{1}{\sqrt{2}} \phi \ . \tag{3.26}$$

Let us choose $\epsilon = \epsilon_+$. Then we have

$$g_{mn} = e^{\sqrt{2}\phi} \delta_{mn} \qquad H_{mnp} = -\varepsilon_{mnpq} \partial_q \big(e^{\sqrt{2}\phi} \big) \ . \tag{3.27}$$

The remaining equation $\delta\chi = 0$ is solved by any Euclidean instanton solution for the $SO(3)$-vector potential $A^a_m(y)$ $(a = 1, 2, 3)$. Let us choose $\epsilon = \epsilon_+$ and the one instanton solution of size ρ,

$$A^a_m(y) = \frac{2}{(r^2 + \rho^2)} \eta^a{}_{mn} y^n \ , \tag{3.28}$$

where r is the radial distance (in the Euclidean metric) from the YM core of the fivebrane and $\eta^a{}_{mn}$ is 't Hooft's 3-vector valued self-dual tensor [45].

It remains to check that H as given in (3.27) satisfies the Bianchi identity (3.20). Let us first pause to consider some general implications of this identity. Integrating it over the four-dimensional transverse space, and using the definition of Q in (3.16), we find that

$$Q = \frac{1}{16\pi^2} \int_{T_4} \mathrm{tr}(F \wedge F) \tag{3.29}$$

which is just the instanton number, and therefore an integer (this is the correct normalization of the instanton number for the trace in the vector representation of $SO(3)$; the often seen normalization of one over $8\pi^2$ is for the fundamental representation of $SU(2)$). This should make us happy because we saw earlier that Q had to be an integer because of a Dirac-like quantization condition in string theory, but before we are overcome by euphoria we

should recall that we have yet to establish a relation between the constant α' that appears in the action (3.17) (and in the definition of H) and the constant α' that appears in the string action (3.15). The connection between them is established by comparison of (3.17) with the low-energy effective action obtained by requiring conformal invariance of the two-dimensional sigma-model defined by the string action. A two-loop calculation is needed to fix the coefficient of F^2 in this action but a one-loop sigma-model anomaly calculation [46] suffices to fix the coefficient of the anomalous term in the Bianchi identity (3.20). The result of the latter calculation is precisely (3.20), so the two *a priori* different constants α' are actually the same. Moreover, since we chose a solution with instanton number one we find that

$$Q = 1 \ . \tag{3.30}$$

(I therefore disagree with the $Q = 8$ value given in the literature.) Now we substitute our result for H_{mnp} into the Bianchi identity (3.20). This gives the equation

$$\Box\big(e^{\sqrt{2}\phi}\big) = -\frac{1}{16} \alpha' \varepsilon_{mnpq} \mathrm{tr}(F_{mn} F_{pq}) \ . \tag{3.31}$$

This is the four-dimensional version of Poisson's equation with a non-singular source of size ρ centered at the origin. Clearly there is a solution and, if we impose the boundary condition that $\phi \to 0$ as $r \to \infty$, its asymptotic form is

$$e^{\sqrt{2}\phi} \sim 1 + \frac{\alpha' Q}{r^2} \ . \tag{3.32}$$

Thus, there is a non-singular fivebrane solution of the Chapline-Manton theory that breaks half the supersymmetry. By taking the $\rho \to 0$ limit one can find an explicit 'elementary fivebrane' solution [11] for which the asymptotic result (3.32) is exact. This corresponds to a singular source at $r = 0$. The metric $g_{mn} = e^{\sqrt{2}\phi} \eta_{mn}$ is also singular in this limit, with the result that the singularity is removed to the 'end' of an infinite wormhole throat with cross-section S^3. This might be considered acceptable in supergravity but in string theory e^ϕ is the effective string coupling constant and this diverges as $r \to 0$ if the asymptotic form (3.32) is exact. Consequently, the elementary fivebrane solution cannot be expected to survive quantum corrections.

To extend the non-singular solution of Chapline-Manton theory to string theory we must, in particular, further modifify the Bianchi identity (3.20) to

$$dH = X_4 \equiv \frac{1}{4} \alpha' \big[\mathrm{tr}(F \wedge F) - \mathrm{tr}(R_- \wedge R_-) \big] \ , \tag{3.33}$$

as required by the Green-Schwarz anomaly cancellation [47]. Observe that the curvature two-form occurring in this formula is the one for the connection ω_-. For the solution of the Chapline-Manton theory found above, R_- is a self-dual two-form. This can be seen as follows: Firstly, it can be verified that ω_+ is self-dual on the $SO(4)$ group indices, which implies that R_+ is too. Secondly, $R_{abcd}(\omega_+) = R_{cdab}(\omega_-)$ so that R_- is self-dual on the form indices. As a confirmation of this note that the supersymmetry transformation of the supercovariant gravitino curvature ψ_{ab} (in a purely bosonic background) is [44]

$$\delta_\epsilon \psi_{ab} = -\frac{1}{4} \gamma_{mn} R_{mnab}(\omega_-) \epsilon \ . \tag{3.34}$$

which indeed vanishes for $\epsilon = \epsilon_+$ if R_- is a self-dual two-form. Clearly, the additional R_--dependent term in (3.33) leads to a modification of the previous fivebrane solution because the source for the dilaton equation is now different (a difference that was not taken into account in [5]). Leaving aside, for the moment, the issue of higher-derivative corrections in R_-, we are now faced with a problem of self-consistency. Given R_- we can solve Poisson's equation for ϕ, but it is precisely this solution that determines R_-. One self-consistent solution is found [6] by identifying ω_+ with the YM connection (this is possible because the self-duality of ω_+ in its $SO(4)$ indices means that it actually takes values in $SO(3)$). In this case $\Box(e^{\sqrt{2}\phi}) = 0$ and, as for the elementary fivebrane, the asymptotic solution (3.32) is exact. This, 'symmetric', solution has the additional merit that all higher-order corrections vanish, so it is an exact solution of (classical) heterotic string theory. However, it also has the defect that it is unlikely to survive as an approximate solution of the quantum theory.

The exactness of the 'symmetric' fivebrane solution can be deduced from the fact that the action (in conformal gauge) for a string in this background is a sigma model with (4,4) supersymmetry, and it is known that such theories are conformally invariant to all orders of perturbation theory [48]. All other fivebrane solutions must correspond to sigma models with at least (4,0) supersymmetry, because this is the sigma-model equivalent of the condition of half-breaking of supersymmetry that we used to find these solutions [6]. In particular, the solution envisaged by Strominger must be in this class so further progress on this front can be made by investigation of the conditions for the finiteness (strictly speaking, conformal invariance) of $(4,0)$ sigma models. There exist arguments that purport to prove the finiteness of all (4,0) sigma models [49], but they could be vitiated by chiral (worldsheet) anomalies. Also, there was initially some puzzlement as to how the finiteness of $(4,0)$ sigma models could be compatible with corrections to the spacetime supersymmetry transformations, but it has recently been shown [50], up to three-loop order, that *finite* local counterterms are needed at each order to maintain (4,0) supersymmetry (which lead to modifications of the spacetime supersymmetry transformations), and that these are precisely the counterterms required for conformal invariance. This would appear to be strong evidence that there does exist a 'non-symmetric' solution of heterotic string theory generalizing the fivebrane solution of the Chapline-Manton theory.

We now turn to the question of the effective worldvolume action for fivebranes. The effective action for the elementary fivebrane is expected to be that of the standard $d = 10$ super-fivebrane of the brane-scan. The evidence for this is that the linearized limit of the gauge-fixed super fivebrane is the free six-dimensional hypermultiplet, for which the field content is four scalars and one complex spinor (the supersymmetry of this action is to be expected from the fact that the fivebrane configuration preserves precisely half of the original $d = 10$ supersymmetry). This is known to be the physical field content of the effective action for the 'elementary' fivebrane [7]. It is believed, however, that this theory is anomalous, and if this is the case then additional worldvolume fields will be needed. Strominger's fivebrane solution does have additional 'heterotic' worldvolume fields because the Atiyah-Singer index theorem implies that the gluino equation has zero-modes, one (for a one-instanton solution) for each gaugino triplet. From (3.24) we see that there are a total of 30 gluino triplets so there will be a total of 30 gluino zero modes. One of these, the $SO(29)$ singlet is the zero mode associated with the partial breaking of supersymmetry which produces the fermion content of the, non-heterotic, super-fivebrane. This means that there are 29 additional zero

modes, the coefficients of which will appear as additional 'heterotic' worldvolume fermions. Supersymmetry implies additional bosonic variables and the final result of the zero mode analysis is that there are an *additional 29 'heterotic hypermultiplets'* [7]. We don't know whether this ensures anomaly freedom but at least there is a chance. Part of the difficulty in answering this question is that the fully relativistic and spacetime supersymmetric *heterotic fivebrane* action that includes these additional fields has not yet been found.

The fact that the fivebrane is the magnetic dual of the string suggests that it should be possible to reformulate string theory as a fivebrane theory. A start on this program can be made by considering what the effects of such a string/fivebrane duality should be at the level of the effective supergravity theory [10]. It has been appreciated for a long time that there is a dual formulation of the supergravity/YM action in which the two-form potential B is replaced by a six-form potential $\tilde{B}$ [51,52]. This dual form may be found from (3.17) by introducing $\tilde{B}$ as a Lagrange multiplier for the Bianchi identity of H. This amounts, after an integration by parts, to the addition of the term

$$\int G \wedge \left[H - \frac{1}{4}\alpha' K_3 \right] , \tag{3.35}$$

where $G = d\tilde{B}$ and K_3 is the Chern-Simons (CS) three-form potential for $X_4 = dK_3$ (although, in the Chapline-Manton action the Lorentz CS contribution is absent). The three-form H may now be treated as an independent auxiliary field which can be eliminated by its field equation $H = e^{2\phi} \star G$. This results in a G^2 term appearing with a factor $e^{2\phi}$ rather than $e^{-2\phi}$, spoiling the factorization of the exponential of ϕ. To remedy this we rescale the metric: $g_{MN} \to e^{2\phi/3} g_{MN}$. This leads to the action (note the absence of a $(\partial\phi)^2$ term)

$$\begin{aligned} S = \int d^{10}x \sqrt{-g} \Big\{ e^{\frac{2}{3}\phi} \big[R - \frac{1}{2.7!} G^{MNPQRST} G_{MNPQRST} \big] - \alpha' \mathrm{tr} F^{MN} F_{MN} \Big\} \\ + \alpha' \int \tilde{B} \wedge X_4 \, . \end{aligned} \tag{3.36}$$

Although we have now arranged for a common factor of $e^{(2/3)\phi}$ in the pure supergravity sector, this has been at the expense of uniformity with the YM sector. However, at the one *string*-loop level we should include F^4 terms in the action. After rescaling the metric these appear precisely with a factor of $e^{(2/3)\phi}$. Taking this into account we may rewrite the Lagrangian as

$$e^{\frac{2}{3}\phi} L_o + L_1 + \dots , \tag{3.37}$$

where L_0 has the form $(R + G^2 + F^4)$. Duff and Lu now interpret $e^{-(2/3)\phi}$ as a *fivebrane-loop* counting parameter [10]. From this point of view $e^{(2/3)\phi} L_0$ is the classical Lagrangian and L_1 a fivebrane one-loop correction. Evidence for this interpretation is that $e^{(2/3)\phi} L_0$, now to be considered as the field-theory limit of a hypothetical fundamental fivebrane theory, admits *string* solutions [12], as one would expect from string/fivebrane duality. Duff and Lu further argue that the *complete* action as a double expansion in two-dimensional sigma-model and string loops can be re-interpreted as a similar expansion in six-dimensional sigma-model and fivebrane loops.

At the string one-loop level we should also take into account the effects of $d = 10$ chiral anomalies. Recall [53] that these are encoded in a 12-form X_{12} which, provided the gauge group is $SO(32)$ or $E_8 \times E_8$, factorizes as $X_{12} = X_4 \wedge X_8$ where X_4 is the four-form of (3.19) and

$$\begin{aligned} X_8 \equiv dK_7 = \Big[& \mathrm{tr}(F \wedge F \wedge F \wedge F) - \frac{1}{8}\mathrm{tr}(F \wedge F)\mathrm{tr}(R \wedge R) \\ & + \frac{1}{32}\mathrm{tr}(R \wedge R)\mathrm{tr}(R \wedge R) + \frac{1}{8}\mathrm{tr}(R \wedge R \wedge R \wedge R) \Big] \ . \end{aligned} \tag{3.38}$$

Because of the anomalous Bianchi identity for H the two-form B acquires an anomalous YM and Lorentz transformation. The non-invariance of the one string-loop effective action is then cancelled [47] by the anomalous variation of certain local terms, in particular the term $c\int B \wedge X_8$, where c is a calculable constant (but one which I have not calculated for the conventions used here). This is taken into account in the dual theory by modifying G to $G = d\tilde{B} - cK_7$, so that G now has the anomalous Bianchi identity

$$dG = -cX_8 \ , \tag{3.39}$$

while, as we see from (3.34), the effect of the anomalous Bianchi identity for H is to produce the term $\int \tilde{B} \wedge X_4$ in the action. This term is no longer YM and Lorentz invariant, however, because of the anomalous transformation of $\tilde{B}$ required by (3.37), and this fact is responsible for anomaly cancellation in the dual theory [51].

These facts, and a number of others that I haven't mentioned, support the conjecture that string theory has a dual formulation as a fivebrane theory. Here I wish to make a further observation in this connection. The fivebranes discussed here are all of infinite extent. It is this assumption that allows the possibility of finding *static* solutions. This greatly simplifies the analysis but it is a physical idealization unless the five spatial dimensions of the worldvolume are periodically identified. This means that the natural framework for the discussion of static fivebranes solutions is a toroidally compactified spacetime where space is at least a five-torus and the fivebrane is wrapped around five compact spatial dimensions. In this case the fivebranes appear as *particles* in the lower dimension. Consider a toroidal compactification to $d = 4$ on a six-torus. In addition to the fivebranes wrapped around the homology 5-cycles of the six-torus, which are now to be viewed as magnetic monopoles, we also have the strings wrapped around the homology 1-cycles. String fivebrane duality would require the spectra of these two objects to be the same. They are, by *Poincaré duality*!

As mentioned previously, the anomalous Bianchi identity for the two-form H can be derived from worldsheet considerations [46]. Specifically, the worldsheet chiral fermions, which include 32 'heterotic' fermions that couple to the background $SO(32)$ YM potentials, lead to anomalous one-loop diagrams with two insertions of either the background spin connection ω_- or the YM potential. Because the two vertices are associated with *background* gauge potentials that are functions of the worldsheet fields X^μ, rather than with independent 'fundamental' gauge potentials, the anomaly is called a 'sigma-model anomaly'. Its physical interpretation differs from that of the usual non-abelian gauge anomaly (in two dimensions), but its form is the same. In particular, it is characterized by a four form. One of the miracles of string theory is that this four form is the same as the four form X_4 appearing in the factorization of the spacetime anomaly twelve-form X_{12}, as a result of which the

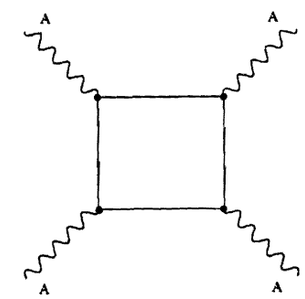
anomalous transformation of the two-form potential B required by the GS anomaly cancellation mechanism is exactly what is required for cancellation of the worldsheet sigma-model anomaly. String/fivebrane duality would suggest that the anomalous transformation of the dual six-form potential $\tilde{B}$, required by the Bianchi identity (3.39), should be precisely what is required for the cancellation of *six-dimensional sigma-model anomalies*. In order for this to be true, the eight-form characterizing these anomalies must be the eight form X_8 of (3.38).

A partial check of this can be made on the basis of the little information we already have about the heterotic fivebrane. Each of the heterotic hypermultiplets of the heterotic fivebrane contains an $SU(2)$-Majorana *chiral* spinor λ^r $(r = 1, 2)$ [54]. The YM fields in the $SO(3) \times SO(29)$ subgroup of $SO(32)$ couple to these fermions via the Lagrangian density

$$\mathcal{L} = \frac{i}{2}\bar{\lambda}^i_I \Gamma^\mu \lambda^j_J \left(\varepsilon_{ij} A^{IJ}_\mu + \delta^{IJ} A_{\mu ij} \right) \ . \tag{3.40}$$

The chirality of the six-dimensional spinors means that these vertices will produce a sigma-model anomaly from the square diagram of Fig. 3.2 in which the external lines at the four vertices are background YM potentials (there are more anomalous diagrams but they need not be computed as their contribution is determined by consistency conditions, as for the usual non-abelian chiral anomalies).

Fig. 3.2: Worldvolume sigma-model anomaly



The anomaly from this diagram is encoded in the eight form $\mathrm{tr}(F^4)$. Fortunately for the string/fivebrane duality conjecture, this is precisely the purely YM part of X_8. The agreement is non-trivial since X_8 might have contained a $[\mathrm{tr}(F^2)]^2$ term, but in fact does not. Thus we recover from fivebrane worldvolume considerations part of the anomalous YM transformation of $\tilde{B}$. Further progress will require the construction of the complete heterotic fivebrane action. I leave this as an exercise for the student.

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