GRUPO DE TEORIAS CUANTICAS RELATIVISTAS Y GRAVITACION
GTCRG-91-7
August 1991

Non-linear Realisations and Extended Objects

C. F. Yastremiz

Instituto de Astronomía y Física del Espacio
Casilla de Correo 67 - Sucursal 28, 1428 Buenos Aires – Argentina
and
DAMTP, Silver Street, Cambridge CB3 9EW, UK

ABSTRACT: The method of non-linear realisations is set in a mathematical framework with a view to interpreting the terms that appear in the effective description of extended objects in relation to the geometry of the theory. The rôle played by central extensions in the case of one-dimensional theories is also studied, as well as the extension to supersymmetric objects.

ROUTE TO
LOCATION
FERMILAB
LIBRARY
APR 20 1992
FERMILAB

DAMTP-R-91-10
Fermilab Library
0 1160 0007942 0

1: Introduction

In theories where some of the symmetries are 'broken' or 'hidden', the 'vacuum manifold' or the space of classical minimal energy solutions is the homogeneous space G/H, where G is the full symmetry group and H is the unbroken subgroup, since by definition the latter is the isotropy group of any of the minimal energy solutions. The fields that characterise the zero modes, the Goldstone fields, correspond to translations in this manifold, and therefore determine a set of coordinates on it [C1]. This means, in turn, that the Goldstone fields transform linearly under H, and non-linearly under the 'rest' of G. Any effective action written in terms of the zero mode fields must take these transformation properties into account [CWZ][CCWZ][O][IO].

In this paper, we shall analyse the so-called method of non-linear realisations that gives, up to field redefinitions, all the possible G-invariant functions of the fields, thus giving only the actions with strictly invariant Lagrangians. Quasi-invariant Lagrangians, which change by a total derivative under the action of G, are related to non-trivial elements of the Chevalley-Eilenberg cohomology group of G. In principle, these actions are not obtainable directly through this method. Nevertheless, we shall see that in the case of a one dimensional system (particles and superparticles) it is possible to exploit the relation between the second Chevalley-Eilenberg cohomology class of the group and the possible central extensions to extract the action. We shall also see that when the quasi-invariant action has a geometrical interpretation, either the action itself or its equations of motion can be obtained from the non-linear realisations, as in the case of extended and superextended objects.

Our main motivation is to apply later the method to theories of extended objects as topological defects. The imbedding of an extended object in a target space-time breaks invariance under translation and some rotations; thus, extended objects are an example of a system with partial symmetry breaking and can be treated by this method. In particular, we shall apply it to finding the form of higher derivative

corrections to the effective action. To this end, we must first understand the mathematical structure that underlies the method of non-linear realisations described in [CWZ][CCWZ][O][V], etc. Without this necessary geometrical understanding, it is not possible to interpret the results of the method or to apply them correctly to the theory of extended objects.

In section 2, we shall endeavour to provide a mathematical basis for the method in its general form, and for the particular case of external symmetries being included in the symmetry group. We shall see that the method is based mainly on the theories of induced representations and invariant connections on homogeneous spaces. Having in sight the application of the method to extended objects, in section 3 we shall study some submanifold theory and indicate its relation to non-linear realisations of affine groups.

Section 4 gives a short account of the cohomological properties of theories with quasi-invariant Lagrangians, and its relation to central extensions in the one-dimensional case. Section 5 gives an indication of a possible extension of the results of the previous sections to the supersymmetric case.

2: General formalism

Let G be a connected Lie group, and H a closed subgroup. Let G and H be the Lie algebras of G and H respectively, and M a linear subspace of G such that $\mathcal{G} = \mathcal{H} \oplus \mathcal{M}$ (as linear subspaces).

The Mackey decomposition theorem [BR] in its most general form says that if G is a separable locally compact group and H a closed subgroup, then there exists a Borel set S in G such that every element of $g \in G$ can be uniquely represented in the form

$$g = s(g)\, h(g) \quad , \tag{2·1}$$

with $s(g) \in S$ and $h(g) \in H$. The hypotheses are automatically fulfilled if G is a Lie group and H a Lie subgroup of G. We shall write $s(g) = s_g$, $h(g) = h_g$, so that $g = s_g h_g$. The set S is not unique; for example, if $h_0 \colon S \to H$ is a continuous function, $S' = \{s h_0(s) \,:\, s \in S\}$ is also a suitable set, and the decomposition (which is still unique in terms of the new set S') becomes $g = s'(g)h'(g)$, where $s'(g) = s(g)h_0(s(g)) \in S'$, $h'(g) = h_0^{-1}(s(g))h(g) \in H$.

Even though S need not itself be a manifold, we shall make use of this freedom to take as S, at least locally, an integral manifold of M (where M is seen as a set of left invariant vector fields on G). We will generally restrict our attention to a neighbourhood $\mathcal{U}$ of a point in G where S can be taken in this manner. This will suffice for the physical applications.

As we explained in the previous section, we are interested in $G/H = \{gH : g \in G\}$, the homogeneous space consisting of all the left H cosets. This space is the same as $G/\mathcal{R}$, where $\mathcal{R}$ is the equivalence relation $g_1 \,\mathcal{R}\, g_2 \iff \exists\, h \in H$ such that $g_1 = g_2 h$. This implies that $g_1 \,\mathcal{R}\, g_2 \iff s_{g_1} = s_{g_2}$. Therefore there is a 1-1 correspondence between the left cosets gH and elements of S, and in what follows we will use this correspondence freely, speaking of S and G/H interchangeably.

This equivalence relation determines a principal fibre bundle $G(G/H, H)$, where G is the bundle space, G/H is the base and H is the structure group, and the Mackey decomposition theorem gives a local trivialisation. The bundle projection π is given by the natural projection $\pi \colon G \to G/H$, $\pi(g) = gH$. G acts naturally on the left (through the group product) on this bundle, and, in particular, on G/H.

We can take local coordinates on G/H by taking local coordinates on G in $\mathcal{U}$ and then restricting them to S; if they are to be independent, they should be adapted coordinates Ξ^i (in which, if dim(M)= n, and dim(G) = N, the defining equations for S are locally $\Xi^{n+1}(g) = \ldots = \Xi^N(g) = 0$) [1]. In particular, it would be useful to consider, in a neighbourhood of the identity, canonical coordinates of

[1] Actually, for our applications, it is enough that the corresponding tangent frame be adapted, $\partial_1, \ldots, \partial_n \in \mathcal{M}$ (i.e., be tangent to S).

the second kind subordinated to the decomposition G=H+M [SW], which realise the Mackey decomposition,

$$g = \exp(iX)\exp(iY) \quad , \qquad X \in \mathcal{M}, \quad Y \in \mathcal{H} \quad . \tag{2·2}$$

The coordinates on G/H will constitute our basic non linear realisation of G, so any change of coordinatisation will be equivalent to a field redefinition (and vice versa).

We will assume that (G, H) form a reductive pair; that is $\mathrm{Ad}(H)(\mathcal{M}) \subset \mathcal{M}$. In terms of Lie algebras, this means $[\mathcal{H}, \mathcal{M}] \subset \mathcal{M}$. This is a physically reasonable assumption if we are going to apply the method to theories with spontaneous symmetry breaking; if H is the group of unbroken symmetries, and M is generated by the broken infinitesimal symmetries, one wants the broken symmetries to be stable under the unbroken ones. This will be needed in what follows, notably, in the definition of a G-invariant connection. A typical example of this situation is when G is a compact semisimple group, and H is semisimple and connected. Then a suitable choice is $\mathcal{M} = \mathcal{H}^{\perp}$ with respect to the Killing form of G [SW], [CWZ]. Another example is when G is a semidirect product, $G = S \otimes H$.

To construct our basic non-linear realisation, we take a set of coordinates Ξ^i, $i = 1, \ldots, n$ on G/H. These are elements of $\mathcal{M}^*$, the dual of M. Then the action of G is defined as $g: \Xi \mapsto \Xi'$, $\Xi'(\xi) = \Xi(g^{-1}\xi)\ \forall \xi \in G/H$. This action is in general non-linear ('non-linear' includes non homogeneous), but if we take normal coordinates of the second kind, and write $\xi \in S$ as $\xi = \exp(i\Xi\cdot\mathcal{B})$ ($\mathcal{B}$ is a set of generators for M), then the action becomes linear when restricted to the subgroup H. This is due to the property of the exponentials, $h(\exp X)h^{-1} = \exp[\mathrm{Ad}(h)(X)]$ [SW], and the invariance of M under the adjoint action of H. It may seem a disadvantage that for H to be represented linearly the choice of coordinatisation must be restricted, but actually this restriction turns out to be irrelevant. Indeed, only objects that form a linear representation of H can enter the invariant actions we want to construct. Therefore, we will devote the rest of this section to show how to extend linear

representations of H to realisations of G and to construct a particular realisation out of fields with values in G/H and their derivatives.

Induced representations.[MK] Let $h \mapsto L(h)$ be a unitary representation of H in a separable Hilbert space $\mathcal{H}$, $L(h) \in \mathrm{End}[\mathcal{H}]$, and $\mathcal{H}^L = \{u: S \to \mathcal{H}\}$ the set of all square integrable functions from S into $\mathcal{H}$. The induced representation U^L of G on $\mathcal{H}^L$ is given by

$$U^L(g)\, u(\xi) = L^{-1}(h_{g^{-1}\xi})\, u(s_{g^{-1}\xi}) \quad , \tag{2·3}$$

where h and s refer to the Mackey decomposition (2·1)[2]. More explicitly, if we define ξ' and $h(\xi, g)$ via

$$g\xi = \xi' h(\xi, g) \quad , \tag{2·4}$$

then the representation (2·3) is

$$u'(\xi') = L\,(h(\xi, g))\; u(\xi) \quad . \tag{2·5}$$

It is clear that if $g \in H$ then the action becomes the original representation, with $\xi' = g\xi g^{-1}$ and $h(\xi, g) = g$ (because M is invariant under $\mathrm{Ad}(H)$). For any vector bundle $E(G/H, \mathcal{H}, H, G)$ associated to $G(G/H, H)$ with fibre $\mathcal{H}$, the induced representation gives an action of G on the sections of E, $\Sigma(E)$.

As an important example, one can take $\mathcal{H} = \mathcal{M}$ and L=Ad if one introduces an inner product in M that makes this action a unitary representation. This coincides with the action induced on M (seen as $T_e(S)$) by the left action of G on S when restricted to H, but $U^L \neq$Ad for the whole group. Then E can be seen as the tangent bundle to S but with reduced structure group (from $GL(n)$ to $L(H)$).

[2] Some technical subtleties about the possible nonexistence of an invariant measure in G/H have been disregarded (in the case when the measure is only quasiinvariant, its Radon-Nikodym derivative has to be included in the definition of the induced representation, to keep it unitary [BR]).

Connections and covariant derivatives.[KN] The canonical or Maurer-Cartan 1-form θ of a group G is defined as the left-invariant G-valued 1-form uniquely determined by $\theta_g(\tilde{A}_g) = A \quad \forall A \in \mathcal{G}$, where $\tilde{A}$ and A are related by $\tilde{A}_g = (L_g)_* A$ (for matrix groups, this definition gives the usual $\theta = g^{-1}dg$ in the matrix coordinate system [GS]). This 1-form obeys the Maurer-Cartan equation,

$$d\theta = -[\theta, \theta] \quad . \tag{2·6}$$

Since (G, H) form a reductive pair, the H-component ω of θ with respect to the decomposition $\mathcal{G} = \mathcal{H} + \mathcal{M}$ defines a connection in the bundle $G(G/H, H)$ which is invariant under left translations of G, and such that the horizontal space at the identity e is M (under the identification $T_e(G) \simeq \mathcal{G}$)[KN][3]. This connection 1-form can now be used to define a covariant derivative on the sections of any of associated bundle E, that is, on any fields on G/H. These covariant derivatives ('G-covariant derivatives') transform in the same way as the sections under the action of G.

Let now W be a p+1-manifold which we will take as our spacetime, and let $\Xi: G/H \to \mathbf{R}^n$ a coordinatisation of G/H as above. Given a map $\xi: W \to G/H$, $x \mapsto \xi(x)$, we define the fundamental or Goldstone fields as $\Xi \circ \xi$, and call them $\Xi(x)$. There may be more fields in the theory than simply the Goldstone fields. It is possible to take two different approaches to the treatment of these fields. One way, they depend only on x; the second approach is more inspired by the original definition (2·3) of the induced representation. The other fields in the theory are functions not only on W but on G/H as well. These approaches are mathematically equivalent but slightly different in the philosophy behind them. The relations between different fibre bundles in these two approaches are represented diagramatically in the Appendix.

In the first approach, one defines the Goldstone fields directly as $\mathbf{R}^n$-valued, and then determines their transformation properties under G by working only with

[3] The converse also holds: any connection in $G(G/H, H)$ invariant under left translations of G determines such a decomposition $\mathcal{G} = \mathcal{H} + \mathcal{M}$ and is obtainable in the same way as the $\mathcal{H}$-component of θ [KN].

elements in S of the form $\exp(i\Xi(x)\cdot\mathcal{B})$; this is the usual procedure in physical applications. This is possible because $\mathcal{M}^* \simeq \mathbf{R}^n$. In this case, unless G is both compact and connected, one could speak properly only of infinitesimal transformations, to ensure that one remains in the image of the exponential map[4]. In the same spirit, for the other fields in the theory, one can work directly with $\mathcal{H}$-valued functions on W, $\Theta: W \to \mathcal{H}$, where $\mathcal{H}$ is some unitary representation of H. One can then substitute $\exp(i\Xi(x)\cdot\mathcal{B})$ for ξ in the definition of the induced representation (2·3), $g: \Theta(x) \mapsto \Theta'(x) = L\big(h(\exp(i\Xi(x)\cdot\mathcal{B}), g)\big)\Theta(x)$ in the notation of (2·5). Since Θ take values in $\mathcal{H}$, they are sections of some vector bundle $E'(W, \mathcal{H}, H, Q)$ associated to a principal bundle $Q(W, H)$.

The transformation under G given by the induced representation shows that there must exist some principal bundle $P(W, G)$ reducible to $Q(W, H)$. There is a theorem [KN] which states that this reduction $P \to Q$ is possible if and only if the associated bundle $\widetilde{E}(W, G/H, G, P)$ admits a global section $\hat{\xi}$, which here is induced by $\xi = \exp(i\Xi\cdot\mathcal{B})$. This map is either defined globally or is extendable if W is paracompact and G/H is finite dimensional. The correspondence between sections and subbundles is 1-1[5].

To construct actions which are invariant under G from these fields, we need to define their G-covariant derivatives. Given a vector bundle $E'(W, \mathcal{H}, H, Q)$ associated to Q, the definition (2·3) shows how to extend the structure group to G for each Q, or, equivalently, for each section; thus the extension depends on the value of the Goldstone fields. G-invariant connections on these bundles Q are given by the projection onto H of the corresponding pullback of the Maurer-Cartan 1-form. For a trivial bundle $P = W \times G$, if $f: W \times G \to G$ is the natural projection, the canonical flat connection is given by $\omega = f^*\theta$. Nevertheless, the connection induced on these subbundles is not flat, unless M is a subalgebra of G, because the curvature 2-form

[4] This might not be necessary for particular cases [Hel]. In any case, this is not too strong a restriction, since conservation laws are related to infinitesimal transformations.

[5] If $f: Q \to P$ is the injection and $\mu: P \to \widetilde{E} = P/H$ the projection, the section $\hat{\xi}: M \to \widetilde{E}$ is defined by $\hat{\xi}(x) = \mu(f(u))$, where $u \in Q$ is any element such that $\pi(f(u)) = x$. Reciprocally, given $\hat{\xi}$, Q is the set of points $u \in P$ such that $\mu(u) = \hat{\xi}(\pi(u))$ [KN] (See the Appendix).

Ω of the connection is given by the projection onto H of $\tilde{\Omega}(X,Y) = -\frac{1}{2}[X,Y]$ for X, Y arbitrary left invariant vector fields on G belonging to M. If Ω vanishes, there is a canonical imbedding of $S \simeq G/H \overset{i}{\hookrightarrow} G$ and $G = S \otimes H$. Thus the non flatness of the induced connection reflects the non triviality of the imbedding of G/H in G.

In the second approach, one can notice that $\tilde{E}(W, G/H, G, P) = P(W,G)/H$; then $P(\tilde{E}, H)$ is a principal fibre bundle with base $\tilde{E}$ and structure group H. Given any vector bundle $\hat{E}(\tilde{E}, \mathcal{H}, H, P)$ associated to $P(\tilde{E}, H)$, we can define the action of G on it via (2·3). Then the rest of the fields in the theory are sections of these associated bundles, and hence functions of the points of spacetime and of the Goldstone fields themselves, $\Theta = \Theta(x, \xi)$. The transformation law is defined as

$$g : \Theta \mapsto \Theta'(x) = L(\xi, g)\Theta(x) \quad . \tag{2·7}$$

The construction of the first approach is reobtained when we choose a section of $\tilde{E}$, or equivalently an imbedding of W in $\tilde{E}$. If we restrict Θ to the image of the section, then all the fields only depend on x.

In general, the pullback of the canonical 1-form from $G(G/H, H)$ to $Q(W, H)$ will be given by

$$\tilde{\xi}^* \theta = \omega' + \varphi, \quad \omega' \in \mathcal{H}, \ \varphi \in \mathcal{M} \quad , \tag{2·8}$$

where $\tilde{\xi}$ is the lifting to Q of ξ. Then ω' gives the G-invariant H-connection. Sometimes, as in the previous section, φ (or part of it) is written as $\mathcal{D}\Xi \cdot \mathcal{B}$, and the $\mathcal{D}\Xi$ referred to as 'the covariant derivatives of the Goldstone fields' because they contain derivatives of Ξ and transform 'covariantly' under the induced representation of G, although they are not the proper covariant derivatives formed with the connection ω' [O][CCWZ]. Higher covariant derivatives are formed in the usual way with ω'; relations among these higher covariant derivatives of the Goldstone fields ('generalised Bianchi identities') can be obtained using the fact that θ obeys the Maurer-Cartan equation (2·6).

Invariants and Lagrangians. Since the representation spaces $\mathcal{H}$ are Hilbert spaces and the representations of H are unitary, one obvious way to construct invariants under H is to construct inner products of $\mathcal{H}$-valued fields. Invariants formed from the Goldstone fields and their derivatives will necessarily enter as functions of φ, since it has the correct transformation properties. Actually, the invariance of the bilinears will be a good guide to find the appropriate inner product in M. If Θ denotes a section on E ($\mathcal{H}$-valued field), then an invariant Lagrangian will be any H-invariant function $\mathcal{L}(\Theta, \nabla\Theta, \varphi) = \mathcal{L}(\Theta, \nabla\Theta, \mathcal{D}\Xi)$. The Lagrangian $\mathcal{L}$ can include higher covariant derivatives of Θ and Ξ. Since the action of G on these fields is the induced representation action, H-invariance implies G-invariance.

Spacetime symmetries. What we said up to now assumes that G is a purely internal symmetry group. If G includes spacetime symmetries, then the map $\xi: W \to G/H$ is an imbedding, and $\xi_*(T_x(W))$ is some subspace of M, with a basis $\mathcal{P} = \{\mathcal{P}_A\}$; we choose a basis of M such that the first $p+1$ elements are the $\mathcal{P}_A$, and we shall still call $\mathcal{B}$ the remaining generators. The components of Ξ in the $\mathcal{P}$ subspace are no longer independent fields, and can be used to determine local normal coordinates on W; then the Goldstone fields are the remaining components of Ξ, and they transform linearly under $\mathcal{P}$ (they live in the left regular representation of the translation group seen as $\mathbf{R}^{p+1}$). For this reason, we speak of $\mathcal{P}$ as the 'unbroken translations', although they do not belong to the unbroken subgroup H because they are not elements of the isotropy subgroup of any point in spacetime, which is imbedded in G/H. In accord with this distinction between $\mathcal{P}$ and $\mathcal{B}$, φ splits into two parts:

$$\varphi = E \cdot \mathcal{P} + D\Xi \cdot \mathcal{B} \quad . \tag{2·9}$$

As we shall see later, E is a soldering 1-form, so the matrix of its components in some coordinate basis on W, which we shall denote by the same letter E, is a p+1-bein. Then $\det E$ is a G-invariant measure on W, and if we want to write the covariant derivatives of the Ξ in the corresponding coordinate basis we have to define $\mathcal{D}\Xi = E^{-1}\mathcal{D}\Xi$. In that case the invariant Lagrangian densities are $\mathcal{L}(\Theta, \nabla\Theta, \mathcal{D}\Xi) \det E$.

3: Extended objects

A relativistic extended object or 'p-brane' is essentially a p-dimensional 'object' manifold $\widetilde{W}$ moving in a d-dimensional 'spacetime' manifold M, determining the imbedding i of a $p+1$ dimensional manifold W in M. In general we shall take both W and M to be Lorentzian, and though in principle both can be arbitrary metric manifolds, in the applications M will be flat (partly for convenience, but mainly because it is only in that case that we know how to describe them in terms of non-linear realisations).

There is an equivalent description of extended objects as 'σ-models', a $p+1$-dimensional field theory on W (as a spacetime) where the fields take values in M (as a target space). In this description, symmetries in M are regarded as 'internal', while reparametrisations on W are 'external'. We shall switch back and forth between these two interpretations depending on which is the most convenient at each point.

The action for such a physical system is taken to be the $p+1$-volume of W, as a generalisation of the usual action for a relativistic particle and the Nambu-Goto action for strings [Ac]. The volume element is the square root of the determinant of the induced metric, or the determinant of the $p+1$-bein. Since the volume element is invariant under reparametrisations (coordinate changes) of W, not all the components of the imbedding i are independent. One way of eliminating the spurious coordinates is to fix coordinates on W ('fix a gauge for the local reparametrisation invariance') with relation to the imbedding. As we shall see, this gauge-fixing breaks the affine (Poincaré) symmetry in M; for a particular choice of gauge, the 'static' or 'physical' gauge that corresponds to the choice of adapted coordinates on W, there is a 'residual' global affine invariance on W (all that is left of the reparametrisation invariance) and invariance under rotations that leave $i(\widetilde{W})$ invariant.

If, starting from the other end, all one knows about these objects is this set of symmetries (the affine group), and which of them are broken and which unbroken,

one can apply the method of non-linear realisations to find a set of invariants from which to form actions from a connection. These actions will be obtained as a set of gauge fixed Lagrangians, not fully invariant under reparametrisations. From this method one can obtain, not only the free action (the worldvolume), but a complete set of self-interaction terms, with higher derivatives of the independent components of the imbedding i. This will be useful for finding effective low energy actions for topological defects, where the interaction between the zero mode and the higher order ones can give rise to these higher derivative terms. An affine connection is also a condensed way of writing a linear connection and a field of orthonormal bases, so it provides a link between the method of non-linear realisations and the theory of submanifolds which will allow us to interpret physically the results of the former.

Affine connections. [KN] The affine orthogonal group $AO(r,s)$ ($r+s=d$) is defined as the semidirect product $\mathbf{R}^d \otimes SO(r,s)$, the group of rigid motions of $\mathbf{R}^d$ considered as having a metric with signature (r,s). The corresponding relation holds for the Lie algebras, replacing the semidirect product by the semidirect sum: $\mathcal{AO}(r,s) = \mathbf{R}^d \oplus \mathcal{SO}(r,s)$.

If one considers the tangent space at each point of a manifold W as an affine space instead of a vector space, the bundle of orthogonal frames $O(W)$ can be extended to a bundle $A(W)$ of affine frames. An affine frame consists of a frame and an origin, i.e., a vector, and there is a canonical imbedding γ of $O(W)$ into $A(W)$ which takes the vector to zero, and so the origin is the origin of $\mathbf{R}^d$ seen as a vector space. Given a generalised connection on a principal affine bundle, the pullback of its connection 1-form $\tilde{\omega}$ by γ is an $\mathcal{AO}(r,s)$-valued 1-form on the orthogonal bundle. This form is naturally split by the semidirect sum structure of the algebra, $\gamma^*\tilde{\omega} = \hat{\omega} + \omega$, with $\hat{\omega}$ $\mathbf{R}^d$-valued and ω $\mathcal{SO}(r,s)$-valued. Using the theorem on invariant connections on reductive spaces mentioned in the previous section [KN], since any semidirect product defines a reductive pair, ω defines a metric connection on W and $\hat{\omega}$ is an $\mathbf{R}^d$-valued 1-form that determines a map between the tangent space at each point and $\mathbf{R}^d$. This 1-form is sometimes called

a soldering 1-form[6]. The canonical choice for a 1-1 correspondence between affine and orthogonal connections identifies this map with a field of orthonormal frames, such that the map is the identity at each point but the basis in the domain and the image are different. These are often called 'moving frames' or 'd-beins'.

Submanifolds. [PT] Let W be a $p+1$-dimensional immersed submanifold of a d-dimensional semi-Riemannian manifold (M, g)[7], and ∇ the covariant derivative of the Levi-Civita connection of g. Let $T_x(W)^\perp$ denote the orthogonal complement of $T_x(W)$ in $T_x(M)$, $\nu(W)$ the normal bundle of W in M, i.e., $\nu(W)_x = T_x(W)^\perp$, and $i: W \to M$ the inclusion.

There are three basic local invariants associated to W: the first fundamental form $h = i^*g$ is the induced metric on W; the second fundamental form or extrinsic curvature is a section of $(T^*(W) \otimes_{sym} T^*(W)) \otimes \nu(W)^*$, defined by

$$K(u_1, u_2, v) = -g\left((\nabla_{u_1} v)^T, u_2\right), \tag{3·1}$$

where the superscript T means the orthogonal projection onto $T(W)$; it is not difficult to prove that the definition is symmetric in the two first arguments. The third invariant is the induced normal connection $\nabla^\perp$ on $\nu(W)$, defined as the orthogonal projection of ∇ onto $\nu(W)$. The second fundamental form can also be seen as a section of $(T^*(W) \otimes_{sym} T^*(W)) \otimes \nu(W)$, using the canonical mapping given by the metric. We shall use both forms as equivalent.

A section on $O(M)$ (a field of orthonormal frames) is adapted to W if the first $p+1$ vectors form a section on $O(W)$ (i.e., are tangent to W on W). Let us denote such a section by $e_1, \ldots e_d$, and its dual coframe by $\hat\omega^1, \ldots, \hat\omega^d$. Such a coframe and

[6] The concept can be generalised slightly, by taking a subalgebra of the appropriate dimension in the Lie algebra of any group and not necessarily the affine one, giving then one possible mechanism to transform 'internal' symmetries into 'external' ones [CY].

[7] The definition of a semi-Riemannian submanifold requires that i^*g be a metric; this is to say, the tangent space of the submanifold is a non-degenerate subspace of the full tangent space at each point, i.e., the metric is non-degenerate when restricted to it. This condition eliminates a few pathological situations and implies that the normal subspace is well defined [O'N]. It also rules out null surfaces.

the Levi-Civita connection 1-form ω define an affine connection as in the previous subsection; we take as a basis of $\mathrm{ao}(d)$ the translations $\mathcal{P}$ plus the rotations $\mathcal{M}$; it is convenient to write the components of ω with a pair of antisymmetric indices. It is useful to distinguish between components in $T(W)$ and $\nu(W)$, so we split the indices on M as $0 \le A, B \ldots \le p \le i, j, \ldots \le d-1$. To obtain the components of the three local invariants in these bases, we pull back the semi-Riemannian structure on M onto W via the inclusion i. In what follows the i^* is understood on all forms; then

$$\hat\omega^j = 0, \tag{3·2}$$

$$h = \mathrm{tr}(\hat\omega \otimes \hat\omega), \tag{3·3}$$

and the structure equations translate as[8]

$$d\hat\omega^A = \omega^A{}_B \wedge \hat\omega^B \tag{3·4}$$

and

$$d\hat\omega^i = \omega^i{}_A \wedge \hat\omega^A = 0, \tag{3·5}$$

which implies that the $\omega^i{}_A$ are linear combinations of the $\hat\omega^A$ with some coefficients,

$$\omega^i{}_B \equiv -\hat\omega^A K_{AB}{}^i. \tag{3·6}$$

Substituting (3·6) back into (3·5), one obtains that the coefficients K are symmetric in AB, and comparing (3·6) with the definition of the second fundamental form (3·1), it can be seen that they give the components of the extrinsic curvature tensor

$$K = K_{AB}{}^i \hat\omega^A \otimes \hat\omega^B \otimes e_i. \tag{3·7}$$

Equation (3·4) gives the Levi-Civita connection on W as the pullback of that on M, and the remaining components of ω, $\omega^i{}_j$, define the normal connection. The pullback of the curvature equations gives the Gauss-Codazzi and Ricci equations for W. In this way, $\hat\omega^A$ and $\omega^A{}_B$ define an affine connection on W, and $\hat\omega^A$ is a $p+1$-bein (cf. (3·3)).

[8] Using that pullbacks commute with exterior derivatives and exterior products.

The mean curvature vector $\hat{K}$ is defined as the trace of the extrinsic curvature tensor,

$$\hat{K} = \hat{K}^i e_i \equiv h^{AB} K_{AB}{}^i e_i. \tag{3·8}$$

An immersion is called minimal if it satisfies $\hat{K} = 0$, since this implies that the volume of W has a stationary point with respect to deformations of the immersion with compact support. This is, $\hat{K} = 0$ are the Euler-Lagrange equations of a variational principle for the volume functional[9].

Gauge fixing and non-linear realisations. If one now looks at the immersion as a set of fields over W with values in M, these fields transform linearly by reparametrisations on W. These reparametrisations act as 'external symmetries', being general coordinate transformations, or diffeomorphisms. On the other hand, if M is assumed flat, its isometries are the rigid affine transformations (Poincaré group), which act as 'internal symmetries', because they transform the fields without touching W. Once an adapted orthonormal frame field in M has been fixed, one can see that the symmetry under translations in the directions normal to W is broken, and so are the rotations and Lorentz boosts that mix parallel and normal directions. One can then fix a coordinate system in W with reference to this frame field, or the coordinates on M, that will reduce the free components of the imbedding i to the independent ones; in particular, to the coordinates that correspond to the perpendicular directions, X^i. The fields in this smaller set will transform non-linearly under the broken translations and rotations, but linearly under the translations parallel to W and the remaining rotations. They form a representation of the rotation subgroup that leaves $T(W)$ invariant, and the parallel translations can be written via the gauge fixing as rigid coordinate transformations on W. Thus, the gauge fixing mixes internal and external symmetries, defining certain quantities with internal indices in terms of other quantities with external ones, and so changing their transformation properties under the symmetry groups of the theory [JR][H'tH][C3].

[9] Note that the volume is always invariant under reparametrisations of W, and that $\hat{K}$ is a normal vector.

The above analysis depends only on the symmetries of the physical situation, and not on the particular action chosen. Therefore all possible terms can be obtained from the method of non-linear realisations with $G = AO(d-1,1)$ and $H = SO(p,1) \times SO(d-p-1)$. The correspondence can be made via the identifications

$$\begin{aligned} \hat{\omega}^A &\to E^A \\ \hat{\omega}^i &\to \mathcal{D}X^i \\ K_{AB}{}^i &\to \mathcal{D}_A \Omega_B{}^i, \end{aligned} \tag{3·9}$$

where the X^i and the $\Omega_A{}^i$ are the Ξ-fields corresponding to the perpendicular translations and the broken rotations respectively. These identifications stem from comparing the rôle of $i^*(\hat{\omega})$ and $i^*(\omega)$ for a flat affine connection and the decompositions (2·8) and (2·9) of the same connection in terms of the Goldstone fields. Then $\mathcal{D}X^i = 0$ is a constraint, while setting the trace of $\mathcal{D}\Omega_A{}^i$ to zero is equivalent to the equations of motion for the area functional taken as an action.

4: Quasi-invariant Lagrangians; cohomology and central extensions

The de Rham cohomology group of a manifold is defined as the classes of equivalence $[\alpha]$ of closed differential forms modulo exact forms, $[\alpha] \equiv \{\beta : d\beta = 0,\ \alpha - \beta = d\gamma\}$. A closed form α is called a cocycle, and it is called non-trivial if it is not exact, $\alpha \neq d\beta$. If the manifold is a Lie group G, the Chevalley-Eilenberg cohomology is similarly defined, but with the restriction to left invariant forms. The CE cohomology classes are the closed left-invariant forms modulo exact left-invariant forms.

In many cases of physical interest, the spacetime manifold M is a homogeneous space, G/H, for some group G and a closed subgroup H. We shall assume that this is the case, and denote by $\pi : G \to G/H = M$ the projection. Using the notation and constructions of the previous sections, let us further assume that M has trivial

de Rham cohomology, as is the case if $M = \mathbf{R}^n$, to avoid clouding the issue. If the pull back of a closed $n+1$-form ω on M determines a non-trivial CE cocycle $\pi^*\omega$, then there must exist a form β such that $\omega = d\beta$, but necessarily $\pi^*\beta$ is not left invariant. Then β can be used as a quasi-invariant Lagrangian on M, since its variation under the projection of the group left action will be a total derivative.

The classes of equivalent one-dimensional central extensions of a group are in one to one correspondence with the non-trivial elements of the second Chevalley-Eilenberg cohomology group [10][deA]. Therefore, for one-dimensional field theories, the existence of a Wess-Zumino term, which corresponds to a closed 2-form as above, is equivalent to the existence of a central extension of the symmetry group. In that case, if the extension is maximal and the extended group has a trivial second cohomology group, i.e., it cannot be further extended, it is possible to enlarge the system with a new variable such that its 'conjugate momentum', the generator of translations along this new variable, corresponds to the central charge [MMSS]. This variable can be identified with the Wess-Zumino action seen as a function of its endpoint.

Let us represent our original generalised coordinates or one-dimensional fields collectively by $q(t)$, and the original Lagrangian 1-form by $\mathcal{L}(q,\dot{q})$. We suppose that the corresponding action is invariant under some symmetry group with generators X_i [11], but the Lagrangian changes by a total derivative,

$$\delta_i \mathcal{L}(q,\dot{q}) = L_{X_i}\mathcal{L}(q,\dot{q}) \equiv d\Delta_i = \dot{\Delta}_i\, dt \quad . \tag{4·1}$$

L_{X_i} represents the Lie derivative in the direction of X_i. We could now try adding a new term of the form $-\dot{\chi}$ to the Lagrangian, and defining its variations such that

$$\delta_i \chi = \Delta_i \quad . \tag{4·2}$$

[10] The central extension is one-dimensional because the Chevalley-Eilenberg cohomology groups are equivalent to the $\mathbf{R}$-valued cohomology groups of the algebra; extensions by an abelian algebra $\mathfrak{a}$ are classified by the $\mathfrak{a}$-valued cohomology groups.

[11] We shall represent the vector fields that realise the group on the tangent configuration space with the same letter as the corresponding generators of the abstract Lie algebra.

The new extended Lagrangian $\overline{\mathcal{L}}(q,\dot{q},\chi) = \mathcal{L}(q,\dot{q}) - \dot{\chi}dt$ would be then fully invariant. The set of equations (4·2) are actually a set of differential equations, and their integrability conditions are equivalent to saying that the Chevalley-Eilenberg cocycle determined by $d\Delta_i$ is trivial. If this cocycle is not trivial, it determines a central extension; for this centrally extended group the cocycle is trivial and the Lagrangian $\overline{\mathcal{L}}$ is fully invariant. The central charge is represented up to a constant as the vector $(\partial/\partial\chi)$ on the tangent of the extended configuration space [deA], and it is in that sense that the central charge and the variable χ are conjugate; it is clear from the definition that actually the momentum canonically conjugate to χ is -1, so χ is not a dynamical variable.

It is possible to define

$$\chi(t) = \int^t \mathcal{L}(q,\dot{q}) \quad , \tag{4·3}$$

since of course χ is determined up to any invariant quantity, giving $\overline{\mathcal{L}} = 0$. This result is useful when applying the method of non-linear realisations to a one-dimensional theory, such as particles or superparticles. If the method is applied to the extended group, and the central charge is considered as a broken generator (because it does not annihilate the vacuum), χ can be identified as the Goldstone field corresponding to that generator (in the same way as the Goldstone fields that correspond to translations are identified with coordinates in the direction of the translation), and equation (4·3) will be found as a solution of the constraint $\mathcal{D}\chi = 0$ (which actually says that χ determines a cocycle). For the relativistic case, it is not strictly necessary to include the central extension to find the lowest order action, since this action corresponds to the length of the worldline and can be obtained from the determinant of the 1-bein in the bosonic case or through its equations of motion in the supersymmetric case. For the Galilean particle, though, this is apparently the only possibility.

5: Supersymmetry

By formally extending the definitions in the previous sections to systems with Grassmann coordinates, we can construct a theory of supersymmetric non-linear realisations, and apply it to superextended objects.

If some of the supersymmetries are broken, there will be Goldstone fermions along with the Goldstone bosons, one such fermion for each broken supersymmetry. If any supersymmetries remain unbroken, then the Goldstone fields must transform linearly under these, and are therefore superfields. Superfields can be thought of as fields in superspace, which means we can consider W as a supermanifold with the unbroken supersymmetries as the generators of the supertranslations. In this way, the broken supersymmetries must be included in $\{\mathcal{B}\}$, whilst the unbroken supersymmetries are in $\{\mathcal{P}\}$.

For rigid supersymmetry, there is a simple argument to show that in the case of a topologically trivial theory with $d \geq 3$ either all the supersymmetries are broken or none of them is. Therefore we can have a partial breaking of supersymmetry either for low dimensions, or when space is non-trivial [HP].

Super p-branes. There are in principle three ways of supersymmetrising extended objects: having rigid supersymmetry in the target space M, 'super p-branes'; local supersymmetry on the object W itself, 'spinning p-branes'; or both at the same time, 'super spinning p-branes'. The last two do not appear to be consistent in general for $p \geq 2$. Moreover, for $p=1$, it is known that the gauge-fixed superstring and the spinning string are indeed equivalent; it has been shown explicitly in [GY] that the gauge-fixed superparticle also possesses a rigid supersymmetry on the worldline. This is true in general: gauge-fixed super p-branes have a rigid supersymmetry 'on the brane'. This is necessary if they are going to be any use as models of topological defects in supersymmetric theories. It is known that solitons of supersymmetric theories that saturate the Bogomol'nyi bound are supersymmetric [Ru]. For this

relation to be possible the original action for the supersymmetric extended object must have κ-invariance: a local worldvolume invariance with a set of parameters which are worldvolume scalars but spacetime spinors.

In the full reparametrisation and κ-invariant form, a superextended object describes an ordinary manifold W imbedded in a rigid superspace M. Seen as a field theory on W, the coordinates of the imbedding transform as scalars under reparametrisations; since M is flat, for $N=1$ they divide into two sets that transform respectively as a vector X^i and as a spinor ψ, or a vector and a set of spinors ψ^a for $N>1$, under the N-extended superPoincaré group corresponding to the isometries of M. The κ-symmetry is like a spacetime supersymmetry with a wrong sign, but its parameter is local on W, and it involves a projection which leaves effectively only half as many independent parameters. Gauge fixing the reparametrisations eliminates $p+1$ components of the set of coordinates X^i that transform as a vector; fixing the κ-invariance eliminates half of the components of the spinors ψ^a. As in the bosonic case, there is a residual rigid invariance, which corresponds again to the rotations among the transverse coordinates but this time to the superPoincaré group on W. What remains from the spacetime spinors, which before gauge fixing transformed as scalars under transformations on W, now transforms as a set of worldvolume spinors [12]. A similar phenomenon occurs for gauge theories with broken symmetries, and non-linear field theories like the sine-Gordon equation and its equivalence with the massive Thirring model, where solutions of a purely bosonic theory obey Fermi statistics.

After gauge fixing, therefore, the field theory looks as if it were defined on a supermanifold of which W is the bosonic part; all the remaining fields can be arranged into superfields on W, although this might not be a trivial algebraic task if the dimension of the spinors is large. Moreover, we see that by this mechanism the supersymmetries have been partially broken to half of their original number.

[12] This construction will make sense only when half of the dimension of the spinors in M is an acceptable value for some N' number of spinors on W. This is a necessary but not sufficient condition; a more stringent requirement is the existence of a non-zero Wess-Zumino term to build an action with κ invariance (which also needs the Wess-Zumino term to have a fixed coefficient).

For each particular case, it can be seen explicitly how half of the supersymmetries are linearly represented on the gauge-fixed fields, and the other half is non-linearly realised. This partial breaking also shows up in the algebra as the presence of a central charge, which implies that only half of the supersymmetries (the linearly realised ones) annihilate the vacuum.

Cohomology. The theory of the previous section carries over to the supersymmetric theory with little change, except that care must be taken when dealing with supermanifolds, and the symmetry properties of exterior products of Grassmann-valued forms.

Acknowledgements

I would like to thank Trinity College, Cambridge, the Overseas Research Scheme, and the Directorate General for Science Research and Development of the Commission of the European Communities under contract No. C11-0540-M(TT) for financial support.

I would also like to thank M Castagnino and the Instituto de Astronomía y Física del Espacio for their kind hospitality while this work was being completed, and J Gauntlett, A Kent, N Landsman, S Majid, and G Watts for helpful comments and suggestions.

Appendix A

In this appendix we collect some diagrams that may help to follow the reasoning of section 2.

We shall use a somewhat non-standard notation but we hope it is clear. We shall denote a fibre bundle of total space B, base M, fibre F and projection π_B by

$$\begin{array}{ccc} F & \longrightarrow & B \\ & & \big\downarrow{\scriptstyle \pi_B} \\ & & M \end{array} .$$

The projection can be omitted if there is no ambiguity, and if necessary the fibre will be written on the right. Arrows going upwards from M to B will represent sections.

In this notation, the different fibre bundles that appear in the discussion on the 'matter' fields, i.e., the fields that appear in the theory apart from the Goldstone fields, can be summarised as follows:

Principal bundles

$$\begin{array}{ccc} H \longrightarrow Q & G \longrightarrow P & H \longrightarrow P \\ \big\downarrow & \big\downarrow & \big\downarrow \\ W & W & \widetilde{E} \\ \vdots & \vdots & \vdots \\ \downarrow & \downarrow & \downarrow \end{array}$$

Associated bundles

$$\begin{array}{ccc} \mathcal{H} \longrightarrow E' & G/H \longrightarrow \widetilde{E} & \mathcal{H} \longrightarrow \widehat{E} \\ \big\downarrow\big\uparrow{\scriptstyle \Theta_{1,\xi}} & \big\downarrow\big\uparrow{\scriptstyle \hat{\xi}} & \big\downarrow\big\uparrow{\scriptstyle \Theta_2} \\ W & W & \widetilde{E} \end{array}$$

The relation between the two approaches to the treatment of these fields is given by the composition

$$\begin{array}{ccc} \mathcal{H} & \longrightarrow & \widehat{E} \\ & & \big\downarrow\big\uparrow{\scriptstyle \Theta_2} \\ G/H & \longrightarrow & \widetilde{E} \\ & & \big\downarrow\big\uparrow{\scriptstyle \hat{\xi}} \\ & & W \end{array}$$

with $\Theta_{1,\xi}(x) = \Theta_2(x, \xi(x))$ in some local trivialisation of $\widetilde{E}$, since $\hat{\xi}$ also induces an imbedding of E' into $\widehat{E}$.

The relation between sections in $\widetilde{E}$ and subbundles $Q(W,H)$ of $P(W,H)$ corresponds to the commutativity of the following diagram:

$$\begin{array}{ccc} Q & \overset{f}{\hookrightarrow} & P \\ {\scriptstyle \pi\circ f}\big\downarrow & {\scriptstyle \pi}\swarrow & \big\downarrow{\scriptstyle \mu} \\ W & \xrightarrow{\hat{\xi}} & \widetilde{E}. \end{array}$$

We also have the following diagram,

$$\begin{array}{ccccccc} H & \longrightarrow & Q & \xrightarrow{\tilde{\xi}} & G & \longleftarrow & H \\ & & \big\downarrow & & \big\downarrow & & \\ & & W & \xrightarrow{\xi} & G/H & & \end{array}$$

where $\tilde{\xi}$ is the lifting of ξ to the principal bundles Q and G. The connection on Q is the pullback by $\tilde{\xi}$ of the H-component of the Maurer-Cartan 1-form on G, $\omega' = \tilde{\xi}^*(\theta_{\mathcal{H}})$.

References

[BR] A.O. Barut and R. Raczka, *Theory of group representations and applications* (World Scientific, 1986).

[C1] S.Coleman, *Secret symmetry*, in *Laws of Hadronic matter* (Academic Press, 1975).

[C3] S. Coleman, *Quantum sine-Gordon equation as the massive Thirring model*, Phys. Rev. **D11** (1975), 2088.

[CCWZ] S. Coleman, C. Callan, J. Wess and B. Zumino, *Structure of Phenomenological Lagrangians II*, Phys. Rev. **177** (1969), 2247.

[CWZ] S. Coleman, J. Wess and B. Zumino, *Structure of Phenomenological Lagrangians I*, Phys. Rev. **177** (1969), 2239.

[CY] M. Castagnino and C. Yastremiz, *Gauge Theory of Conformal Group*, in *Quantum Mechanics of Fundamental Systems 1*, C. Teitelboim ed. (Plenum, 1988), p. 73.

[deA] J.A. de Azcárraga, *Lie Groups, Lie Algebras and Cohomology*, unpublished lecture notes, preprint DAMTP 89-10.

[GY] J.P. Gauntlett and C.F. Yastremiz, *Massive superparticle in d=2+1 and sigma model solitons*, Class. Quantum Grav. **7** (1990), 2089.

[He] S. Helgason, *Differential Geometry, Lie Groups and Symmetric Spaces* (Academic Press, 1978).

[HP] J. Hughes and J. Polchinski, *Partially broken global supersymmetry and the superstring*, Nucl. Phys. **B278** (1986), 147.

[H'tH] P. Hasenfratz and G. 't Hooft, *Fermion-boson puzzle in a gauge theory*, Phys. Rev. Lett. **36** (1976), 1119.

[IO] E.A. Ivanov and V.I. Ogievetsky, *The inverse Higgs phenomenon in nonlinear realizations*, Teor. Mat. Fiz. **25** (1975), 164.

[JR] R. Jackiw and C. Rebbi, *Spin from isospin in a gauge theory*, Phys. Rev. Lett. **36** (1976), 1116.

[KN] S. Kobayashi and K. Nomizu, *Foundations of Differential Geometry*, vol. I (Interscience, 1963).

[MK] G. Mackey, *Unitary group representations — in Physics, probability and number theory* (Addison-Wesley, 1989).

[MMSS] G. Marmo, G. Morandi, A. Simoni and E.C.G. Sudarshan, *Quasi invariance and central extensions*, Phys. Rev. **D37** (1988), 2196.

[O] V.I. Ogievetsky, *Non-linear realizations of Internal and Spacetime Symmetries*, in *Proceedings of the 10'th Karpacz Winter School of Theoretical Physics* (Acta Universitatis Wratislaviensis, No207, 1974), vol. 1, p. 117.

[O'N] B. O'Neill, *Sémi-Riemannian Geometry — with applications to Relativity* (Academic Press, 1983).

[PT] R.S. Palais and C-l. Terng, *Critical point theory and submanifold theory*, Lecture Notes in Mathematics 1353 (Springer-Verlag, 1988).

[Ru] P. J. Ruback, *Sigma model solitons and their moduli space metrics*, Comm. Math. Phys. **116** (1988), 645.

[SW] A. Sagle and R.E. Walde, *Introduction to Lie Groups and Lie Algebras* (Academic Press, 1973).

[V] D. V. Volkov, *Phenomenological Lagrangians*, Soviet J. Particles Nucl. **4** (1973), 1.