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SOME REMARKS ON THE q-POINCARE ALGEBRA IN R-MATRIX FORM

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Abstract The braided approach to q-deformation (due to the author and collaborators) gives natural algebras $R_{21}\mathbf{u}_1R\mathbf{u}_2 = \mathbf{u}_2R_{21}\mathbf{u}_1R$ and $R_{21}\mathbf{x}_1\mathbf{x}_2 = \mathbf{x}_2\mathbf{x}_1R$ for q-Minkowski and q-Euclidean spaces respectively. These algebras are covariant under a corresponding background 'rotation' quantum group. Semidirect product by this according to the bosonisation procedure (also due to the author) gives the corresponding Poincaré quantum groups. We review the construction and collect the resulting R-matrix formulae for both Euclidean and Minkowski cases in both enveloping algebra and function algebra form, and the duality between them. Axioms for the Poincaré quantum group $*$-structure and the dilaton problem are discussed.

1 Introduction

The programme of q-deforming the basic geometrical notions of spacetime has been extensively studied in recent years and by several groups. Of the various approaches, two have been pushed quite far. One, which is the 'minimal deformation', involves a non-commutative time co-ordinate but the space co-ordinates remain unchanged. The corresponding Poincaré quantum group is the so-called 'κ-Poincaré' studied by Lukierski *et al*[1] and others, obtained by a contraction procedure. Its semidirect product structure in terms of (usual) Lorentz rotations and a κ-deformed 4-momentum is due to the author and Ruegg[2] and came later, along with the identification of the correct (non-commutative) Minkowski space such that the κ-Poincaré algebra acts covariantly on it. The semidirect structure is an example of a *bicrossproduct Hopf algebra* as introduced in[3]. We don't want to say too much about this here except that this approach uses the standard theory of Hopf algebras (or quantum groups) and not the more novel braided group theory.

The second deformation, which is the one that concerns us in this paper, is a programme introduced first by Carow-Watamura *et al.* [4][5] from consideration of the tensor product of

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two quantum planes (the 2-spinorial or twistor point of view). At about the same time in 1990, the present author introduced independently a theory of braided groups[6][7], including braided matrices $B(R)$[8] with relations

$$\mathbf{u} = \{u^i{}_j\}, \quad R_{21}\mathbf{u}_1 R\mathbf{u}_2 = \mathbf{u}_2 R_{21}\mathbf{u}_1 R \tag{1}$$

which for the 2×2 case also provides a natural definition of q-deformed Minkowski space. There is a hermitian $*$-structure whenever R is of real type[9]. This is the approach which we consider in the present paper. For the standard $SL_q(2)$ or Jones-polynomial solution R of the Quantum Yang-Baxter Equations (QYBE) we have the algebra $BM_q(2)$[8][5]

$$\begin{gathered}\mathbf{u} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad ba = q^2 ab, \quad ca = q^{-2}ac, \quad da = ad \\ bc = cb + (1-q^{-2})a(d-a), \quad db = bd + (1-q^{-2})ab, \quad cd = dc + (1-q^{-2})ca\end{gathered} \tag{2}$$

which means that our braided matrix approach includes and extends the approach of Carow-Watamura *et al.* We define its $*$-algebra of co-ordinate functions as $\mathrm{R}_q^{1,3} = BM_q(2)$. It has time direction central and the space directions mutually non-commuting. Note that the braided approach replaces such explicit relations by R-matrix formulae, which are easier to work with and more general.

The chacteristic feature of this braided approach, as well as being always of a general R-matrix form, is that the underlying objects are not Hopf algebras or quantum groups in any usual sense. Instead, we require the new concept, introduced in [6][8][7], of a *braided group.* This is like a quantum group but the coproduct map $\Delta : B \to B\underline{\otimes}B$, say, is a homomorphism with respect to the *braided tensor product* algebra in which the two tensor factors in $B\underline{\otimes}B$ do not commute. Instead they enjoy mutual *braid statistics*, given in our examples by R-matrices (and any q or other parameters in them). Long introductions to the general theory of braided groups are in [10] and [11]. See also chapter 10 of my textbook on quantum groups[12].

We begin in the preliminary Section 2 by recalling from [8][13] the multiplicative and additive braided group structures in the braided matrices $B(R)$. This indeed justifies this name for the algebra (1), for it corresponds when $q = 1$ to the multiplication and addition of usual matrices. For $\mathrm{R}_q^{1,3}$ in the twistor viewpoint, the first is needed to pick out a natural q-determinant or square-radius function (which determines the quantum metric) and the second (which is due to U. Meyer[13]) gives the linear structure of spacetime (such as the addition of 4-momentum).

The general framework of *braided coaddition* for linear braided group structures was introduced in [14], where we showed, quite generally, how to build from this a Poincaré quantum group by a ***bosonisation*** construction[15][14]. We adjoin the q-Lorentz generators by a kind of semidirect product which is different, however, from the bicrossproducts needed for the κ-deformation mentioned above. The abstract picture is summarized in the Appendix A. The calculation for the specific case of braided matrices $B(R)$ was done in [16] and gave the q-Poincaré (function algebra) quantum group in R-matrix form, for the first time. We recall this is Section 3. We also explained at the end of [16] that once in R-matrix form, it is an easy matter to dualise and obtain the Poincaré quantum group in enveloping algebra form. The modest contribution in the present paper is to give the resulting formulae explicitly for those who do not want to carry out the exercise. The result originates, however, in [16] and the formulae are very similar. This is covered in Section 4. The standard R-matrix gives the Minkowski space q-Poincaré quantum enveloping algebra which can be compared with explicit generators and relations in [17].

Next we turn our attention to q-Euclidean space. There is a parallel R-matrix theory for this, introduced (in the R-matrix form) in [18] based on the algebra $\bar{A}(R)$

$$\mathbf{x} = \{x^i{}_j\}, \quad R_{21}\mathbf{x}_1\mathbf{x}_2 = \mathbf{x}_2\mathbf{x}_1 R \tag{3}$$

in place of (1). The standard $SU_q(2)$ or Jones polynomial R-matrix gives an algebra $\bar{M}_q(2)$

$$\begin{gathered} \mathbf{x} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}; \qquad ba = qab, \quad ca = q^{-1}ac, \quad da = ad \\ bc = cb + (q - q^{-1})ad \qquad db = q^{-1}bd \quad dc = qcd \end{gathered} \tag{4}$$

which is (by an accident) isomorphic to the usual FRT bialgebra $A(R) = M_q(2)$ of quantum matrices. The latter was proposed as q-Euclidean space in [5] so, once again, the braided approach includes and extends that pioneering work in a general R-matrix form (3) introduced in this context in [18]. We define the non-commuting co-ordinate functions as $\mathrm{R}_q^4 = \bar{M}_q(2)$. There is a natural unitary-like $*$-structure with corresponding Euclidean-type norm defined by the natural q-determinant. Also in [18] is a theory of twisting or 'quantum Wick rotation' which says that the $\bar{A}(R)$ algebra (3) is 'gauge equivalent' (in a sense generalising ideas of Drinfeld[19]) to our first $B(R)$ algebra (1). If we did not have a different $*$ then we would have something strictly equivalent (via twisting) to the already-established q-Minkowski space above, which would not be very interesting. Instead, we put the algebra to good use as a complement to the q-Minkowski above[18].

The treatment of (3) as an additive braided group (or braided covector space) is in Section 5. We also give its multiplicative structure, which is like the braided group $B(R)$ but the non-commutativity required does not obey the Artin braid relations (or QYBE), so it is something a bit beyond even the concept of a braided group. In Section 6 we recall the corresponding q-Poincaré quantum group in R-matrix form as obtained by the general bosonisation construction, applied now to (3). This is also from[18]. Once again, we emphasized there the function algebra theory and left the dualisation for the enveloping algebra q-Poincaré quantum group as an easy exercise. For completeness, we do this now in Section 7. The standard R-matrix gives the Euclidean group of motions and can be compared with the $n = 4$ case of computations based on $SO_q(n)$-covariant quantum planes in [20] and elsewhere. We also give explicitly the duality pairing and, accordingly, the action of the Poincaré quantum group on spacetime co-ordinates $B(R)$ and $\bar{A}(R)$. Note that we will use the term 'Poincaré' quantum group to cover all dimensions and signatures since our construction is quite uniform.

Let us stress that this paper for the most part collects results and formulae already obtained in some form in [14][16][18] modulo conventions and elementary dualisation. Nevertheless, it is hoped that a self-contained account now will be useful as an overview and introduction. It complements an extensive 50-page introduction to our 'braided geometry' approach to q-Minkowski space in [21]. The extensive further literature on this topic, including works by the author[22][23] and U. Meyer[24], as well as by the Munich and Berkeley groups are covered there. It could be said that the underlying geometry is fairly well understood by now, though not the full story regarding the $*$-structure and the construction of actual q-deformed quantum field theories on these spacetimes, which remain a goal and motivation for the subject. Such a q-deformation would surely have an application either as a tool for regularising infinities (as poles at $q = 1$)[25] or as a model of quantum or other effects on the structure of geometry at the Planck scale[26][27].

Finally, Section 8 contains some newer material. We point out that the bosonisation point of view does suggest a natural solution to the problem of what should be the correct axioms for the $*$-structure on our Poincaré quantum groups. This makes contact with some preliminary ideas in the preprint of Fiore[28]. We also observe that one can avoid the dilatonic extension needed in the constructions above by only partially bosonising. In this case the Poincaré algebra

is a braided group not a quantum group, but of a fairly mild kind where the braid-statistics are given by a phase factor as in [29].

To be clear about normalisation, what we call the standard $SU_q(2)$ or Jones-polynomial R-matrix is

$$R = \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & 1 & q-q^{-1} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & q \end{pmatrix} \tag{5}$$

where the rows and columns label the two copies of M_n in $M_n \otimes M_n$, wherein R lives. Our formulae are quite general and not at all limited to this particular R-matrix.

2 Preliminaries I: q-Minkowski spaces in R-matrix form

The braided-matrix approach to q-Minkowski space was presented at the Zdikov Winter School in January 1993, at the Guadeloupe Spring School[30] in May 1993 and in Clausthal[31] in July 1993. So this preliminary section and the beginning of the next will recall some of that basic material from [9][13].

The idea is that in classical geometry one can take the space of hermitian matrices with norm given by the determinant, as Minkowski space. So let us build a convincing q-deformation of the concept of a hermitian matrix. To be a matrix, we need to be able to linearly add, and matrix multiply (with an identity for the multiplication). In our algebraic language we indeed have this on the braided matrices $B(R)$ as a *braided matrix comultiplication* and *counit*

$$\Delta \mathbf{u} = \mathbf{u} \otimes \mathbf{u}, \quad \epsilon \mathbf{u} = \mathrm{id} \tag{6}$$

where Δ is an algebra homomorphism $B(R) \to B(R) \underline{\otimes} B(R)$. The *multiplicative braid statistics* needed for it to extend as an algebra homomorphism are[8]

$$\mathbf{u}'' = \mathbf{u}\mathbf{u}'; \qquad R^{-1}\mathbf{u}_1' R \mathbf{u}_2 = \mathbf{u}_2 R^{-1} \mathbf{u}_1' R \tag{7}$$

whereby $\mathbf{u}''$ obeys the relations (1) of $B(R)$ if $\mathbf{u}, \mathbf{u}'$ do.

We also have a hermitian $*$-structure $u^i{}_j{}^* = u^j{}_i$ whenever R is of real-type[9], obeying the axioms (introduced there) of a $*$-braided group of real type,

$$\Delta \circ * = (* \otimes *) \circ \tau \circ \Delta, \quad \epsilon \circ * = \overline{\epsilon(\ )}. \tag{8}$$

where τ is transposition. When there is an 'inverse' or braided antipode S (which is not the case for braided matrix multiplication in $B(R)$ of course) we demand also $* \circ S = S \circ *$. The astute reader may wonder, by the way, about these axioms (8). They are quite different from the usual axioms of a Hopf $*$-algebra. Indeed, usual hermitian matrices do not form a group under composition, which is why in algebraic terms Δ is not a $*$-algebra homomorphism. But if A, B are hermitian then $(AB)^* = BA$, which is why Δ commutes with $*$ when we put in the extra transposition τ. This means that the algebraic notion of $*$-braided groups or $*$-quantum groups with the transposed axiom (8) is useful even for $R = 1$, where it allows us to view the space of hermitian matrices (e.g. the mass-shell in Minkowski space) as a 'group' in this generalised sense.

Finally, we have, at least when R is q-Hecke, a *braided coaddition*[13]

$$\Delta \mathbf{u} = \mathbf{u} \otimes 1 + 1 \otimes \mathbf{u}, \quad \epsilon \mathbf{u} = 0, \quad S\mathbf{u} = -\mathbf{u} \tag{9}$$

again extended as an algebra homomorphism $\Delta : B(R) \to B(R) \underline{\otimes} B(R)$. This time we use Meyer's *additive braid statistics*[13]

$$\mathbf{u}'' = \mathbf{u} + \mathbf{u}'; \qquad R^{-1}\mathbf{u}_1' R \mathbf{u}_2 = \mathbf{u}_2 R_{21} \mathbf{u}_1' R. \tag{10}$$

When R is of real type, we have again a $*$-braided group with our $*$ as above and obeying the same axioms (8). In this case the transposition τ in the axioms would not be visible when $R = 1$ because in this case the coaddition Δ would be cocommutative. This is correct because there is no problem adding hermitian matrices.

These basic properties amply justify the term 'braided matrices' for the algebra (1). We also frequently write all the four R-matrices in these equations on one side as 'big' multiindex matrices $\mathbf{R}$, $\mathbf{R}'$ etc. where we consider $u^i{}_j = u_I$ as a vector with multiindex $I = (i_0, i_1)$. This is how in [8] we wrote equations such as (1), which occur in other contexts too[32], in 'Zamalodchikov' or *braided covector* form

$$\mathbf{u}_1 \mathbf{u}_2 = \mathbf{u}_2 \mathbf{u}_1 \mathbf{R}', \qquad \mathbf{u}'' = \mathbf{u} + \mathbf{u}'; \qquad \mathbf{u}_1' \mathbf{u}_2 = \mathbf{u}_2 \mathbf{u}_1' \mathbf{R} \tag{11}$$

for suitable $\mathbf{R}', \mathbf{R}$. They were introduced in [8],[13] under the names Ψ', R_L respectively. Likewise for (7) introduced in [8] as Ψ. We assume that the reader can freely transfer back and forth between the multiindex $\mathbf{R}, \mathbf{R}'$ braided covector form and the previous matrix form (1) etc.,

without imagining that they are anything but different notations. The reason we use both forms is that the matrix form emphasizes the braided comultiplication structure while the braided covector form emphasises the linear structure as like a quantum plane. There is a detailed theory of braided covectors $V^{\vee}(\mathbf{R}',\mathbf{R})$ introduced in [14][33] which we can now use for $B(R)$ as a linear braided space. This includes covariance and Poincaré quantum groups[14], a theory of braided-differentiation, integration[34], epsilon tensor and electromagnetism[22], and so on – all the constructions we are familiar with for R^n.

For our standard example where R is (5), we have the multiplicative braid statistics[8]

$$\begin{gathered} a'a = aa' + (1-q^2)bc', \quad a'b = ba', \quad a'c = ca' + (1-q^2)(d-a)c' \\ a'd = da' + (1-q^{-2})bc', \quad b'a = ab' + (1-q^2)b(d'-a'), \quad b'b = q^2bb', \quad \text{etc.} \end{gathered} \tag{12}$$

for $\begin{pmatrix} a'' & b'' \\ c'' & d'' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$ to obey the relations (2), and additive braid statistics[13]

$$\begin{gathered} a'a = q^2aa', \quad a'b = ba', \quad a'c = q^2ca' + (q^2-1)ac', \quad c'a = ac' \\ a'd = da' + (q^2-1)bc' + q^{-2}(q^2-1)^2aa', \quad b'a = q^2ab' + (q^2-1)ba', \quad b'b = q^2bb', \quad \text{etc.} \end{gathered} \tag{13}$$

for $\begin{pmatrix} a'' & b'' \\ c'' & d'' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$ to obey (2).

The braided comultiplication is needed (as for usual matrices) to fix the *braided determinant* as group-like or 'multiplicative'. It comes out as [8]

$$\underline{\det}(\mathbf{u}) = ad - q^2cb \tag{14}$$

and is central, as well as bosonic with respect to the multiplicative braid statistics. We use is as a square-distance function on $BM_q(2)$.

Finally, we have the hermitian $*$,

$$\begin{pmatrix} a* & b* \\ c* & d* \end{pmatrix} = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \tag{15}$$

so that these matrices are indeed naturally hermitian, and hence obviously provide a natural definition $\mathrm{R}_q^{1,3} = BM_q(2)$, as explained in [9]. The $*$-structure is needed to determine what should be the 'real' or self-adjoint space-time co-ordinates under $*$. The required linear combinations are

$$t = \frac{q^{-1}a + qd}{2}, \quad x = \frac{b+c}{2}, \quad y = \frac{b-c}{2i}, \quad z = \frac{d-a}{2} \tag{16}$$

and the braided determinant becomes[9]

$$\underline{\det}(\mathbf{u}) = \frac{4q^2}{(q^2+1)^2}t^2 - q^2x^2 - q^2y^2 - \frac{2(q^4+1)q^2}{(q^2+1)^2}z^2 + \left(\frac{q^2-1}{q^2+1}\right)^2 2qtz \tag{17}$$

which justifies indeed the interpretation as Minkowski length from the braided approach. From it we can extract a *quantum metric* tensor by braided differentiation[22] which in our matrix basis is

$$\eta^{IJ} = \begin{pmatrix} q^{-2}-1 & 0 & 0 & 1 \\ 0 & 0 & -q^{-2} & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}; \quad \underline{\det}(\mathbf{u}) = (1+q^{-2})^{-1}\eta^{IJ}u_J u_I. \tag{18}$$

3 Minkowski q-Poincaré quantum group in function algebra form

One of the first consequences of writing the braided matrices $B(R)$ algebra (1) in the braided covector form (11) is that we know at once from [14] what its associated Poincaré quantum group looks like. For in [14] was introduced a completely general R-matrix construction for such objects based on a theory of 'bosonisation' in [15]. The formulae are as follows. For the 'rotation group' we take the usual FRT bialgebra[32] $A(\mathbf{R})$ with generator ${\Lambda^I}_J$ say, and relations

$$\mathbf{R}\Lambda_1\Lambda_2 = \Lambda_2\Lambda_1\mathbf{R}, \quad \Delta\Lambda = \Lambda\otimes\Lambda, \quad \epsilon\Lambda = \mathrm{id} \tag{19}$$

to which we add relations needed to give us a Hopf algebra with antipode. They include such things as a metric relation

$$\Lambda_1\Lambda_2\eta_{21} = \eta_{21}, \quad \text{i.e.,} \quad S{\Lambda^I}_J = {\Lambda^A}_B\eta^{BI}\eta_{AJ} \tag{20}$$

where we let η_{IJ} be the transposed inverse of η^{IJ}. This is the vectorial approach to the Lorentz quantum group in [13]. Next, we extend this by adjoining a central group-like element ς (the dilaton) in such a way that our braided covectors are fully covariant under the extended transformation

$$\mathbf{u} \to \mathbf{u}\Lambda\varsigma \tag{21}$$

with additive braid statistics correctly induced by this covariance. We need ς to achieve this because $\mathbf{R}$ as given is not in the 'quantum group normalisation' and we have to compensate for this[14]. We now use the braided covector generators $\mathbf{u}$ for the momentum sector of our q-Poincaré quantum group, and denote them as such by $\mathbf{p}$ to avoid confusion with spacetime

co-ordinates. We now make a coalgebra semidirect coproduct by the above coaction. At the same time, the dual quasitriangular structure[13] (universal R-matrix functional) of our extended Lorentz quantum group converts this coaction to an action, and we make an algebra semidirect product by this. The theory of bosonisation ensures that the result is necessarily a Hopf algebra. Some of this general theory is recalled in Appendix A. All we need to know for now is the resulting R-matrix formula[14]

$$\begin{gathered}\mathbf{p}_1\mathbf{p}_2 = \mathbf{p}_2\mathbf{p}_1\mathbf{R}', \quad \mathbf{p}_1\Lambda_2 = \lambda\Lambda_2\mathbf{p}_1\mathbf{R}, \quad \mathbf{p}\varsigma = \lambda^{-1}\varsigma\mathbf{p}, \quad [\Lambda,\varsigma] = 0 \\ \Delta\varsigma = \varsigma\otimes\varsigma, \quad \epsilon\varsigma = 1, \quad S\varsigma = \varsigma^{-1} \\ \Delta\mathbf{p} = \mathbf{p}\otimes\Lambda\varsigma + 1\otimes\mathbf{p}, \quad \epsilon\mathbf{p} = 0, \quad S\mathbf{p} = -\mathbf{p}\varsigma^{-1}\Lambda^{-1}\end{gathered} \tag{22}$$

where λ is the *quantum group normalisation constant* of $\mathbf{R}$ defined such that $\lambda\mathbf{R}$ is in the quantum group normalisation[35]. Some special cases of this construction for $ISO_q(n)$, etc., were first studied in [36], though without the above general construction. Finally, no proposal for a q-Poincaré quantum group is complete without a covariant action of it on the q-spacetime co-ordinates. The general construction[14] introduced just this, in the coaction form

$$\mathbf{u} \to \mathbf{u}\Lambda\varsigma + \mathbf{p} \tag{23}$$

extended as an algebra homomorphism. The $\mathbf{u}$ commute with the Poincaré generators. This covariance was one of the main achievements of the braided approach in [14] and is a general feature of the bosonisation theory recalled in Appendix A.

This is the *vectorial form* of the q-Poincaré quantum group when we apply it to (1) in the form (11). There is also a *spinorial form* obtained when we unwind the above construction back in terms of R rather than $\mathbf{R}', \mathbf{R}$. Firstly, we replace the quantum group $A(\mathbf{R})$ by $A(R)$ and make this into a Hopf algebra A. Two copies of it are needed to play the role of $A(\mathbf{R})$ above, with generators $\mathbf{s}, \mathbf{t}$ say, forming a double cross product Hopf algebra $A{\bowtie}A$[37]

$$\begin{gathered}R\mathbf{s}_1\mathbf{s}_2 = \mathbf{s}_2\mathbf{s}_1R, \quad R\mathbf{t}_1\mathbf{t}_2 = \mathbf{t}_2\mathbf{t}_1R, \quad R\mathbf{t}_1\mathbf{s}_2 = \mathbf{s}_2\mathbf{t}_1R \\ \Delta\mathbf{s} = \mathbf{s}\otimes\mathbf{s}, \quad \Delta\mathbf{t} = \mathbf{t}\otimes\mathbf{t}, \quad \epsilon\mathbf{s} = \mathrm{id}, \quad \epsilon\mathbf{t} = \mathrm{id}\end{gathered} \tag{24}$$

among the further relations needed to give $\mathbf{s}, \mathbf{t}$ antipodes. The vectorial form is realised in terms of the spinorial form by

$$\Lambda^I{}_J = (Ss^{j_0}{}_{i_0})t^{i_1}{}_{j_1} \tag{25}$$

and the covariance (21) takes the form cf[4]

$$\mathbf{u} \to \mathbf{s}^{-1}\mathbf{u}\mathbf{t}\varsigma. \tag{26}$$

We again adjoin a central grouplike dilaton ς to take care of the fact that R is not in the quantum group normalisation. The general construction of the quantum group $A{\bowtie}A$ was developed in [14, Sec. 4], as well as the isomorphism with Drinfeld's quantum double[38] and the identification (relevant later) as a twisting of $A \otimes A$. It connects in the $SU_q(2){\bowtie}SU_q(2)$ case with the proposal for q-Lorentz group in [4] and with the proposal as the quantum double in [39]. There is a $*$-structure

$$s^i{}_j{}^* = St^j{}_i, \quad t^i{}_j{}^* = Ss^j{}_i, \quad \varsigma^* = \varsigma \tag{27}$$

as in [4], which now works for general R of real-type[14]. The diagonal case $\mathbf{t} = \mathbf{s}$ (without the dilaton) is covariance under the spacetime spinor rotation group $SU_q(2)$ and preserves the multiplicative structure, the distance function $\underline{\det}(\mathbf{u})$ and the time co-ordinate t[8].

The formulae (22) then become the spinorial q-Poincaré quantum group[16]

$$\begin{gathered} R_{21}\mathbf{p}_1 R\mathbf{p}_2 = \mathbf{p}_2 R_{21}\mathbf{p}_1 R, \quad \mathbf{p}_1\mathbf{s}_2 = \mathbf{s}_2 R^{-1}\mathbf{p}_1 R, \quad \mathbf{p}_1\mathbf{t}_2 = \lambda\mathbf{t}_2 R_{21}\mathbf{p}_1 R \\ \mathbf{p}\varsigma = \lambda^{-1}\varsigma\mathbf{p}, \quad [\mathbf{s},\varsigma] = [\mathbf{t},\varsigma] = 0, \quad \Delta\varsigma = \varsigma\otimes\varsigma, \quad \epsilon\varsigma = 1, \quad S\varsigma = \varsigma^{-1} \\ \Delta\mathbf{p} = \mathbf{p}\otimes\mathbf{s}^{-1}(\ )\mathbf{t}\varsigma + 1\otimes\mathbf{p}, \quad \epsilon\mathbf{p} = 0, \quad S\mathbf{p} = -\mathbf{p}S(\mathbf{s}^{-1}(\ )\mathbf{t}\varsigma) \end{gathered} \tag{28}$$

where $\mathbf{s}^{-1}(\ )\mathbf{t}$ has a space for the matrix indices of $\mathbf{p}$ to be inserted. The constant λ is the square of the quantum group normalisation constant of R. Its value for our standard example is $\lambda = q^{-1}$. One can again derive this construction by the more abstract bosonisation construction (84) in Appendix A, knowing only the full covariance (26). The two methods give the same answer. Finally, the coaction of the q-Poincaré quantum group on the spacetime co-ordinates is the algebra homomorphism

$$\mathbf{u} \to \mathbf{s}^{-1}\mathbf{u}\mathbf{t}\varsigma + \mathbf{p} \tag{29}$$

where the $\mathbf{u}$ commute with the Poincaré generators.

Recall that we also have a multiplicative braided group structure on $B(R)$. The 'mass shell' in q-Minkowski space is the $*$-braided group $BSU_q(2)$ in [8] with (braided) antipode. If we bosonise by the above rotational $SU_q(2)$ covariance, we obtain this time[9] the quantum double of $SU_q(2)$ which, as noted above, is isomorphic to the q-Lorentz group $SU_q(2){\bowtie}SU_q(2)$ in spinorial form. Such an isomorphism has no classical counterpart, being singular at $q = 1$.

4 Minkowski q-Poincaré quantum group in enveloping algebra form

Once we have our Poincaré quantum groups in R-matrix form, it is a pretty easy exercise to dualise them and find the corresponding Poincaré quantum group enveloping algebras. We just dualise the individual pieces of the bosonisation or semidirect construction. We suppose that the dual quantum group to the Lorentz quantum group function algebra can be put in FRT form with generators $\mathbf{L}^\pm$ say and relations[32]

$$\mathbf{L}_1^\pm\mathbf{L}_2^\pm\mathbf{R} = \mathbf{R}\mathbf{L}_2^\pm\mathbf{L}_1^\pm, \quad \mathbf{L}_1^-\mathbf{L}_2^+\mathbf{R} = \mathbf{R}\mathbf{L}_2^+\mathbf{L}_1^-, \quad \Delta\mathbf{L}^\pm = \mathbf{L}^\pm\otimes\mathbf{L}^\pm, \quad \epsilon\mathbf{L}^\pm = \mathrm{id} \tag{30}$$

among others needed in the dual. We also define the dual of the dilaton generator ς to be the generator ξ of the enveloping algebra $U_q(1)$, which we define with a non-standard universal R-matrix[40]. We extend our Lorentz enveloping algebra by this. Finally, for the dual of our previous momentum co-ordinates p_I as a braided covector space $V^{\vee}(\mathbf{R}',\mathbf{R})$ we need look no further than the corresponding *braided vector algebra* $V(\mathbf{R}',\mathbf{R})$ which is arranged carefully in [14][33] to be the braided group dual to the braided covectors. Its generators P^I, say, have upper indices. The pairing between our objects in the last section and our new enveloping algebra generators is then

$$\langle\Lambda_1,\mathbf{L}_2^+\rangle = \lambda\mathbf{R}, \quad \langle\Lambda_1,\mathbf{L}_2^-\rangle = \lambda^{-1}\mathbf{R}_{21}^{-1}, \quad \langle P^I,p_J\rangle = \delta^I{}_J, \quad \langle\varsigma,\xi\rangle = 1 \tag{31}$$

with the trivial pairing provided by the counits between the different quantum or braided groups.

Using this pairing we can then deduce the Poincaré quantum group in enveloping algebra form in the standard way by dualising. The resulting structure is[14]

$$\begin{gathered}\mathbf{P}_1\mathbf{P}_2 = \mathbf{R}'\mathbf{P}_2\mathbf{P}_1, \quad \mathbf{L}_1^+\mathbf{P}_2 = \lambda^{-1}\mathbf{R}_{21}^{-1}\mathbf{P}_2\mathbf{L}_1^+, \quad \mathbf{L}_1^-\mathbf{P}_2 = \lambda\mathbf{R}\mathbf{P}_2\mathbf{L}_1^-, \quad \lambda^\xi\mathbf{P} = \lambda^{-1}\mathbf{P}\lambda^\xi \\ [\xi,\mathbf{L}^\pm] = 0, \quad \Delta\xi = \xi\otimes 1 + 1\otimes\xi, \quad \epsilon\xi = 0, \quad S\xi = -\xi \\ \Delta\mathbf{P} = \mathbf{P}\otimes 1 + \lambda^\xi\mathbf{L}^-\otimes\mathbf{P}, \quad \epsilon\mathbf{P} = 0, \quad S\mathbf{P} = -\lambda^{-\xi}(S\mathbf{L}^-)\mathbf{P}.\end{gathered} \tag{32}$$

We can also obtain the above formulae by starting with covariance under the Hopf algebra obtained from (30) and dilatonic extension, namely

$$\mathbf{L}_1^+\triangleright\mathbf{P}_2 = \lambda^{-1}\mathbf{R}_{21}^{-1}\mathbf{P}_2, \quad \mathbf{L}_1^-\triangleright\mathbf{P}_2 = \lambda\mathbf{R}\mathbf{P}_2, \quad \lambda^\xi\triangleright\mathbf{P} = \lambda^{-1}\mathbf{P} \tag{33}$$

and applying it to the enveloping-type bosonisation theorem (85) in Appendix A. Finally, we can dualise our coaction (23) on the spacetime co-ordinates by evaluating the relevant part of

the output of the coaction against our q-Poincaré enveloping algebra generators. It obviously becomes now an action of the q-Poincaré enveloping algebra

$$\mathbf{L}_1^+ \triangleright \mathbf{u}_2 = \mathbf{u}_2 \lambda \mathbf{R}_{21}, \quad \mathbf{L}_1^- \triangleright \mathbf{u}_2 = \mathbf{u}_2 \lambda^{-1} \mathbf{R}^{-1}, \quad \lambda^{\xi} \triangleright \mathbf{u} = \lambda \mathbf{u}, \quad P^I \triangleright u_J = \delta^I{}_J \tag{34}$$

necessarily making our position co-ordinates $\mathbf{u}$ a module algebra. This means that it extends to products using the coproduct in (22). For example, we see easily that

$$P^I \triangleright (u_{I_1} \cdots u_{I_m}) = u_{J_2} \cdots u_{J_m} \left[m; \mathbf{R}_{21}^{-1} \right]^{I J_2 \cdots J_m}_{I_1 I_2 \cdots I_m} \tag{35}$$

where $[m; \mathbf{R}_{21}^{-1}]$ is a *braided integer matrix* as introduced in [33] in the theory of braided differentiation on linear braided spaces. It means that the realisation of the P^I generators acting on our braided spacetime is exactly by braided differentiation. We concentrated in [33] on braided differentials ∂^I with matrix $[m; \mathbf{R}]$; the other case $\bar{\partial}^I$ say, is equivalent by a symmetry principle[33, Sec. V]. If $V^{\vee}(\mathbf{R}', \mathbf{R})$ is a braided covector algebra then so is $V^{\vee}(\mathbf{R}', \mathbf{R}_{21}^{-1})$. It lives in the 'opposite' category with inverse-transposed braiding. So $P^I = \bar{\partial}^I$ is our natural realisation of the momentum sector.

One of the main results in [33] is that ∂^I (and hence also $\bar{\partial}^I$) always obey the relations of the braided vector algebra $V(\mathbf{R}', \mathbf{R})$, which confirms their role now as representing the P^I. The ∂^I extend to products with the inverse braiding Ψ^{-1} and the $\bar{\partial}^I$ with the usual braiding Ψ, i.e. by a *braided Leibniz rule*[33]. If we consider u_I also as an operator by left multiplication then the corresponding braided-Leibniz rules are expressed in general as the commutation relations

$$\partial_1 \mathbf{u}_2 - \mathbf{u}_2 \mathbf{R}_{21} \partial_1 = \mathrm{id}, \quad \bar{\partial}_1 \mathbf{u}_2 - \mathbf{u}_2 \mathbf{R}^{-1} \bar{\partial}_1 = \mathrm{id}. \tag{36}$$

This generalised and included considerations for specific algebras[41][42][17] to the completely general setting of differentiation on any braided covector space.

This is the vectorial form of the q-Poincaré enveloping algebra according to the general construction for any braided covector space [14]. Next we note that when there is quantum metric η, it provides[13] an isomorphism between the braided vectors and covectors as braided groups. This is in fact the abstract definition of a quantum metric in braided geometry. Hence in this case we don't have to work with braided covectors (though that is more canonical) but can instead use the quantum metric to lower the indices and work everywhere with braided covectors. To do this one needs the standard quantum metric identities

$$\eta_{IA} \mathbf{R}^{-1A}{}_J{}^K{}_L = \lambda^2 \mathbf{R}^A{}_I{}^K{}_L \eta_{AJ}, \quad \eta_{KA} \mathbf{R}^I{}_J{}^A{}_L = \lambda^{-2} \mathbf{R}^{-1I}{}_J{}^A{}_K \eta_{AL}, \quad \text{etc.} \tag{37}$$

deduced automatically from (20) by evaluating it against the universal R-matrix functional. If we make this change of basis to new covector generators $P_I = \eta_{IA}P^A$, then the vectorial q-Poincaré enveloping algebra with braided covectors $\mathbf{P}$ is clearly

$$\begin{gathered}\mathbf{P}_1\mathbf{P}_2 = \mathbf{P}_2\mathbf{P}_1\mathbf{R}', \quad \mathbf{L}_1^+\mathbf{P}_2 = \lambda\mathbf{P}_2\mathbf{R}_{21}\mathbf{L}_1^+, \quad \mathbf{L}_1^-\mathbf{P}_2 = \lambda^{-1}\mathbf{P}_2\mathbf{R}^{-1}\mathbf{L}_1^-, \quad \lambda^\xi\mathbf{P} = \lambda^{-1}\mathbf{P}\lambda^\xi \\ [\xi, \mathbf{L}^\pm] = 0, \quad \Delta\xi = \xi\otimes 1 + 1\otimes\xi, \quad \epsilon\xi = 0, \quad S\xi = -\xi \\ \Delta P_I = P_I\otimes 1 + \lambda^\xi SL^{-A}{}_I\otimes P_A, \quad \epsilon\mathbf{P} = 0, \quad SP_I = -\lambda^{-\xi}(S^2L^{-A}{}_I)P_A.\end{gathered} \tag{38}$$

The structure is a bosonisation according to (85) in Appendix A by $\mathbf{L}^\pm$ acting as in (34) since the P_I transform like the u_I under these, and $\lambda^\xi\triangleright P_I = \lambda^{-1}P_I$. The pairing (31) with the function algebra Poincaré generators from the last section and the action (34) on the spacetime co-ordinates of course get modified to

$$\langle P_I, p_J\rangle = \eta_{IJ}, \quad P_I\triangleright u_J = \eta_{IJ}. \tag{39}$$

One can do the same lowering of indices for the braided differentiation operators to $\partial_I = \eta_{IA}\partial^A$ and $\bar\partial_I = \eta_{IA}\bar\partial^A$, in which case these obey the braided covector relations and braided-Leibniz rule

$$\begin{gathered}\partial_1\partial_2 = \partial_2\partial_1\mathbf{R}', \quad \bar\partial_1\bar\partial_2 = \bar\partial_2\bar\partial_1\mathbf{R}' \\ \partial_1\mathbf{u}_2 - \lambda^{-2}\mathbf{u}_2\partial_1\mathbf{R}_{21}^{-1} = \eta, \quad \bar\partial_1\mathbf{u}_2 - \lambda^2\mathbf{u}_2\bar\partial_1\mathbf{R} = \eta\end{gathered} \tag{40}$$

by the same elementary rearrangements using the quantum metric identities (37). So our lower-index momentum generators are represented on spacetime by these lowered index braided differentials $P_I = \bar\partial_I$.

This is the vectorial Poincaré group enveloping algebra with lower indices. As in the start of Section 3, all the above is quite general. Now we apply it to the specific form of $\mathbf{R}', \mathbf{R}$ in (11) in Section 2, and unwind in terms of R and $\mathbf{u}$ as a braided matrix $B(R)$. This gives the q-Poincaré enveloping algebra in spinorial form. So we let $\mathbf{l}^\pm$ be dual to the generator $\mathbf{s}$ and $\mathbf{m}^\pm$ dual to the generator $\mathbf{t}$ for each copy of a quantum enveloping algebra H dual to A, with pairing given by R as in [32]. In the standard case they are each copies of $U_q(su_2)$ in FRT form. The required Lorentz enveloping algebra is a *double cross coproduct* $H \bowtie H$ and does *not* contain the copies

of H as sub-Hopf algebras. Instead they have a twisted coproduct[43]

$$\begin{gathered}\mathbf{l}_1^\pm\mathbf{l}_2^\pm R = R\mathbf{l}_2^\pm\mathbf{l}_1^\pm,\quad \mathbf{l}_1^-\mathbf{l}_2^+ R = R\mathbf{l}_2^+\mathbf{l}_1^-,\quad \mathbf{m}_1^\pm\mathbf{m}_2^\pm R = R\mathbf{m}_2^\pm\mathbf{m}_1^\pm\\ \mathbf{m}_1^-\mathbf{m}_2^+ R = R\mathbf{m}_2^+\mathbf{m}_1^-,\quad [\mathbf{l}_1^\pm,\mathbf{m}_2^\pm] = [\mathbf{l}_1^\pm,\mathbf{m}_2^\mp] = 0\\ \Delta\mathbf{l}^\pm = \mathcal{R}^{-1}(\mathbf{l}^\pm\otimes\mathbf{l}^\pm)\mathcal{R},\quad \Delta\mathbf{m}^\pm = \mathcal{R}^{-1}(\mathbf{m}^\pm\otimes\mathbf{m}^\pm)\mathcal{R},\quad \epsilon\mathbf{l}^\pm = \mathrm{id},\quad \epsilon\mathbf{m}^\pm = \mathrm{id}\end{gathered}\tag{41}$$

where $\mathcal{R}$ is the universal R-matrix or quasitriangular structure[38] of H viewed as $\mathcal{R}_{21}\in H\,\underline{\otimes}\,H$. There is of course an appropriate antipode and $*$, correspondingly twisted. We can realise the vectorial form above in terms of these spinorial generators by cf. [13]

$$L^{+I}{}_J = (\mathbf{l}^-\mathbf{m}^+)^{i_1}{}_{j_1}((S_0^{-1}\mathbf{m}^+)(S_0^{-1}\mathbf{l}^+))^{j_0}{}_{i_0},\quad L^{-I}{}_J = (\mathbf{l}^-\mathbf{m}^-)^{i_1}{}_{j_1}((S_0^{-1}\mathbf{m}^+)(S_0^{-1}\mathbf{l}^-))^{j_0}{}_{i_0}\tag{42}$$

where S_0 is the usual 'matrix inverse' antipode of H. The covariance of the spacetime generators is expressed now as $B(R)$ a module algebra under[18]

$$\mathbf{l}_1^+\triangleright\mathbf{u}_2 = \lambda^{-\frac12}R_{21}^{-1}\mathbf{u}_2,\quad \mathbf{l}_1^-\triangleright\mathbf{u}_2 = \lambda^{\frac12}R\mathbf{u}_2,\quad \mathbf{m}_1^+\triangleright\mathbf{u}_2 = \mathbf{u}_2\lambda^{\frac12}R_{21},\quad \mathbf{m}_1^-\triangleright\mathbf{u}_2 = \mathbf{u}_2\lambda^{-\frac12}R^{-1}\tag{43}$$

which is also the action we use on the lowered momentum generators $P_I = P^{i_0}{}_{i_1}$ regarded now as a braided matrix $B(R)$. We add the dilaton ξ as before. Semidirect product by this action (and the induced semidirect coproduct) according to the bosonisation construction (85) in Appendix A, or just proceeding from (38) for the specific form of $\mathbf{R}$ from (11), yields the Poincaré quantum group in spinorial form as

$$\begin{gathered}R_{21}\mathbf{P}_1R\mathbf{P}_2 = \mathbf{P}_2R_{21}\mathbf{P}_1R,\quad \mathbf{l}_1^+\mathbf{P}_2\mathbf{l}_2^- = \lambda^{-\frac12}R_{21}^{-1}\mathbf{P}_2\mathbf{l}_2^-\mathbf{l}_1^+,\quad \mathbf{l}_1^-\mathbf{P}_2\mathbf{l}_2^- = \lambda^{\frac12}R\mathbf{P}_2\mathbf{l}_2^-\mathbf{l}_1^-\\ \mathbf{m}_1^+\mathbf{P}_2\mathbf{l}_2^- = \mathbf{P}_2\mathbf{l}_2^-\lambda^{\frac12}R_{21}\mathbf{m}_1^+,\quad \mathbf{m}_1^-\mathbf{P}_2\mathbf{l}_2^- = \mathbf{P}_2\mathbf{l}_2^-\lambda^{-\frac12}R^{-1}\mathbf{m}_1^-,\quad \lambda^\xi\mathbf{P} = \lambda^{-1}\mathbf{P}\lambda^\xi\\ [\xi,\mathbf{l}^\pm] = [\xi,\mathbf{m}^\pm] = 0,\quad \Delta\xi = \xi\otimes 1 + 1\otimes\xi,\quad \epsilon\xi = 0,\quad S\xi = -\xi\\ \Delta\mathbf{P} = \mathbf{P}\otimes 1 + \lambda^\xi\mathbf{l}^-\mathbf{m}^+(\ )(S_0\mathbf{m}^-)(S_0\mathbf{l}^-)\otimes\mathbf{P},\quad \epsilon\mathbf{P} = 0\\ S\mathbf{P} = -\lambda^{-\xi}S_0(\mathbf{m}^+\mathbf{l}^-(\ )(S_0\mathbf{l}^-)(S_0\mathbf{m}^-))\,\mathbf{P}\end{gathered}\tag{44}$$

where ( ) is a space for the matrix entries of $\mathbf{P}$ to be inserted, and S_0 is the usual matrix antipode in either copy of H. As far as I know, these are new formulae presented here for the first time, although following directly from [14] as above, or by dualisation of [16]. Finally, the action on the spacetime generators becomes, in this spinorial form

$$\begin{gathered}\mathbf{l}_1^+\triangleright\mathbf{u}_2 = \lambda^{-\frac12}R_{21}^{-1}\mathbf{u}_2,\quad \mathbf{l}_1^-\triangleright\mathbf{u}_2 = \lambda^{\frac12}R\mathbf{u}_2,\quad \mathbf{m}_1^+\triangleright\mathbf{u}_2 = \mathbf{u}_2\lambda^{\frac12}R_{21},\quad \mathbf{m}_1^-\triangleright\mathbf{u}_2 = \mathbf{u}_2\lambda^{-\frac12}R^{-1}\\ \lambda^\xi\triangleright\mathbf{u} = \lambda\mathbf{u},\quad \mathbf{P}_1\triangleright\mathbf{u}_2 = \eta.\end{gathered}\tag{45}$$

The action of the lowered P_I is by the lowered $\bar\partial_I$ automatically obeying relations and braided-Leibniz rule

$$R_{21}\bar\partial_1 R\bar\partial_2 = \bar\partial_2 R_{21}\bar\partial_1 R, \quad R^{-1}\bar\partial R\mathbf{u}_2 - \lambda^2\mathbf{u}_2 R_{21}\bar\partial_1 R = R^{-1}\eta^{(1)}R\eta^{(2)} \tag{46}$$

where $\eta = \eta^{(1)}\otimes\eta^{(2)}$ is $\eta_{IJ} = \eta^{i_0}{}_{i_1}{}^{j_0}{}_{j_1}$ as an element of $M_n\otimes M_n$. Some authors[44] have recently considered Leibniz rules for $\mathrm{R}_q^{1,3}$ of similar form on the left hand side. We stress, however, that this is not a new equation but just the usual (36) or (40) for the specific $\mathbf{R}$ introduced in [13]. The rearrangement between the two notations is just as in (10) and (11), and quite routine since[45]. Moreover, our treatment here works for general R of q-Hecke type, with λ the square of its quantum group normalisation constant.

5 Preliminaries II: q-Euclidean spaces in R-matrix form

Now we make the same constructions as above for the $\bar A(R)$ algebra (3) which was introduced in this context in[18]. As an algebra it is exactly gauge equivalent to (1) by the comodule algebra twisting theory in [18]. We take, however, a non-hermitian $*$-structure which ends up in the standard case as more like Euclidean space than Minkowski.

To describe the multiplicative structure, we need to generalise the concept of braided group B slightly. We still require the coproduct $\Delta : B \to B\underline{\otimes}B$ to be an algebra homomorphism, but don't insist that $\underline{\otimes}$ is a braided tensor product. The most general concept (which is certainly general enough, but probably too general) is that $B\underline{\otimes}B$ should now be some algebra which contains the two copies of B as subalgebras, and uniquely factorises into them in the sense that the map $B\otimes B \to B\underline{\otimes}B$ given by including the subalgebras and multiplying, is a linear isomorphism. This is the algebra part of the theory of Hopf algebra factorisations in [3, Sec. 3.2], and more recently in [46] and elsewhere.

In this sense, $\bar A(R)$ does have a matrix comultiplication

$$\Delta\mathbf{x} = \mathbf{x}\otimes\mathbf{x}, \quad \epsilon\mathbf{x} = \mathrm{id} \tag{47}$$

where $\Delta : \bar A(R) \to \bar A(R)\underline{\otimes}\bar A(R)$ is an algebra homomorphism. So this is like a braided group (and indeed is gauge equivalent to (6)) but the non-commutation relations describing $\underline{\otimes}$ do not obey the Artin braid relations of QYBE; rather some more general (but not completely general)

algebra factorisation. The *multiplicative statistics* are[18]

$$\mathbf{x}'' = \mathbf{x}\mathbf{x}'; \qquad \mathbf{x}_1'\mathbf{x}_2 = \mathbf{x}_2 R^{-1}\mathbf{x}_1' \tag{48}$$

whereby $\mathbf{x}''$ obeys the same relations (3) if $\mathbf{x}, \mathbf{x}'$ do.

We also have a natural $*$-structure on $\bar{A}(R)$ whenever its more familiar cousin $A(R)$ has a $*$-structure with real-type universal R-matrix functional. By a theorem in [18] we can take the same operation on the $\mathbf{x}$ as we would on the $\mathbf{t}$ of $A(R)$ in this case. It tends to be of the type which is unitary in the quotient Hopf algebra when R is of real-type, namely of the form

$$x^i{}_j{}^* = \epsilon_{ai} x^a{}_b \epsilon^{bj} \tag{49}$$

where ϵ^{ij} is invariant under the quotient Hopf algebra and ϵ_{ij} the transposed inverse.

Finally, we have a normal braided coaddition, under which $\bar{A}(R)$ remains an additive braided group, at least when R is q-Hecke[18]. We have

$$\Delta\mathbf{x} = \mathbf{x}\otimes 1 + 1\otimes\mathbf{x}, \quad \epsilon\mathbf{x} = 0, \quad S\mathbf{x} = -\mathbf{x} \tag{50}$$

extended as a braided group with *additive braid statistics*[18]

$$\mathbf{x}'' = \mathbf{x} + \mathbf{x}'; \qquad \mathbf{x}_1'\mathbf{x}_2 = R\mathbf{x}_2\mathbf{x}_1' R. \tag{51}$$

This coaddition is typically compatible with the $*$ to give a $*$-braided group as in (8).

Clearly, we can also move the R's to one side and write $\bar{A}(R)$ as a braided covector algebra $x^{i_0}{}_{i_1} = x_I$ with (now in covector form as in (11)) the relations

$$\mathbf{x}_1\mathbf{x}_2 = \mathbf{x}_2\mathbf{x}_1\mathbf{R}', \qquad \mathbf{x}'' = \mathbf{x} + \mathbf{x}'; \quad \mathbf{x}_1'\mathbf{x}_2 = \mathbf{x}_2\mathbf{x}_1'\mathbf{R} \tag{52}$$

for suitable $\mathbf{R}', \mathbf{R}$. These are given explicitly in [18]. Thus we can equally well write this algebra with its linear braided group structure in the form $V^{\vee}(\mathbf{R}', \mathbf{R})$ needed for our general constructions in [14][33].

For our standard example where R is (5), we have the multiplicative statistics[18]

$$\begin{gathered} a'a = q^{-1}aa' + (q^{-1}-q)bc', \quad a'b = ba', \quad a'c = q^{-1}ca' + (q^{-1}-q)dc', \quad a'd = da' \\ b'a = q^{-1}ab' + (q^{-1}-q)bd', \quad b'b = bb', \quad b'c = q^{-1}cb' + (q^{-1}-q)dd', \quad \text{etc.} \end{gathered} \tag{53}$$

whereby the matrix product of the unprimed and primed matrices obeys the same relations (4). We also have the additive braid statistics

$$a'a = q^2aa', \quad a'b = qba', \quad a'c = qca' + (q^2-1)ac', \quad a'd = da' + (q-q^{-1})bc'$$
$$b'a = qab' + (q^2-1)ba', \quad b'b = q^2bb', \quad b'c = cb' + (q-q^{-1})(da'+ad') + bc'(q-q^{-1})^2, \quad \text{etc.} \tag{54}$$

whereby the sum of the unprimed and primed matrices obey the same relations (4).

From the generalised comultiplication, or from the close connection of $\bar{M}_q(2)$ with usual quantum matrices, one has a natural quantum determinant

$$\underline{\det}(\mathbf{x}) = ad - qcb \tag{55}$$

which is central, as well as bosonic with respect to the multiplicative. We use is as the square-distance function on $\bar{M}_q(2)$.

Finally, it is easy to see that the standard 2×2 quantum matrices $M_q(2)$ for real q are a $*$-bialgebra with

$$\begin{pmatrix} a^* & b^* \\ c^* & d^* \end{pmatrix} = \begin{pmatrix} d & -q^{-1}c \\ -qb & a \end{pmatrix} \tag{56}$$

and hence from theory in [18] we know that our algebra $\bar{M}_q(2)$ is also a $*$-algebra with the operation (56). Under the coaddition it forms a $*$-braided group obeying (8). Such a 'unitary' form provides us a natural definition $\mathrm{R}_q^4 = \bar{M}_q(2)$, as explained in [18]. The $*$-structure determines 'real' or self-adjoint (under $*$) spacetime co-ordinates

$$t = \frac{a-d}{2\imath}, \quad x = \frac{c-qb}{2}, \quad y = \frac{c+qb}{2\imath}, \quad z = \frac{a+d}{2} \tag{57}$$

and the q-determinant above becomes

$$\underline{\det}(\mathbf{x}) = (\frac{1+q^2}{2})t^2 + x^2 + y^2 + (\frac{1+q^2}{2})z^2 \tag{58}$$

which justifies the interpretation as Euclidean length in this approach[18]. From it we can extract a *quantum metric* tensor by braided differentiation[22] which in our matrix basis is

$$\eta^{IJ} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -q^{-2} & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}; \quad \underline{\det}(\mathbf{x}) = (1+q^{-2})^{-1}\eta^{IJ}x_Jx_I. \tag{59}$$

6 Euclidean q-Poincaré quantum group in function algebra form

For the construction of the q-Poincaré quantum group appropriate to (3), we start with just the same formulae (22) in Section 3. For the construction there (from [14]) applies to any braided covector space, and we have just seen in (52) that $\bar{A}(R)$ can be put in this braided covector form. So the vectorial form of the q-Poincaré quantum group for this case is just the same (22) with $\mathbf{R}$ now determined from (52). The formula (23) is now

$$\mathbf{x} \to \mathbf{x}\Lambda\varsigma + \mathbf{p} \tag{60}$$

where $\mathbf{p}$ is the copy of algebra $\bar{A}(R)$ being used for momentum rather than position $\mathbf{x}$.

Because the form of $\mathbf{R}$ for our particular example $\bar{A}(R)$ is simpler than before, we have this time a simpler spinorial form. Firstly, we replace $A(\mathbf{R})$ by the tensor product Hopf algebra $A \otimes A$ where each A is a quantum group obtained from $A(R)$, with generators $\mathbf{s}, \mathbf{t}$ say. So the spinorial spacetime rotation group is

$$\begin{gathered} R\mathbf{s}_1\mathbf{s}_2 = \mathbf{s}_2\mathbf{s}_1 R, \quad R\mathbf{t}_1\mathbf{t}_2 = \mathbf{t}_2\mathbf{t}_1 R, \quad [\mathbf{t}_1, \mathbf{s}_2] = 0 \\ \Delta\mathbf{s} = \mathbf{s} \otimes \mathbf{s}, \quad \Delta\mathbf{t} = \mathbf{t} \otimes \mathbf{t}, \quad \epsilon\mathbf{s} = \mathrm{id} = \epsilon\mathbf{t} \end{gathered} \tag{61}$$

and further relations needed to have an antipode. The realisation of the vectorial form in terms of the spinorial form is the same as (25) and the coaction takes the same form

$$\mathbf{x} \to \mathbf{s}^{-1}\mathbf{x}\mathbf{t}\varsigma. \tag{62}$$

For the standard example (5) this fits with considerations for q-Euclidean space in the appendix of [5]. As far as I know, the general R-matrix setting using (3) is, however, due to [18]. We take the tensor product $*$-structure

$$s^i{}_j{}^* = S^3 s^j{}_i, \quad t^i{}_j{}^* = S t^j{}_i, \quad \varsigma^* = \varsigma \tag{63}$$

again according to general theory in [18]. There one sees that an extra automorphism S^2 in the definition of the $*$-structure in the first copy of $A \otimes A$ is needed for the coaction [18, Eq. (11)] to be a $*$-algebra map. In our case it becomes an extra S^2 on the $\mathbf{s}$ generator for compatibility of (62) with the Euclidean $*$-structure (49) on the space-time co-ordinates $\bar{A}(R)$, as well as for $\Lambda^I{}_J{}^* = S\Lambda^J{}_I$. If we kept the original form (27) with hermitian co-ordinates then such a system would be just our previous q-Minkowski example in a twisted form.

The formulae (22) then become the spinorial q-Poincaré quantum group[18]

$$R_{21}\mathbf{p}_1\mathbf{p}_2 = \mathbf{p}_2\mathbf{p}_1 R, \quad \mathbf{p}_1\mathbf{s}_2 = \mathbf{s}_2\lambda^{-\frac{1}{2}}R^{-1}\mathbf{p}_1, \quad \mathbf{p}_1\mathbf{t}_2 = \lambda^{\frac{1}{2}}\mathbf{t}_2\mathbf{p}_1 R$$
$$\mathbf{p}\varsigma = \lambda^{-1}\varsigma\mathbf{p}, \quad [\mathbf{s},\varsigma] = [\mathbf{t},\varsigma] = 0, \quad \Delta\varsigma = \varsigma\otimes\varsigma, \quad \epsilon\varsigma = 1, \quad S\varsigma = \varsigma^{-1} \tag{64}$$
$$\Delta\mathbf{p} = \mathbf{p}\otimes\mathbf{s}^{-1}(\ )\mathbf{t}\varsigma + 1\otimes\mathbf{p}, \quad \epsilon\mathbf{p} = 0, \quad S\mathbf{p} = -\mathbf{p}\varsigma^{-1}\mathbf{t}^{-1}(\ )\mathbf{s}$$

where λ is the quantum group normalisation constant of $\mathbf{R}$, which is the square of that of R. Its value in the standard example is $\lambda = q^{-1}$. As usual, one can derive both vectorial and spinorial forms by the abstract bosonisation construction (84) in Appendix A. Finally, the coaction on the spacetime generators in the spinorial form is of course

$$\mathbf{x} \to \mathbf{s}^{-1}\mathbf{x}\mathbf{t}\varsigma + \mathbf{p}. \tag{65}$$

7 Euclidean q-Poincaré quantum group in enveloping algebra form

For the enveloping algebra form for this q-Poincaré quantum group we use (32) since this was a completely general construction for any braided covector space[14]. We dualise each of the ingredients of the semidirect product just as before. We use the pairing $\langle P^I, x_J\rangle = \delta^I{}_J$ between braided vectors and covectors in (31), or $\langle P_I, p_J\rangle = \eta_{IJ}$ in (39) for the lowered index form of the Poincaré enveloping algebra (38). The only difference from this part of Section 4 (up to and including (40)) is that the quantum metric tensor and $\mathbf{R}', \mathbf{R}$ now come from (52) in Section 5, where we cast $\bar{A}(R)$ as a braided covector space. The action of the vectorial q-Poincaré enveloping algebra on the spacetime co-ordinates takes the same form

$$\mathbf{L}_1^+ \triangleright \mathbf{x}_2 = \mathbf{x}_2\lambda\mathbf{R}_{21}, \quad \mathbf{L}_1^- \triangleright \mathbf{x}_2 = \mathbf{x}_2\lambda^{-1}\mathbf{R}^{-1}, \quad \lambda^{\xi} \triangleright \mathbf{x} = \lambda\mathbf{x}, \quad P^I \triangleright x_J = \delta^I{}_J \tag{66}$$

as before and necessarily makes the spacetime co-ordinates $\mathbf{x}$ into a module algebra under it. The action of P^I is by the braided differentials $\bar{\partial}^I$ as before defined with $[m; \mathbf{R}_{21}^{-1}]$ and obeying the braided Leibniz rule (36) with respect now to x_I. Equivalently, when there is a quantum metric η the action of the lowered P_I is by lowered differentials $\bar{\partial}_I$ obeying the braided covector algebra and Leibniz rule

$$\bar{\partial}_1\mathbf{x}_2 - \lambda^2\mathbf{x}_2\bar{\partial}_1\mathbf{R} = \eta \tag{67}$$

as before. Indeed, everything for the vectorial q-Poincaré enveloping algebras in Section 4 was for a general braided covector space.

We then use the specific form of $\mathbf{R}$ to give the spinorial description in terms of $\mathbf{x}$ as a quantum matrix $\bar{A}(R)$. As in the preceding section, it looks simpler than the braided matrix case in Section 4 because of the simpler form of $\mathbf{R}$. This time the spinorial form of the spacetime rotation enveloping algebra is the tensor product Hopf algebra $H \otimes H$ where each H is dual to A. So we take the two copies with FRT generators $\mathbf{l}^\pm$ and $\mathbf{m}^\pm$ as in (41) but now the matrix coproducts

$$\Delta \mathbf{l}^\pm = \mathbf{l}^\pm \otimes \mathbf{l}^\pm, \quad \Delta \mathbf{m}^\pm = \mathbf{m}^\pm \otimes \mathbf{m}^\pm, \quad \epsilon \mathbf{l}^\pm = \mathrm{id}, \quad \epsilon \mathbf{m}^\pm = \mathrm{id}. \tag{68}$$

The $*$-structure for the Euclidean picture dual to (63) is

$$l^{\pm i}{}_j{}^* = S^{-1} l^{\mp j}{}_i, \quad m^{\pm i}{}_j{}^* = S m^{\mp j}{}_i. \tag{69}$$

The realisation of the vectorial spacetime rotation generators in the spinorial ones is

$$L^{\pm I}{}_J = (S^{-1} l^{\pm j_0}{}_{i_0}) m^{\pm i_1}{}_{j_1}. \tag{70}$$

The spacetime-coordinates become a module algebra under the spacetime rotation generators just as in (43) by[18]

$$\mathbf{l}_1^+ \triangleright \mathbf{x}_2 = \lambda^{-\frac{1}{2}} R_{21}^{-1} \mathbf{x}_2, \quad \mathbf{l}_1^- \triangleright \mathbf{x}_2 = \lambda^{\frac{1}{2}} R \mathbf{x}_2, \quad \mathbf{m}_1^+ \triangleright \mathbf{x}_2 = \mathbf{x}_2 \lambda^{\frac{1}{2}} R_{21}, \quad \mathbf{m}_1^- \triangleright \mathbf{x}_2 = \mathbf{x}_2 \lambda^{-\frac{1}{2}} R^{-1} \tag{71}$$

which is also the action we use on the lowered momentum generators $P_I = P^{i_0}{}_{i_1}$ regarded now as a matrix $\bar{A}(R)$. We add a dilaton ξ as before. Semidirect product by this action (and the induced semidirect coproduct) according to the bosonisation construction in Appendix A, or just working from (38) for our particular $\mathbf{R}$ in (52), immediately gives the spinorial form of the q-Poincaré enveloping algebra as

$$\begin{gathered} R_{21}\mathbf{P}_1\mathbf{P}_2 = \mathbf{P}_2\mathbf{P}_1 R, \quad \mathbf{l}_1^+\mathbf{P}_2 = \lambda^{-\frac{1}{2}} R_{21}^{-1}\mathbf{P}_2\mathbf{l}_1^+, \quad \mathbf{l}_1^-\mathbf{P}_2 = \lambda^{\frac{1}{2}} R\mathbf{P}_2\mathbf{l}_1^- \\ \mathbf{m}_1^+\mathbf{P}_2 = \mathbf{P}_2\lambda^{\frac{1}{2}} R_{21}\mathbf{m}_1^+, \quad \mathbf{m}_1^-\mathbf{P}_2 = \mathbf{P}_2\lambda^{-\frac{1}{2}} R^{-1}\mathbf{m}_1^-, \quad \lambda^\xi \mathbf{P} = \lambda^{-1}\mathbf{P}\lambda^\xi \\ [\xi, \mathbf{l}^\pm] = [\xi, \mathbf{m}^\pm] = 0, \quad \Delta\xi = \xi \otimes 1 + 1 \otimes \xi, \quad \epsilon\xi = 0, \quad S\xi = -\xi \\ \Delta\mathbf{P} = \mathbf{P} \otimes 1 + \lambda^\xi(\mathbf{l}^-(\ )S\mathbf{m}^-) \otimes \mathbf{P}, \quad \epsilon\mathbf{P} = 0, \quad S\mathbf{P} = -\lambda^{-\xi} S(\mathbf{l}^-(\ )S\mathbf{m}^-)\mathbf{P} \end{gathered} \tag{72}$$

where $\mathbf{l}^-(\ )S\mathbf{m}^-$ has a space for the matrix indices of $\mathbf{P}$ to be inserted. As far as I know, these are new formulae presented here for the first time, although following directly from [14] as above, or by dualisation of [18]. Finally, our q-Euclidean space $\bar{A}(R)$ becomes a module algebra under

this q-Poincaré enveloping algebra by

$$\mathbf{l}_1^+ \triangleright \mathbf{x}_2 = \lambda^{-\frac{1}{2}} R_{21}^{-1} \mathbf{x}_2, \quad \mathbf{l}_1^- \triangleright \mathbf{x}_2 = \lambda^{\frac{1}{2}} R \mathbf{x}_2, \quad \mathbf{m}_1^+ \triangleright \mathbf{x}_2 = \mathbf{x}_2 \lambda^{\frac{1}{2}} R_{21}, \quad \mathbf{m}_1^- \triangleright \mathbf{x}_2 = \mathbf{x}_2 \lambda^{-\frac{1}{2}} R^{-1}$$
$$\lambda^{\xi} \triangleright \mathbf{x} = \lambda \mathbf{x}, \quad \mathbf{P}_1 \triangleright \mathbf{x}_2 = \eta. \tag{73}$$

The action of the lowered P_I is by the lowered $\bar{\partial}_I$ automatically obeying relations and braided-Leibniz rule

$$R_{21} \bar{\partial}_1 \bar{\partial}_2 = \bar{\partial}_2 \bar{\partial}_1 R, \quad \bar{\partial}_1 \mathbf{x}_2 - \lambda^2 R \mathbf{x}_2 \bar{\partial}_1 R = \eta \tag{74}$$

This is not a new equation but just (67) in the matrix notation according to the correspondence between the notations in (51) and (52).

The constructions in this spinorial setting work for general q-Hecke R, with λ the square of its quantum group normalisation constant. Finally, let us note that there is an isomorphism of algebras between this q-Poincaré enveloping algebra and the braided matrix case in Section 4 by mapping $\mathbf{P}$ to $\mathbf{Pl}^-$. The two quantum groups are related by the 'quantum wick rotation' in [18], namely by twisting (cf. [19]) via the same quantum cocycle $\mathcal{R}^{-1}$ which relates the spacetime rotation quantum groups, but viewed now in the Euclidean Poincaré enveloping algebra.

8 Concluding remarks: $*$-structure and the dilaton problem

We conclude here with two unconnected observations concerning problems of current interest. Firstly, we know that our spacetime co-ordinates and Lorentz algebras in all the above sections have reasonable $*$-structures and that the Lorentz transformations preserve them. Hence it is natural to ask if the Poincaré quantum group also has a natural $*$-structure. If we use the usual axioms of a Hopf $*$-algebra then the answer appears in general to be no.

It does seem that one needs new axioms, albeit reducing to the usual ones when $q = 1$. A first step to formulating the correct axioms is in [23] where we study systematically $*$-structures on braided covector spaces. We confirmed the axioms (8) introduced in [9] and classify the situations when they arise in the linear setting. We also explained there that the construction of Poincaré groups by bosonisation would then inherit natural $*$ properties with axioms to be elaborated elsewhere. Between the release of [23] and the present paper, there appeared a preprint by Fiore[28] in which the one-dimensional case of case of a Poincaré algebra with two coproducts connected by $*$ was considered directly.

In fact, the question of when bosonisations have natural structures as $*$-algebras was already studied in [9] in order to have an interpretation of the quantum double as quantum mechanics. For the enveloping algebra Poincaré quantum group to be a $*$-algebra we need that the action $\triangleright$ of the 'Lorentz' quantum group H on the 'momentum' braided group C is unitary in the standard Hopf algebra sense $(h\triangleright c)^* = (Sh)^*\triangleright c^*$ for h in the Lorentz sector (including the dilaton) and c in the momentum sector. This *is* the case in all our examples above, for it just corresponds to the coaction of the function algebra Lorentz quantum group being a $*$-algebra homomorphism. The coalgebra however, is a semidirect product by the action induced by the universal R-matrix of H (see Appendix A). We assume the latter is of real-type in the sense $\mathcal{R}^{*\otimes *} = \mathcal{R}_{21}$, which again is the case for our examples when R is of real-type in the sense $R^{\dagger\otimes\dagger} = R_{21}$. This is true for (5) when q is real. One can easily see that in general the bosonisation $C\rtimes H$ in this case will *not* obey the usual axioms of a Hopf $*$-algebra. Instead we find

Proposition 8.1 *In the general bosonisation theory[15] as in Theorem A.2 in the appendix, if the action of H is 'unitary' and its universal R-matrix real-type as explained above then the coproduct and antipode of $C\rtimes H$ obey*

$$(*\otimes *)\circ\Delta\circ * = \mathcal{R}^{-1}(\tau\circ\Delta\ )\mathcal{R},\quad \overline{\epsilon(\ )} = \epsilon\circ *,\quad *\circ S\circ * = \vartheta^{-1}(S^{-1}\ )\vartheta \tag{75}$$

where $\vartheta = \mathcal{R}^{(1)}S\mathcal{R}^{(2)}$ and $\mathcal{R}$ are viewed in the bigger algebra. We propose to call a $*$-algebra with such an $\mathcal{R}$ a quasi-$*$ *Hopf algebra.*

The proof is easy by Hopf algebra techniques. Thus

$$(*\otimes *)\circ\Delta c = \mathcal{R}^{(2)*}c_{\underline{(1)}}{}^*\otimes(\mathcal{R}^{(1)}\triangleright c_{\underline{(2)}})^* = \mathcal{R}^{-(1)}c_{\underline{(1)}}{}^*\otimes\mathcal{R}^{-(2)}\triangleright(c_{\underline{(2)}}{}^*) = \bar{\Delta}(c^*)$$

where $\bar{\Delta}$ is the second 'conjugate bosonisation' coproduct (90), the second equality is our reality and unitarity assumption and the third is the $*$-axiom (8) for braided groups. Here $c_{\underline{(1)}}\otimes c_{\underline{(2)}}$ is the braided coproduct of C and $\mathcal{R}^{(1)}\otimes\mathcal{R}^{(2)}$ the universal R-matrix of H (summations implicit). On the other hand, $\bar{\Delta}$ is always twisiting equivalent to $\tau\circ\Delta$ by cocycle $\mathcal{R}^{-1}$. Since H is a sub-Hopf algebra, we also have $(\Delta\otimes\mathrm{id})\mathcal{R} = \mathcal{R}_{13}\mathcal{R}_{23}$ and $(\mathrm{id}\otimes\Delta)\mathcal{R} = \mathcal{R}_{13}\mathcal{R}_{12}$ when viewed in our bigger Hopf algebra. So $C\rtimes H$ is like a quasitriangular Hopf algebra except that in one axiom we replace Δ by $(*\otimes *)\circ\Delta\circ *$.

The second coproduct $\bar{\Delta}$ is given for free in our braided approach as an automatic feature of the theory: as explained in [10] every braided construction has a conjugate one where we reverse the braid crossings. In the present case Δ and $\bar{\Delta}$ coincide on the quantum group part H, where they are both its usual coproduct. But on the braided group part C they are more like opposite (transposed) coproducts and indeed become that when $\mathcal{R} = 1$. This is how we interpolate between the axioms (8) for a braided group (with a transposition τ) and for a usual quantum group (without τ)!

How does this second coproduct look for our q-Poincaré enveloping algebras above? For the vectorial setting (32) with upper indices P^I it is

$$\bar{\Delta}\mathbf{P} = \mathbf{P} \otimes 1 + \lambda^{-\xi}\mathbf{L}^+ \otimes \mathbf{P}, \quad S\mathbf{P} = -\lambda^{\xi}(S\mathbf{L}^+)\mathbf{P}. \tag{76}$$

If we use lowered covector indices P_I then (38) becomes

$$\bar{\Delta}P_I = P_I \otimes 1 + \lambda^{-\xi} S L^{+A}{}_I \otimes P_A, \quad SP_I = -\lambda^{\xi}(S^2 L^{+A}{}_I)P_A. \tag{77}$$

One can confirm (75) directly for $\mathbf{R}$ of the appropriate reality type and $*$ on the braided covectors as in our paper [23]. For example, in the Euclidean case it is $P_I^* = P^I$ while the Lorentz generators obey $L^{\pm I}{}_J{}^* = SL^{\mp J}{}_I$. For the Minkowski $*$ in vectorial form see [13]. This all works generally for any braided covector space with the correct reality properties.

For the Minkowski spinorial Poincaré enveloping algebra (44) in Section 4, where $\mathbf{P}$ is a braided matrix $P^i{}_j$ in $B(R)$, the conjugate quantum group structure is

$$\bar{\Delta}\mathbf{P} = \mathbf{P} \otimes 1 + \lambda^{-\xi}\mathbf{l}^+\mathbf{m}^+(\ )(S_0\mathbf{m}^+)(S_0\mathbf{l}^-) \otimes \mathbf{P}, \quad \bar{S}\mathbf{P} = -\lambda^{\xi}S_0(\mathbf{m}^+\mathbf{l}^+(\ )(S_0\mathbf{l}^+)(S_0\mathbf{m}^+))\mathbf{P} \tag{78}$$

where the space is for the matrix indices of $\mathbf{P}$ to be inserted and S_0 is the usual 'matrix inverse' antipode. For the Euclidean one in (72) in Section 7, where $P^i{}_j$ is in $\bar{A}(R)$, the conjugate structure is

$$\bar{\Delta}\mathbf{P} = \mathbf{P} \otimes 1 + \lambda^{-\xi}\mathbf{l}^+(\ )S\mathbf{m}^+ \otimes \mathbf{P}, \quad \bar{S}\mathbf{P} = -\lambda^{\xi}S(\mathbf{l}^+(\ )S\mathbf{m}^+)\mathbf{P}. \tag{79}$$

Thus, the astute reader who was wondering why only $\mathbf{L}^-$ appeared in the coproduct of $\mathbf{P}$ in Section 4, etc., in the construction of [14], sees now that the symmetry is restored with the corresponding $\mathbf{L}^+$ appearing in the conjugate Poincaré quantum group. This is why they are connected by $*$ as in (75). This suggests that these are indeed very reasonable axioms for our setting. Finally, while the Poincaré generators before were represented on spacetime co-ordinates

by means of $\bar{\partial}$ braided differentials, it is easy to see that with the conjugate coproduct the same algebra acts covariantly with the usual braided differentials ∂ from [33]. We saw already in [33, Sec. V] that if $V(\mathbf{R}', \mathbf{R})$ is a braided vector space of differentials (our momentum sector) then its 'conjugate' or opposite braided group is $V(\mathbf{R}', \mathbf{R}_{21}^{-1})$ obeying the same algebra with reversed braiding. When we bosonise, it means that our one q-Poincaré enveloping algebra extends to products of spacetime generators in *two* different ways (36), one with the Leibniz rule for $\bar{\partial}$ and the other with the conjugate-braiding Leibniz rule for ∂. The two coincide in the triangular or unbraided case so the distinction is not visible classically. In physical terms it means that q-deformed geometry in the braided approach is naturally 'split' into two geometries related by braid-crossing reversal symmetry in the constructions.

It is clear that Proposition 8.1 also solves in principle the important problem of how to tensor product 'unitary' representations of the q-Poincaré group, which we need for physics. That the conjugated coproduct $(*\otimes *)\circ\Delta\circ *$ is twisting equivalent to $\tau\circ\Delta$ means that the tensor product of two representations which are antiunitary in the sense $\rho(h^*)^\dagger = \rho(Sh)$ will remain so, *but up to isomorphism.* In our case the isomorphism is given by the action of $\mathcal{R}^{-1}$ and is a new physical effect which is absent when $\mathcal{R} = 1$. For example, we can define a *braided-unitary* representation to be a pair consisting of V on which our Poincaré quantum group acts and a semilinear form $(\ ,\ )_V$ on it such that $(h^*\triangleright v, v')_V = (v, (Sh)\triangleright v')_V$ for all $v, v' \in V$ and h in our Poincaré (or other quasi-$*$ Hopf) algebra. Then one can see that

$$V\otimes W, \quad (v\otimes w, v'\otimes w')_{V\otimes W} = (\mathcal{R}^{-(2)}\triangleright v, v')_V(\mathcal{R}^{-(1)}\triangleright w, w')_W \tag{80}$$

is again a braided-unitary representation, where we act on tensor products in the usual way via the coproduct Δ. We do not assume that the semilinear forms are conjugate symmetric (like a usual Hilbert space inner product) since this is not in general preserved by (80). The problem of constructing and perhaps classifying such braided-unitary representations for our particular Poincaré algebras will be addressed elsewhere.

For our second topic we recall that the appearance of the dilaton ς or ξ in the q-Poincaré quantum group is an unexpected feature noted already in [36][17], and explained in terms of the quantum group normalisation constant in the general construction [14]. We observe now that, while not solving anything, we can systematically remove it from all our q-Poincaré quantum group formulae in our constructions above if we pay a certain price. The price is that we obtain

then not a quantum group in the usual sense but a $\mathbb{Z}$-graded or $\mathbb{C}$-statistical braided group[29]. This is a braided group of the kind where the statistics are just a power of a generic factor, say λ. This class includes as special cases superquantum groups and anyonic quantum groups[47][48]. If B is such a braided group, then its bosonisation consists of adjoining a new group-like generator g, say, with commutation relations which remember the grading $|b| \in \mathbb{Z}$. It has structure[15][10]

$$\begin{gathered} gb = \lambda^{|b|} bg, \quad \Delta g = g \otimes g, \quad \epsilon g = 1, \quad Sg = g^{-1} \\ \Delta b = b_{\underline{(1)}} g^{|b_{\underline{(2)}}|} \otimes b_{\underline{(2)}}, \quad Sb = g^{-|b|} \underline{S} b \end{gathered} \tag{81}$$

where $b_{\underline{(1)}} \otimes b_{\underline{(2)}}$ is the braided coproduct and $\underline{S}$ the braided antipode of B, and $|\ |$ is the degree of a homogeneous element. We use once again the general theory in Appendix A, either as (a left-handed version of) the bosonisation (84) or in terms of the enveloping algebra version (85) with H the quantum line.

Now, comparing this formula with the formulae for our q-Poincaré enveloping algebras (32),(38),(44),(72) we see that in each case they are of the above form with

$$g = \lambda^{\xi}, \quad |\mathbf{L}^{\pm}| = |\mathbf{l}^{\pm}| = |\mathbf{m}^{\pm}| = 0, \quad |\mathbf{P}| = -1; \quad |\mathbf{u}| = |\mathbf{x}| = 1 \tag{82}$$

and with B defined as the $\mathbb{Z}$-graded braided group given by the same formulae with λ^{ξ} omitted. So these B are braided q-Poincaré groups. They act on the spacetime in a way that preserves grading also, i.e. these become $\mathbb{Z}$-graded module algebras with the grading as shown, etc. Similarly in right-handed conventions for the function algebra quantum Poincaré groups in Sections 3,6 with ς omitted. Everything works as in supersymmetry except that the statistical factor -1 is replaced by λ[29]. In physical terms, the role played mathematically in supersymmetry by fermionic degree is played now by the physical scale dimension. This is another example of the unification of different physical concepts made possible by quantum groups and braided groups.

A Appendix: the abstract bosonisation theory

Here we collect some basic formulae from the abstract theory of bosonisation[15][9]. Conceptually, bosonisation is a generalisation of the Jordan-Wigner transform for turning fermionic systems into bosonic ones. Recall that this is done by adjoining the degree or grading operator. In braided geometry the role of the $\mathbb{Z}_2$ grading of supersymmetry is played by the background

quantum group symmetry, which in the above context is the spacetime rotation quantum group (the concepts of supersymmetry and Lorentz invariance are unified[10]). We assume that the reader is familiar with the definition of a quasitriangular Hopf algebra H in [38] with 'universal R-matrix' $\mathcal{R} = \mathcal{R}^{(1)} \otimes \mathcal{R}^{(2)}$ in $H \otimes H$ (summation of terms implicit), and the dual notion of a dual-quasitriangular Hopf algebra A with dual-quasitriangular structure or 'universal R-matrix functional' $\mathcal{R} : A \otimes A \to \mathbb{C}$ in [49][50]. The latter is characterised by the axioms

$$\begin{gathered}\mathcal{R}(a, bc) = \mathcal{R}(a_{(1)}, c)\mathcal{R}(a_{(2)}, b), \quad \mathcal{R}(ab, c) = \mathcal{R}(a, c_{(1)})\mathcal{R}(b, c_{(2)}) \\ b_{(1)}a_{(1)}\mathcal{R}(a_{(2)}, b_{(2)}) = \mathcal{R}(a_{(1)}, b_{(1)})a_{(2)}b_{(2)}.\end{gathered} \tag{83}$$

The coproducts are denoted $\Delta a = a_{(1)} \otimes a_{(2)}$, etc. (summation implicit). We also assume that the reader is familiar with the basic notion of a braided group B or braided-Hopf algebra[7][50]. Introductions are in [10][11][21].

The first theorem is that if B is a braided group living in the braided category of (say right) A-comodules (i.e. an object which is totally covariant under a coaction of A) then there is an ordinary Hopf algebra $A \triangleright\!\!< B$ constructed as follows[9]. As a coalgebra we make a semidirect coproduct by the right coaction of A. We also use the universal R-matrix functional to turn the right coaction into a right action of A by evaluation in one input, and make an algebra semidirect product by this action. In concrete terms:

Theorem A.1 *[9] cf.[15] The algebra $A \triangleright\!\!< B$ generated by B, A with cross relations, coproduct and antipode*

$$ba = a_{(1)}b^{(\bar{1})}\mathcal{R}(b^{(\bar{2})}, a_{(2)}), \quad \Delta b = b_{\underline{(1)}}{}^{(\bar{1})} \otimes b_{\underline{(1)}}{}^{(\bar{2})}b_{\underline{(2)}}, \quad Sb = (\underline{S}b^{(\bar{1})})Sb^{(\bar{2})} \tag{84}$$

for all $a \in A, b \in B$, is a Hopf algebra.

Here $\underline{\Delta} b = b_{\underline{(1)}} \otimes b_{\underline{(2)}}$ is the braided coproduct of B, $\underline{S}$ its braided antipode and $b^{(\bar{1})} \otimes b^{(\bar{2})} \in B \otimes A$ denotes the output of the coaction of A. The coproduct and antipode of A are not modified (so A is a sub-Hopf algebra).

The second theorem (actually proven first in [15] with the above easily obtained as dual to it) is that if C is some braided group living in the braided category of (say, left) H-modules (i.e. an object which is totally covariant under an action of H) then there is an ordinary Hopf algebra $C \triangleright\!\!< H$ constructed as follows[15]. As an algebra we make a semidirect product by the

left action of H. Also, we use the universal R-matrix to turn the left action into a left coaction by letting its part living in the first factor of H act. In concrete terms:

Theorem A.2 *[15] The algebra $C{\rtimes}H$ generated by H, C with cross relations, coproduct and antipode*

$$hc = (h_{(1)}{\triangleright}c)h_{(2)}, \quad \Delta c = c_{\underline{(1)}}\mathcal{R}^{(2)} \otimes \mathcal{R}^{(1)}{\triangleright}c_{\underline{(2)}}, \quad Sc = (v\mathcal{R}^{(1)}{\triangleright}\underline{S}c)S\mathcal{R}^{(2)} \tag{85}$$

for all $h \in H, c \in C$, is a Hopf algebra.

Here $v = (S\mathcal{R}^{(2)})\mathcal{R}^{(1)}$, $\underline{\Delta}c = c_{\underline{(1)}} \otimes c_{\underline{(2)}}$ the braided coproduct of C, $\underline{S}$ its braided antipode and $\triangleright$ denotes the left action of H. The coproduct and antipode of H are not modified (so H is a sub-Hopf algebra).

It should be perfectly clear that these two constructions are conceptually dual to one another. So if $C = B^*$ (the braided group dual to B) and $H = A^*$ (the quantum group dual to A) then $C{\rtimes}H = (A{\triangleright\!\!<}B)^*$ as usual quantum groups.

Proposition A.3 *cf.[9] The duality pairing of $C{\rtimes}H$ and $A{\triangleright\!\!<}B$ between the various subalgebras is*

$$\langle c, a\rangle = \epsilon(c)\epsilon(a), \quad \langle h, a\rangle = \text{usual}, \quad \langle c, b\rangle = \text{ev}(\underline{S}^{-1}c, b), \quad \langle h, b\rangle = \epsilon(h)\epsilon(b) \tag{86}$$

where ϵ is the counit of the quantum group or braided group, 'usual' means the pairing between H, A as usual quantum groups dual to each other, $\underline{S}$ is the braided antipode (which we assume invertible) of C, and ev *is the braided-group duality evaluation pairing.*

Recall[10] that ev obeys slightly different axioms to a usual quantum group pairing. Indeed, $\text{ev}(\underline{S}^{-1}(\ ),(\ ))$ obeys axioms more like the usual axioms and reduces to them when the braiding is trivial. We have slightly reworked [9] where the duality was given explicitly when $B = C^*$ rather than $C = B^*$ as here. Both statements are true. Also coming out of the bosonisation theory is a canonical coaction of $A{\triangleright\!\!<}B$ on B or, by duality in the setting above, an action of $C{\rtimes}H$ on B. By its very definition in categorical terms, the ordinary (co)representations of the bosonised Hopf algebra are in 1-1 correspondence with the braided (co)-representations of the braided group before bosonisation[15]. Obviously B coacts on itself by its braided coproduct. So the general theory gives at once:

Corollary A.4 *cf.[14] B is a right $A{>\!\!\triangleleft}B$-comodule algebra by*

$$b \to \underline{b_{(1)}}^{(\bar{1})} \otimes \underline{b_{(1)}}^{(\bar{2})} \underline{b_{(2)}} \tag{87}$$

and in the setting above a left $C{>\!\!\triangleleft}H$-module algebra by

$$h{\triangleright}b = b^{(\bar{1})}\langle h, b^{(\bar{2})}\rangle, \quad c{\triangleright}b = \underline{b_{(1)}}^{(\bar{1})}\mathrm{ev}(\underline{S}^{-1}c, \underline{b_{(2)}}^{(\bar{1})})\mathcal{R}^{-1}(\underline{b_{(2)}}^{(\bar{2})}, \underline{b_{(1)}}^{(\bar{2})}) \tag{88}$$

Conceptually, the action of H on B is just the action corresponding to the coaction of A assumed when we said that B was A-covariant to begin with. The action of C in abstract terms is

$$c{\triangleright}b = (\mathrm{ev}(\underline{S}^{-1}c, (\ )) \otimes \mathrm{id}) \circ \Psi^{-1} \circ \underline{\Delta}b \tag{89}$$

which has a braided picture when we write Ψ as a braid crossing and ev = $\cup$. It is a left-handed version of the right coregular representation Reg* introduced and studied in [10][51], and always makes a braided group B a braided module algebra under its dual braided group.

The combination $\bar{\Delta} = \Psi^{-1} \circ \Delta$ is studied in [10] as the *naive opposite coproduct.* It is naive because it does not make the algebra of C into a braided group in our original braided category but rather into a braided group $\bar{C}$, say, living in the 'conjugate' braided category with inverse transposed braiding[10, Lemma 4.6]. $\underline{S}^{-1}$ becomes its braided antipode. In concrete terms it means that the braided group $\bar{C}$ is no longer properly covariant under H (with the correct induced braiding) but under the quantum group H equipped with $\mathcal{R}_{21}^{-1}$ instead for its universal R-matrix. Let us denote the latter by $\bar{H}$. As a Hopf algebra it coincides with H, but has 'conjugate' $\mathcal{R}$.

Corollary A.5 *Every bosonisation $C{>\!\!\triangleleft}H$ has a second 'conjugate' coproduct and antipode on the same algebra,*

$$\bar{\Delta}c = \mathcal{R}^{-(1)}\underline{c_{(2)}} \otimes \mathcal{R}^{-(2)}{\triangleright}\underline{c_{(1)}}, \quad \bar{S}c = (\mathcal{R}^{(2)}\vartheta^{-1}{\triangleright}\underline{S}^{-1}c)\mathcal{R}^{(1)} \tag{90}$$

where $\vartheta = \mathcal{R}^{(1)}S\mathcal{R}^{(2)}$.

This is just the bosonisation $\bar{C}{>\!\!\triangleleft}\bar{H}$ of $\bar{C}$ from Theorem A.2, written (using the algebra relations) in terms of H, C. It has the same algebra as $C{>\!\!\triangleleft}H$ but is generally a different Hopf algebra. Likewise every bosonisation $A{\triangleright\!\!<}B$ has a conjugate $\bar{A}{\triangleright\!\!<}\bar{B}$ with the same coalgebra and a different product. This kind of braid-crossing reversal symmetry is an intrinsic feature of braided group

theory[10]. It is easy to further recognise these second coproducts and products as twisiting equivalent (cf. [19]) to the opposite coproduct or product respectively. Thus $\bar{\Delta}=\mathcal{R}^{-1}(\tau\circ\Delta\,)\mathcal{R}$ by an elementary computation using the algebra relations.

This summarises the relevant parts of the abstract theory of bosonisation of braided groups into ordinary quantum groups. One of the first applications in physics was to the construction of q-Poincaré quantum group function algebras for any braided covector space $B=V\check{}(\mathbf{R}',\mathbf{R})$[14]. The latter lives in the braided category of $\widetilde{A}$-comodules by (21), where A is obtained from $A(\mathbf{R})$ with universal R-matrix functional in [35][52] and we extend by a dilaton with universal R-matrix functional $\mathcal{R}(\varsigma,\varsigma)=\lambda^{-1}$. We obtain from (84) the formula (22) in Section 3 as the Hopf algebra $\widetilde{A}{\triangleright\!\!\!<}V\check{}(\mathbf{R}',\mathbf{R})$; see [14].

For the spinorial examples, we let instead A be a quantum group obtained from $A(R)$ and take $B(R)$ in the category of $\widetilde{A{\bowtie}A}$-comodules by (26) to obtain (28) as $\widetilde{A{\bowtie}A}{\triangleright\!\!\!<}B(R)$; see [16]. Likewise, we take $\bar{A}(R)$ in the category of $\widetilde{A\otimes A}$ comodules under (62) to obtain (64) as $\widetilde{A\otimes A}{\triangleright\!\!\!<}\bar{A}(R)$; see [18].

We also gave the dual construction in [14] to obtain Poincaré quantum enveloping algebras dual by (86) to the above examples. Thus $C=V(\mathbf{R}',\mathbf{R})$ is the braided vector space, dual to the braided covectors above using braided-differentiation[33][34]. The symbol $V\check{}$ denotes the predual, i.e. $V=(V\check{})^*$. The generators are dual spaces to each other (as for usual vectors and covectors). This braided group lives in the same category of $\widetilde{A}$ comodules, or equivalently, $\widetilde{H}$-modules, where $\widetilde{H}$ is dual to $\widetilde{A}$ and A is from $A(\mathbf{R})$. The dilaton contributes $\lambda^{-\xi\otimes\xi}$ to the universal R-matrix. We obtain from (85) the formula (32) in Section 4 as $V(\mathbf{R}',\mathbf{R}){>\!\!\!\triangleleft}\widetilde{H}$; see [14]. We obtain from (86) the explicit duality pairing (31).

Likewise for the spinorial examples, we let H be dual to A obtained from $A(R)$, and obtain (44) as the Hopf algebra $B(R)^*{>\!\!\!\triangleleft}\widetilde{H\,\bowtie\,H}$ and (72) as the Hopf algebra $\bar{A}(R)^*{>\!\!\!\triangleleft}\widetilde{H\otimes H}$. These are canonical constructions independent of any quantum metric. When there is a quantum metric, as in the explicit q-Minkowski and q-Euclidean examples, we have $B(R)^*{\cong}B(R)$ and $\bar{A}(R){\cong}\bar{A}(R)$ in which final form we wrote these examples. They are equivalent to constructing the bosonisations $B(R){>\!\!\!\triangleleft}\widetilde{H\,\bowtie\,H}$ and $\bar{A}(R){>\!\!\!\triangleleft}\widetilde{H\otimes H}$ directly.

Finally, all these examples come equipped with a canonical (co)action from Corollary A.4 on the braided covectors regarded as spacetime co-ordinates. This gives the explicit formulae

(23) and (34) in general and (45), (73) on $B(R), \bar{A}(R)$. We systematically dropped a minus sign coming from the braided antipode on $\mathbf{P}$ in (89). In addition, all our q-Poincaré quantum groups automatically come with conjugates from Corollary A.5, as discussed in Section 8.

We have concentrated here on the applications of bosonisation to the construction of q-Poincaré quantum groups. Other interesting applications are in [53][54] and the theory of differential calculus on quantum groups in [55][56].

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