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DAMTP 93-54

14 September 1993

DAMTP 93-54

Coset Construction for Vertex Operators of the Unitary Discrete Series

Terence M. L. Loke[1]

Department of Applied Mathematics and Theoretical Physics,
University of Cambridge, Silver Street,
Cambridge CB3 9EW, U.K.

Abstract

We study the chiral vertex operators of the unitary discrete series for the Virasoro algebra. We prove an analogue of the coset construction for vertex operators and indicate how the arguments can be modified to prove the fusion rules under the assumption, motivated by Connes fusion, that a certain completeness property characterizes them. We observe that the construction can be used to prove various properties of Virasoro primaries, which are inherited from the corresponding properties of $\hat{su}(2)$ primaries. In particular, we obtain the convergence of correlation functions of Virasoro primaries and indeed their expressions in terms of correlation functions of $\hat{su}(2)$ vertex operators. We also obtain the braiding behaviour of Virasoro primaries. The usefulness of this approach to the study of other coset theories is pointed out.

[1] email:tml11@amtp.cam.ac.uk

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1. Introduction

Our immediate objective is to study the vertex operators (on $\mathbf{P}^1$) in the FQS unitary discrete series [1] for the Virasoro ($\mathcal{L}$) algebra in a constructive approach; and secondarily to extend the approach to an arbitrary coset theory. In a rational conformal field theory, algebraic properties of vertex operators such as the existence and uniqueness of primary fields, and basic analytic properties such as the convergence of correlation functions and the braiding of primaries, are best understood through the study of chiral vertex operators following Tsuchiya-Kanie [2]. If the theory is unitary, the operator theoretic aspects such as establishing Sobolev inequalities for smeared vertex operators and Connes fusion can be studied following Wassermann [3]. In each instance, the canonical examples for the techniques involved have been the $su(2)$-WZW theories at arbitrary level $k \in \mathbf{N}$, and more generally the WZW-theories for any compact group modulo technical difficulties. In this paper, we use the GKO coset construction [4] to extend some of these results to the unitary discrete series. We shall consider the algebraic and basic analytic aspects, leaving the more technically involved operator aspects for separate consideration in a sequel. The methods developed here are not specific to the unitary discrete series and should generalize to any coset theory, modulo technical difficulties. One of our aims is to demonstrate that vertex operators of coset theories are more easily studied using a natural analogue of the GKO construction for primary fields, and that both algebraic and analytic properties do in fact follow from those of the corresponding WZW theories. To this end, we develop the algebraic theory for the unitary discrete series independently from the GKO construction and indicate how the same results follow, more easily, from their manifestation as coset theories.

As far as the algebraic theory is concerned, there are no significant departures from the approach in [2] and we relegate a discussion of the main points to Appendix A. The basic analytic result to consider is the convergence of N-point functions or, equivalently, the composability of vertex operators. To a sequence $h_N, \ldots, h_1$ of unitary highest weights is associated a vector space of N-point functions, considered a priori as formal series, with a preferred basis:

$$\langle 0|\Phi^{h_N}_{k_{N+1}\,k_N}(z_N)\cdots\Phi^{h_1}_{k_2\,k_1}(z_1)|0\rangle, \quad \begin{pmatrix} & h_i & \\ k_{i+1} & & k_i \end{pmatrix} \text{ an allowed vertex, } k_1 = k_{N+1} = 0. \quad (1.1)$$

It is known [5] that these series are formal solutions to a system of $2N+3$ partial differential equations, reflecting the pairs of singular vectors of the corresponding Verma modules that generate the maximal submodule, and the projective invariance of the vacuum. In principle, we can prove the convergence of these formal solutions to bona fide holomorphic functions by showing that this system of equations is equivalent to an integrable Pfaffian system with regular singularities, to which the series in question are formal solutions at a singularity. In practice, however, the algebra is involved for $N > 4$.

In contrast, the problem admits a straightforward solution by exploiting the coset construction for the unitary discrete series. We prove the analogue of the coset construction for primary fields, in particular that every Virasoro primary field of the unitary discrete series, $c = 1 - 6/(N+2)(N+3)$, $N \in \mathbf{N}$, occurs in the decomposition of some linear combination of tensor products of pairs of $\hat{su}(2)$ vertex operators, at levels N and 1 respectively, with respect to the diagonal-$\hat{su}(2)$ at level $N+1$ and the coset Virasoro algebra. When the Virasoro primary is of the type

$$\Phi^{h}_{h\,0}(z), \quad (1.2)$$

the result is immediate from the GKO construction and the 'state-field correspondence'. Indeed, let $\psi^{j,l} = |j\rangle \otimes |h_{p,q}\rangle$ be the highest weight vector for the summand

$$(N+1, j) \otimes (c, h_{p,q}) \tag{1.3}$$

in the GKO decomposition of the tensor product of the irreducible (highest weight, positive energy) $\hat{su}(2)$ modules of spin l at level N, and spin ε at level 1

$$(N, l) \otimes (1, \varepsilon), \tag{1.4}$$

where $p = 2l+1$, $q = 2j+1$, $j + l + \varepsilon \in \mathbf{Z}$. Then $\psi^{j,l}$ is an element of $(N, l) \otimes (1, \varepsilon)$ and corresponds to a vertex operator $\Psi(z)$ descendant from the tensor product of the $\hat{su}(2)$-primaries of spin l at level N, and spin ε at level 1

$$\Phi^{l}_{l\,0}(|l\rangle, z) \otimes \Phi^{\varepsilon}_{\varepsilon\,0}(|\varepsilon\rangle, z). \tag{1.5}$$

It follows that the restriction of $\Psi(z)$ to the summand $(N+1, 0) \otimes (c, h_{1,1})$ in the GKO decomposition of $(N, 0) \otimes (1, 0)$,

$$\Psi(z)|_{(N+1,0)\otimes(c,h_{1,1})} = \Phi^{j}_{j\,0}(|j\rangle, z) \otimes \Phi^{h_{p,q}}_{h_{p,q}\,0}(z). \tag{1.6}$$

There is an obvious generalization. Associated to the lowest energy subspace of an irreducible $\hat{su}(2)$-module $\mathcal{H}_{s,M}$ of spin s at level M is a set of $\hat{su}(2)$ primary fields

$$\Phi^{s}_{s''\,s'}(\,\cdot\,, z) \tag{1.7}$$

corresponding to the allowed vertices of spin s at level M, each of which gives rise to an $\hat{su}(2)$-module of vertex operators isomorphic to $\mathcal{H}_{s,M}$. Then, in the same vein, associated to $\psi^{j,l}$ is also a vertex operator descendant from the primaries

$$\Phi^{l}_{l''\,l'}(|l\rangle, z) \otimes \Phi^{\varepsilon}_{\varepsilon''\,\varepsilon'}(|\varepsilon\rangle, z) \tag{1.8}$$

for each pair of $\hat{su}(2)$ allowed vertices of respective spins l, ε at level N, 1, which we denote as

$$\Psi^{j}\begin{bmatrix} l & \varepsilon \\ l'' \; l' & \varepsilon'' \; \varepsilon' \end{bmatrix}(z). \tag{1.9}$$

We will show that this equates to a sum of tensor products of pairs of $\hat{su}(2)$-primaries and $\mathcal{L}$-primaries, respectively at level $N+1$ and central charge c:

$$\sum_{j'',j'} Y^{j}_{j''\,j'}\begin{bmatrix} l & \varepsilon \\ l'' \; l' & \varepsilon'' \; \varepsilon' \end{bmatrix} \Phi^{j}_{j''\,j'}(|j\rangle, z) \otimes \Phi^{h_{2l+1,2j+1}}_{h_{2l''+1,2j''+1}\;\; h_{2l'+1,2j'+1}}(z) \tag{1.10}$$

where the sums in j'' and j' are given respectively by the GKO decompositions of $(N, l'') \otimes (1, \varepsilon'')$ and $(N, l') \otimes (1, \varepsilon')$, and the branching coefficients

$$Y^{j}_{j''\,j'}\begin{bmatrix} l & \varepsilon \\ l'' \; l' & \varepsilon'' \; \varepsilon' \end{bmatrix} \tag{1.11}$$

do not vanish for *all* allowed vertices

$$\begin{pmatrix} j \\ j'' \; j' \end{pmatrix}. \tag{1.12}$$

When all the vectors $\{\psi^{j,l}\}$ are known explicitly, verification of this proposition, i.e. checking that the branching coefficients do not vanish, is quite immediate. But we have no explicit form for such vectors in general, and so develop a proof that does not require it.

Given this proposition, we easily show that the convergence of correlation functions of Virasoro primaries follows from the convergence of correlation functions of $\hat{su}(2)$ primaries, which is known [2]. Minimally, it is sufficient for the convergence of correlation functions of arbitrary Virasoro primaries that we have convergence of correlation functions of the generating Virasoro primaries. However, the full coset construction for Virasoro primary fields is rather useful for other considerations when the intention is to transfer results, algebraic or analytic, in the $\hat{su}(2)$ theory to the coset theory. In particular, this is the case for fusion rules and Sobolev inequalities for smeared vertex operators.

The generating primary fields, in both the $\hat{su}(2)$ and Virasoro theories, play a fundamental role, given the operator product expansion. More or less equivalently, the $\hat{su}(2)$ or Virasoro modules of the corresponding highest weights play a fundamental role given the fusion product, which for us will ultimately be given by Connes fusion [3]. This is analogous to Weyl's Invariant Theory approach to the construction of irreducible representations of a compact Lie group by decomposing the tensor product of basic representations into irreducible components [6].

The paper is organized as follows. In Section 2, we state some results on 4-point functions involving a generating Virasoro primary that are proved in the appendices. In Section 3, we state and prove the main result. We indicate how the proof can be modified to avoid using prior knowledge of the fusion rules and argue, under a plausible assumption on the characterization of a complete set of primary fields, that this modification amounts to a proof of the fusion rules, and indeed of other properties of Virasoro vertex operators. We emphasize the utility of this approach in obtaining algebraic and analytic properties of vertex operators of a coset theory in terms of the properties of vertex operators of the corresponding affine algebra theories. In Section 4, we apply the main result to prove the convergence of correlation functions of Virasoro primaries and to obtain an alternative proof of the braiding behaviour of Virasoro primaries. In Appendix A, we present with a minimum of formalism a summary of the basic algebraic theory for the unitary discrete series in order to make this paper self-contained, and to clarify our definitions and point of view. We also discuss the solution space of a system of BPZ and projective invariance equations. In the rest of the appendices, we prove some ancillary results required in the main text.

2. 4-point functions involving a generating Virasoro primary field

We establish for later use some results on 4-point functions involving at least one generating Virasoro primary. Let $h_4, \ldots, h_1$ be a sequence of conformal dimensions, such that $h_i = h_{1,2}$ or $h_{2,1}$ for some i. Then the corresponding 4-point functions satisfy the BPZ equation

$$\left[\frac{3}{4h_i+2}\frac{\partial^2}{\partial z_i^2} + \sum_{\substack{j=1\\ j\neq i}}^{4} z_{ji}^{-1}\frac{\partial}{\partial z_j} - h_j\, z_{ji}^{-2}\right] f = 0 \tag{2.1}$$

in addition to the projective invariance equations

$$\left[\sum_{j=1}^{4} z_j^{m+1}\frac{\partial}{\partial z_j} + h_j\,(m+1)\,z_j^{m}\right] f = 0 \quad (m = 0, \pm 1). \tag{2.2}$$

The latter have the general solution

$$\prod_{\substack{j,k=1\\ j<k}}^{4} z_{jk}^{-\gamma_{jk}}\, F(x), \quad x = \frac{z_{12}z_{34}}{z_{13}z_{24}} \tag{2.3}$$

where the γ_{jk} satisfy

$$\sum_{\substack{j=1\\ j<k}}^{4} \gamma_{jk} + \sum_{\substack{j=1\\ j>k}}^{4} \gamma_{kj} = 2h_k \quad (k = 1, \ldots, 4). \tag{2.4}$$

With this Ansatz, we verify in Appendix B that (2.1) reduces to an ordinary differential equation in the variable x, a Fuchsian equation with 3 singular points, at 0, 1 and ∞. In fact, the equation is of second order and hence, with a judicious choice of the γ_{jk}, a hypergeometric equation.

We show in Appendix C that this fact is sufficient for the system of equations given by (2.1) and (2.2) to be equivalent to an integrable system of equations:

$$\frac{\partial \vec{f}}{\partial w_j} = \frac{A_j(w)\vec{f}}{w_j} \quad (j = 1, \ldots, 4); \qquad \vec{f} = \left(\frac{\partial^{r-1} f}{\partial w_1^{r-1}}\right)_{r=1,\ldots,n} \tag{2.5}$$

where each $A_j(w)$ is an $n \times n$ matrix-valued function, holomorphic in a neighbourhood of the origin, and $w_j = z_j/z_{j+1}$ $(j = 1, 2, 3)$, $w_4 = z_4$. Here, $n = 2$. Formal series solutions to such a system

$$\prod_{j=1}^{4} w_j^{s_j} \sum_{\substack{n_j \in \mathbb{N}_0\\ j=1,\ldots,4}} c_{n_4 \ldots n_1}\, w_4^{n_4} \cdots w_1^{n_1} \tag{2.6}$$

are known [6,7] always to converge in some neighbourhood of the origin. It follows that the 4-point functions

$$\langle 0 | \Phi^{h_4}_{k_3\, k_4}(z_4) \cdots \Phi^{h_1}_{k_2\, k_1}(z_1) | 0 \rangle \tag{2.7}$$

are convergent in a corresponding domain in $\{(z_4, \ldots, z_1) \in \mathbb{C}^{*4} : |z_4| > \cdots > |z_1|\}$.

Now let h_3 be one of $h_{1,2}$, $h_{2,1}$. We obtain the following result in Appendix B. For each pair of allowed vertices

$$\begin{pmatrix} h_3 \\ h_4\, h \end{pmatrix}, \quad \begin{pmatrix} h_2 \\ h\, h_1 \end{pmatrix}$$

we have by analytic continuation along some path that

$$\Phi^{h_3}_{h_4\, h}(z)\Phi^{h_2}_{h\, h_1}(w) = \sum_{h'} \Phi^{h_2}_{h_4\, h'}(w)\Phi^{h_3}_{h'\, h_1}(z)\, C^{h_4 h_3 h_2 h_1}_{h' h}, \tag{2.8}$$

where the sum is over all allowed vertices. The braiding matrix $(C^{h_4 h_3 h_2 h_1}_{h' h})_{h' h}$ is an invertible matrix of order 1 or 2 with *no* vanishing entries. As a corollary, the fusion matrix $(F^{h_4 h_3 h_2 h_1}_{h' h})_{h' h}$ appearing in the operator product expansion, $|w| > |z - w| > 0$:

$$\Phi^{h_3}_{h_4\, h}(z)\Phi^{h_2}_{h\, h_1}(w) = \sum_{h'} F^{h_4 h_3 h_2 h_1}_{h' h} \sum_{a_{h'}} \langle a_{h'} | \Phi^{h_3}_{h'\, h_2}(z - w) | h_2 \rangle\, \Phi^{h'}_{h_4\, h_1}(a_{h'}, w) \tag{2.9}$$

is also an invertible matrix of the same order with no vanishing entries. Analogous results hold for the braiding and fusion matrices involving a generating $\hat{su}(2)$-primary (of spin 1/2), i.e. there are no vanishing entries. This is obtained by inspection of explicit formulae in [2].

3. The coset construction for primary fields

The archetypal coset construction is the following theorem of Goddard-Kent-Olive [4]. The tensor product of (N, l), the irreducible highest weight $\hat{su}(2)$-module at level N of spin l, $N \in \mathbb{N}$, $l = 0, 1/2, \ldots, N/2$, and $(1, \varepsilon)$, the irreducible highest weight $\hat{su}(2)$-module at level 1 of spin ε, $\varepsilon = 0, 1/2$, is a direct sum of irreducible highest weight $\hat{su}(2) \times \mathcal{L}$-modules, respectively at level $N + 1$ and with central charge $c = c(N) = 1 - 6/(N+2)(N+3)$, given by

$$(N, l) \otimes (1, \varepsilon) = \bigoplus_q (N + 1, \frac{q-1}{2}) \otimes (c, h_{p,q}) \tag{3.1}$$

where $p = 2l + 1$, and the sum over q is such that $p - q$ is even/odd if 2ε is even/odd, and $1 \le q \le N + 2$. Each unitary highest weight Virasoro module with central charge $c < 1$ appears in the decomposition of precisely 2 choices of $(N, l) \otimes (1, \varepsilon)$. The $\hat{su}(2) \times \mathcal{L}$ action is given by the diagonal $\hat{su}(2)$ subalgebra of $\hat{su}(2) \times \hat{su}(2)$ and the coset Virasoro algebra $\mathcal{L}_c$, i.e. the difference of the Sugawara constructions for $\hat{su}(2) \times \hat{su}(2)$ and its diagonal subalgebra. We write the respective $\hat{su}(2)$ subalgebras as $\hat{su}(2)_N$, $\hat{su}(2)_1$, $\hat{su}(2)_{N+1}$. We want an analogue of these results for the vertex operators.

We briefly recall the 'state-field correspondence': corresponding to an irreducible highest weight, positive energy $\hat{su}(2)$ module (for some spin and level), we can canonically construct spaces of (chiral) vertex operators, each an isomorphic $\hat{su}(2)$-module, spanned by a primary-field of corresponding spin and level and its descendents. The allowable primary fields at a particular spin and level are determined by algebraic conditions known as *fusion rules*. A similar correspondence holds for irreducible highest weight modules of the Virasoro algebra. By the GKO coset construction and this correspondence, we have that a linear combination of tensor products of pairs of $\hat{su}(2)_N$ and $\hat{su}(2)_1$ vertex operators decomposes as a linear combination of tensor products of pairs of $\hat{su}(2)_{N+1}$ and $\mathcal{L}_c$ vertex operators.

Let $\{\psi^{j,l}_a : a = 1, \ldots, \dim V_j\}$ span the lowest energy subspace of the $\hat{su}(2)_{N+1} \times \mathcal{L}_c$-module $(N+1, j) \otimes (c, h_{p,q})$, considered as a summand of the GKO decomposition of $(N, l) \otimes (1, \varepsilon)$, so that $p = 2l + 1$, $q = 2j + 1$ and $j + l + \varepsilon \in \mathbb{Z}$. Here, the $su(2)$-module V_j of spin j is considered as the lowest energy subspace of $(N + 1, j)$. For each pair of allowed vertices of spins l, ε:

$$\begin{pmatrix} l \\ l''\, l' \end{pmatrix}, \quad \begin{pmatrix} \varepsilon \\ \varepsilon''\, \varepsilon' \end{pmatrix}$$

at levels N, 1, let

$$\Psi^j \begin{bmatrix} l & & \varepsilon \\ l''\, l' & & \varepsilon''\, \varepsilon' \end{bmatrix} (\,.\,, z) \tag{3.2}$$

be the corresponding descendents of the $\hat{su}(2)_N \times \hat{su}(2)_1$-primary

$$\Phi^l_{l''\, l'}(|l\rangle, z) \otimes \Phi^\varepsilon_{\varepsilon''\, \varepsilon'}(|\varepsilon\rangle, z). \tag{3.3}$$

By the GKO isomorphism and the 'state-field correspondence', this must be a $\hat{su}(2)_{N+1} \times \mathcal{L}_c$-primary of spin j and conformal dimension $h_{p,q}$. Further, we can decompose this as a sum of primaries corresponding to the various allowed vertices, and thus it can be written as the sum

$$\sum_{j'', j'} Y^j_{j''\, j'} \begin{bmatrix} l & & \varepsilon \\ l''\, l' & & \varepsilon''\, \varepsilon' \end{bmatrix} \Phi^j_{j''\, j'}(\,.\,.\, z) \otimes \Phi^{h_{p,q}}_{h_{p'',q''}\, h_{p',q'}}(z) \tag{3.4}$$

for some branching coefficients

$$Y^{j}_{j'' j'}\left[\begin{smallmatrix} l & & \varepsilon \\ & l'' \; l' & \\ & \varepsilon'' \; \varepsilon' & \end{smallmatrix}\right] \in \mathbf{C}$$

where $p'' = 2l''+1$, $p' = 2l'+1$, $q'' = 2j''+1$, $q' = 2j'+1$, and the sum in j'' and j' is over the values in the GKO decomposition of $(N, l'') \otimes (1, \varepsilon'')$ and $(N, l') \otimes (1, \varepsilon')$. We shall take the convention that the branching coefficients are defined for all such values, but vanishing when

$$\begin{pmatrix} j \\ j'' \; j' \end{pmatrix}$$

is not an allowed vertex. It remains to show that none of the branching coefficients vanish otherwise. Appendix D gives a more careful argument for these assertions.

Distinguished amongst the lowest energy subspaces $\langle \psi_a^{j,l} \rangle \subset (N+1, j) \otimes (c, h_{p,q})$ are those that lie in the lowest energy subspace of the corresponding $(N, l) \otimes (1, \varepsilon)$, whence

$$\Phi^{l}_{l'' \; l'}(\,.\,,z) \odot \Phi^{\varepsilon}_{\varepsilon'' \; \varepsilon'}(\,.\,,z) = \sum_{j=|l\pm\varepsilon|} \Psi^{j}\left[\begin{smallmatrix} l & & \varepsilon \\ & l'' \; l' & \\ & \varepsilon'' \; \varepsilon' & \end{smallmatrix}\right](\,.\,,z). \tag{3.5}$$

Such $\langle \psi_a^{j,l} \rangle$ correspond to spin j and conformal dimension $h_{2l+1,2j+1}$ obeying

$$\Delta(l; N) + \Delta(\varepsilon; 1) = \Delta(j; N+1) + h_{2l+1,2j+1}, \quad j = |l \pm \varepsilon| \tag{3.6}$$

where $\Delta(s; M) = s(s+1)/(M+2)$, which is an identity for each l, and $\varepsilon = 0, 1/2$. The corresponding set of Virasoro primary fields that can be found by decomposing tensor products of $\hat{su}(2)$ primaries are only those of conformal dimension

$$\begin{aligned} h_{p,p}, &\quad p = 1, \ldots, N+1 \\ h_{p,p-1}, &\quad p = 2, \ldots, N+1 \end{aligned} \tag{3.7}$$

bearing in mind that $h_{p,q} = h_{N+2-p,N+3-q}$. This is the set of $h_{p,q}$ that do not exceed 1. We will call these Virasoro primaries the *relevant* operators, following the identification in [8] of precisely this set of operators with the relevant operators of Lagrangian scalar $\phi^{2(N+1)}$-theory at a non-trivial fixed-point. In particular, the generating Virasoro primaries, being of conformal dimension $h_{1,2}$ or $h_{2,1}$, are relevant operators. We have

$$\begin{aligned} h_{1,2} &= h_{2l+1,2j+1} \Leftrightarrow (l, j) = (0, \tfrac{1}{2}), (\tfrac{N}{2}, \tfrac{N}{2}) \\ h_{2,1} &= h_{2l+1,2j+1} \Leftrightarrow (l, j) = (\tfrac{1}{2}, 0). \end{aligned} \tag{3.8}$$

In the following, we will first prove that the branching coefficients

$$Y^{j}_{\cdot\cdot}\left[\begin{smallmatrix} l & & \varepsilon \\ & \cdot\cdot\cdot\cdot & \end{smallmatrix}\right], \quad (l, j) = (0, \tfrac{1}{2}), (\tfrac{1}{2}, 0) \tag{3.9}$$

for allowed vertices, are non-vanishing. These are the cases when a vertex operator corresponding to $\langle \psi_a^{j,l} \rangle$ decomposes as tensor products of trivial and generating primaries, whether regarded as a $\hat{su}(2)_N \times \hat{su}(2)_1$ or $\hat{su}(2)_{N+1} \times \mathcal{L}_c$ primary field.

We then develop an inductive argument. By taking the operator product expansion of the vertex operators

$$\Psi^{j}\left[\begin{smallmatrix} l & & \varepsilon \\ & \cdot\cdot\cdot\cdot & \end{smallmatrix}\right](\,.\,,z), \quad (l, j) \neq (0, 0) \tag{3.10}$$

successively with the primaries

$$\Psi^{\bar{j}}\left[\begin{smallmatrix} \bar{l} & & \varepsilon \\ & \cdot\cdot\cdot\cdot & \end{smallmatrix}\right](\,.\,,z), \quad (\bar{l}, \bar{j}) = (0, \tfrac{1}{2}), (\tfrac{1}{2}, 0), \tag{3.11}$$

we show that all the branching coefficients (for allowed vertices) corresponding to $(l+1, j)$ and $(l, j+1)$ (when defined) are non-vanishing if all those corresponding to (l, j) are non-vanishing. Since the result is first established independently for $(l, j) = (0, 1/2), (1/2, 0)$, we are then finished.

We remark that (3.2), (3.4) illustrate the well-known observation that the fusion rules for the Virasoro theory at $c = 1 - 6/(N+2)(N+3)$ correspond to the fusion rules for the $\hat{su}(2)$ theory at level N and $N+1$, with the correspondence given by (taking $p+p'+p''+q+q'+q'' \in 2\mathbf{Z}$ without loss of generality):

$$\begin{pmatrix} & h_{p,q} & \\ h_{p'',q''} & & h_{p',q'} \end{pmatrix} \text{ at } c = 1 - \frac{6}{(N+2)(N+3)} \tag{3.12}$$

is an allowed vertex if and only if

$$\begin{pmatrix} & \frac{p-1}{2} & \\ \frac{p''-1}{2} & & \frac{p'-1}{2} \end{pmatrix} \text{ at level } N, \quad \begin{pmatrix} & \frac{q-1}{2} & \\ \frac{q''-1}{2} & & \frac{q'-1}{2} \end{pmatrix} \text{ at level } N+1 \tag{3.13}$$

are allowed vertices.

3.1 Branching coefficients for the generating primaries

Corresponding to allowed vertices at level N and level 1 respectively, the tensor product

$$\Phi^{l}_{l'' \; l'}(\,.\,,z) \odot \Phi^{\varepsilon}_{\varepsilon'' \; \varepsilon'}(\,.\,,z) \tag{3.14}$$

of $\hat{su}(2)$-primaries has the decomposition

$$\sum_{j, j', j''} Y^{j}_{j'' \; j'}\left[\begin{smallmatrix} l & & \varepsilon \\ & l'' \; l' & \\ & \varepsilon'' \; \varepsilon' & \end{smallmatrix}\right] \Phi^{j}_{j'' \; j'}(\,.\,,z) \odot \Phi^{h_{p,q}}_{h_{p'',q''} \; h_{p',q'}}(z), \tag{3.15}$$

where $p = 2l+1$, $q = 2j+1$ and similarly for p', p'', q', q''. The sum in j is given by the decomposition of $V_l \otimes V_\varepsilon$ into irreducible $su(2)$-submodules; and in j'' and j' by the GKO decompositions of $(N, l'') \odot (1, \varepsilon'')$ and $(N, l') \odot (1, \varepsilon')$ respectively.

The operator product expansion of a $\hat{su}(2)_M$-primary with its conjugate is, with $|z| > |w - z| > 0$:

$$\begin{aligned} \Phi^{j}_{j' \; j''}(\,.\,,w)\Phi^{j}_{j'' \; j'}(\,.\,,z) &= \sum_{k} F^{j'jjj'}_{M;\,kj''} \sum_{a_k} \langle a_k | \Phi^{j}_{k \; j}(\,.\,,w-z) | \,.\, \rangle \, \Phi^{k}_{j' \; j'}(a_k, z) \\ &= F^{j'jjj'}_{M;\,0j''} \, \langle 0 | \Phi^{j}_{0 \; j}(\,.\,,w-z) | \,.\, \rangle \, \Phi^{0}_{j' \; j'} + \ldots . \end{aligned} \tag{3.16}$$

It follows that the most singular term in the operator product expansion of (3.14) with its conjugate is proportional to the identity operator on $(N,l')\otimes(1,\varepsilon')$:

$$\Phi^0_{l'\,l'}\otimes\Phi^0_{\varepsilon'\,\varepsilon'} \tag{3.17}$$

tensored with the trivial $su(2)\times su(2)$-module

$$\langle 0|\Phi^l_{0\,l}(\,.\,,w-z)|\,.\,\rangle\,\langle 0|\Phi^\varepsilon_{0\,\varepsilon}(\,.\,,w-z)|\,.\,\rangle \cong (V_l\otimes V_l)_{su(2)}\otimes(V_\varepsilon\otimes V_\varepsilon)_{su(2)} \tag{3.18}$$

when

$$F^{l'll'}_{N;\,0l''}\;F^{\varepsilon'\varepsilon\varepsilon\varepsilon'}_{1;\,0\varepsilon''}\neq 0. \tag{3.19}$$

Being at level 1, we always have that $F^{\varepsilon'\varepsilon\varepsilon\varepsilon'}_{1;\,0\varepsilon''}\neq 0$. Also, $F^{l'll'}_{N;\,0l''}\neq 0$ for $l=0$ trivially, and for $l=1/2$ by explicit calculation [2].

It follows from the above considerations that when $(l,j)=(0,1/2)$ or $(1/2,0)$, then for *each* pair of allowed vertices ($\varepsilon\in j+l+\mathbf{Z}$)

$$\begin{pmatrix} l \\ l''\;l'\end{pmatrix},\quad \begin{pmatrix}\varepsilon\\ \varepsilon''\;\varepsilon'\end{pmatrix} \tag{3.20}$$

and for *each* $j'\in l'+\varepsilon'+\mathbf{Z}$, $0\le j'\le(N+1)/2$, that the branching coefficient

$$Y^{j}_{j''\,j'}\begin{bmatrix} l & & \varepsilon \\ l''\;l' & & \varepsilon''\;\varepsilon'\end{bmatrix}$$

is non-vanishing for *some* $j''\in l''+\varepsilon''+\mathbf{Z}$, $0\le j''\le(N+1)/2$. When $(l,j)=(1/2,0)$, there is only the single allowed vertex

$$\begin{pmatrix} j \\ j''\;j'\end{pmatrix}=\begin{pmatrix} 0 \\ j'\;j'\end{pmatrix} \tag{3.21}$$

so that all the branching coefficients for allowed vertices are non-vanishing, and we are done. When $(l,j)=(0,1/2)$, there are either one or two allowed vertices

$$\begin{pmatrix} j \\ j''\;j'\end{pmatrix}=\begin{pmatrix} \frac{1}{2} \\ j'\pm\frac{1}{2}\;j'\end{pmatrix},\quad 0\le j'\pm\tfrac{1}{2}\le\tfrac{N+1}{2}. \tag{3.22}$$

When there is only one (when $j'=0,(N+1)/2$), we are again finished. If there are two, we now show that *both* of the corresponding branching coefficients are non-vanishing. We have ($\varepsilon''=1/2-\varepsilon'$):

$$\Phi^0_{l'\,l'}\otimes\Phi^{\frac{1}{2}}_{\varepsilon''\,\varepsilon'}(\,.\,,z)=\sum_{j'}\sum_{j''=j'\pm\frac{1}{2}} Y^{\frac{1}{2}}_{j''\,j'}\,\Phi^{\frac{1}{2}}_{j''\,j'}(\,.\,,z)\otimes\Phi^{h_{1,2}}_{h_{p',q''}\;h_{p',q'}}(z) \tag{3.23}$$

where

$$Y^{\frac{1}{2}}_{j''\,j'}=Y^{\frac{1}{2}}_{j''\,j'}\begin{bmatrix} 0 & & \frac{1}{2} \\ l'\;l' & & \varepsilon''\;\varepsilon'\end{bmatrix}.$$

Fix a value of $q'=2j'+1$, and let

$$Y^{\frac{1}{2}}_{j_+\,j'}\neq 0,\quad Y^{\frac{1}{2}}_{j_-\,j'}=0 \tag{3.24}$$

where $\{q_+,q_-\}=\{q'\pm 1\}$ and $j_\pm=(q_\pm-1)/2$. To obtain a contradiction, we examine the braiding behaviour of

$$\Phi^0_{l'\,l'}\otimes\Phi^{\frac{1}{2}}_{\varepsilon''\,\varepsilon'}(\,.\,,z)\,P_{p',q'}=Y^{\frac{1}{2}}_{j_+\,j'}\,\Phi^{\frac{1}{2}}_{j_+\,j'}(\,.\,,z)\otimes\Phi^{h_{1,2}}_{h_{p',q_+}\;h_{p',q'}}(z) \tag{3.25}$$

with its conjugate

$$P_{p',q'}\,\Phi^0_{l'\,l'}\otimes\Phi^{\frac{1}{2}}_{\varepsilon'\,\varepsilon''}(\,.\,,w)=\bar{Y}^{\frac{1}{2}}_{j_+\,j'}\,\Phi^{\frac{1}{2}}_{j'\,j_+}(\,.\,,w)\otimes\Phi^{h_{1,2}}_{h_{p',q'}\;h_{p',q_+}}(w) \tag{3.26}$$

where $P_{p',q'}$ is the projection onto the summand $(N+1,j')\otimes(c,h_{p',q'})$. Braiding the left-hand-side, we get, $|z|>|w|$:

$$P_{p',q'}\;\Phi^0_{l'\,l'}\otimes\left[\Phi^{\frac{1}{2}}_{\varepsilon'\,\varepsilon''}(\,.\,,z)\Phi^{\frac{1}{2}}_{\varepsilon''\,\varepsilon'}(\,.\,,w)\right]P_{p'q'}\;C^{\varepsilon'\frac{1}{2}\frac{1}{2}\varepsilon'}_{\varepsilon''\varepsilon''} \tag{3.27}$$

which, by (3.25) and (3.26), has the decomposition, $|z|>|w|$:

$$C^{\varepsilon'\frac{1}{2}\frac{1}{2}\varepsilon'}_{\varepsilon''\varepsilon''}\,|Y^{\frac{1}{2}}_{j_+\,j'}|^2\,\left[\Phi^{\frac{1}{2}}_{j'\,j_+}(\,.\,,z)\Phi^{\frac{1}{2}}_{j_+\,j'}(\,.\,,w)\right]\otimes\left[\Phi^{h_{1,2}}_{h_{p',q'}\;h_{p',q_+}}(z)\Phi^{h_{1,2}}_{h_{p',q_+}\;h_{p',q'}}(w)\right]. \tag{3.28}$$

We examine the operator product expansion of this expression with the $\hat{su}(2)_{N+1}\times\mathcal{L}_c$-primary

$$\Phi^{\frac{1}{2}}_{j_-\,j'}(\,.\,,x)\otimes\Phi^0_{h_{p',q'}\;h_{p',q'}}. \tag{3.29}$$

It is sufficient to look at the $\hat{su}(2)_{N+1}$ part of the operator product expansion. The most singular term in $x-z\to 0$ is proportional to

$$\left[\sum_a\langle\xi^*_a|\Phi^{\frac{1}{2}}_{1\,\frac{1}{2}}(\,.\,,x-z)|\,.\,\rangle\,\Phi^1_{j_-\,j_+}(\xi_a,z)\Phi^{\frac{1}{2}}_{j_+\,j'}(\,.\,,w)\right] \tag{3.30}$$

where $\{\xi_a\}$, $\{\xi^*_a\}$ is a dual basis for V_1, with behaviour

$$\sim(x-z)^{-2\,\Delta(1/2;N+1)+\Delta(1;N+1)}. \tag{3.31}$$

However, on the right-hand-side, braiding gives, $|z|>|w|$:

$$|Y^{\frac{1}{2}}_{j_+\,j'}|^2\left\{\sum_{j'''=j_\pm}\Phi^{\frac{1}{2}}_{j'\,j'''}(\,.\,,z)\Phi^{\frac{1}{2}}_{j'''\,j'}(\,.\,,w)\,C^{j'\frac{1}{2}\frac{1}{2}j'}_{j'''j_+}\right\}$$
$$\otimes\left\{\sum_{q'''=q_\pm}\Phi^{h_{1,2}}_{h_{p',q'}\;h_{p',q'''}}(z)\Phi^{h_{1,2}}_{h_{p',q'''}\;h_{p',q'}}(w)\,C^{h_{p',q'}h_{1,2}h_{1,2}h_{p',q'}}_{h_{p',q'''}h_{p',q_+}}\right\} \tag{3.32}$$

and the $\hat{su}(2)_{N+1}$ part of the most singular term in $x-z\to 0$ in its operator product expansion with the primary field (3.29) is proportional to

$$F^{j_-\frac{1}{2}\frac{1}{2}j_-}_{0j'}\,C^{j'\frac{1}{2}\frac{1}{2}j'}_{j_-j_+}\,\langle 0|\Phi^{\frac{1}{2}}_{0\,\frac{1}{2}}(\,.\,,x-z)|\,.\,\rangle\,\Phi^{\frac{1}{2}}_{j_-\,j'}(\,.\,,w) \tag{3.33}$$

with the behaviour

$$\sim(x-z)^{-2\,\Delta(1/2;N+1)} \tag{3.34}$$

since, by explicit calculation [2], the $\hat{su}(2)$ braiding matrix, and hence also the fusion matrix, involving a generating $\hat{su}(2)$ primary and an arbitrary primary has no vanishing entries. This establishes the contradiction. We conclude that the branching coefficients for allowed vertices are non-vanishing when $(l,j)=(0,1/2)$ and $(1/2,0)$.

3.2 Branching coefficients for other primaries

We consider the operator product expansions of the primaries

$$\Psi^{j}\left[\begin{smallmatrix} l & & \varepsilon \\ & l''' \; l'' & \\ & \varepsilon''' \; \varepsilon'' & \end{smallmatrix}\right](\,.\,,z), \quad (\bar{l},\bar{j}) = (0,\tfrac{1}{2}),\ (\tfrac{1}{2},0) \tag{3.35}$$

with the vertex operator

$$\Psi^{j}\left[\begin{smallmatrix} l & & \varepsilon \\ & l'' \; l' & \\ & \varepsilon'' \; \varepsilon' & \end{smallmatrix}\right](\,.\,,w) \tag{3.36}$$

for fixed but arbitrary $(l,j) \neq (0,0)$.

3.2.1 $h_{p,q} \to h_{p\pm1,q}$

Consider the operator product expansion of

$$\Psi^{0}\left[\begin{smallmatrix} \frac{1}{2} & & \frac{1}{2} \\ & l''' \; l'' & \\ & \varepsilon''' \; \varepsilon'' & \end{smallmatrix}\right](\,.\,,z)\,\Psi^{j}\left[\begin{smallmatrix} l & & \varepsilon \\ & l'' \; l' & \\ & \varepsilon'' \; \varepsilon' & \end{smallmatrix}\right](\,.\,,w). \tag{3.37}$$

Each coefficient of the expansion in powers of $z-w$ is generically a linear combination of $\hat{su}(2)_N \times \hat{su}(2)_1$ vertex operators, each descendant from the primary

$$\Phi_{l''' \, l''}^{l\pm\frac{1}{2}}(\,.\,,w) \odot \Phi_{\varepsilon''' \, \varepsilon'}^{\frac{1}{2}-\varepsilon}(\,.\,,w) \tag{3.38}$$

according to the power of $z-w$ being in

$$-\varepsilon + \frac{-1 \pm p}{2(N+2)} + \mathbf{Z} \tag{3.39}$$

and with dependence on the spin l'' entirely contained in a scalar factor

$$F_{l\pm\frac{1}{2} \, l''}^{l''' \, \frac{1}{2} \, l''} \neq 0; \tag{3.40}$$

the exceptional cases being when the fusion rules are more restrictive and allow only one of the two possibilities above, and this occurs when $l = 0,\ N/2$. On the other hand, in terms of their respective expansions in terms of $\hat{su}(2)_{N+1} \times \mathcal{L}_c$-primaries, we have

$$\sum_{j'',j'} Y^{0}_{j''\,j''}\left[\begin{smallmatrix} \frac{1}{2} & & \frac{1}{2} \\ & l''' \; l'' & \\ & \varepsilon''' \; \varepsilon'' & \end{smallmatrix}\right] Y^{j}_{j''\,j'}\left[\begin{smallmatrix} l & & \varepsilon \\ & l'' \; l' & \\ & \varepsilon'' \; \varepsilon' & \end{smallmatrix}\right]$$
$$\Phi^{j}_{j''\,j'}(\,.\,,w) \odot \left[\Phi^{h_{2,1}}_{h_{p''',q''}\,h_{p'',q''}}(z)\Phi^{h_{p,q}}_{h_{p'',q''}\,h_{p',q'}}(w)\right]. \tag{3.41}$$

Retaining only the terms (in powers of $z-w$) that have coefficients that are also primaries, there are generically 2 terms, with the same exceptions as before:

$$\sum_{\pm}(z-w)^{h_{p\pm1,q}-h_{p,q}-h_{2,1}} \sum_{j'',j'} Z^{j}_{j''\,j'}\left[\begin{smallmatrix} l\pm\frac{1}{2} & & \frac{1}{2}-\varepsilon \\ & l''' \; l' & \\ & \varepsilon''' \; \varepsilon' & \end{smallmatrix}\,;\, l''\right] \Phi^{j}_{j''\,j'}(\,.\,,w) \odot \Phi^{h_{p\pm1,q}}_{h_{p''',q''}\,h_{p',q'}}(w) \tag{3.42}$$

where

$$Z^{j}_{j''\,j'}\left[\begin{smallmatrix} l\pm\frac{1}{2} & & \frac{1}{2}-\varepsilon \\ & l''' \; l' & \\ & \varepsilon''' \; \varepsilon' & \end{smallmatrix}\,;\, l''\right] = Y^{0}_{j''\,j''}\left[\begin{smallmatrix} \frac{1}{2} & & \frac{1}{2} \\ & l''' \; l'' & \\ & \varepsilon''' \; \varepsilon'' & \end{smallmatrix}\right] Y^{j}_{j''\,j'}\left[\begin{smallmatrix} l & & \varepsilon \\ & l'' \; l' & \\ & \varepsilon'' \; \varepsilon' & \end{smallmatrix}\right] F^{h_{p''',q''}\,h_{2,1}\,h_{p,q}\,h_{p',q'}}_{h_{p\pm1,q}\,h_{p'',q''}}. \tag{3.43}$$

The remaining terms in the expansion have coefficients that are descendents of the coefficient of one of these two terms, and they come in integrally larger powers of $z-w$.

Since

$$h_{p+1,q} - h_{p-1,q} = p - q + \frac{p}{N+2} \notin \mathbf{Z}, \tag{3.44}$$

we can identify each of these linear combinations of $\hat{su}(2)_{N+1} \times \mathcal{L}_c$-primaries with the coefficients of the corresponding powers of $z-w$. Now by the 'state-field correspondence', these $\hat{su}(2)_N \times \hat{su}(2)_1$ vertex operators are characterized up to a scalar factor by their decompositions into $\hat{su}(2)_{N+1} \times \mathcal{L}_c$ vertex operators, and are thus, up to scalar multipication, the vertex operators

$$\Psi^{j}\left[\begin{smallmatrix} l\pm\frac{1}{2} & & \frac{1}{2}-\varepsilon \\ & l''' \; l' & \\ & \varepsilon''' \; \varepsilon' & \end{smallmatrix}\right](\,.\,,w) \tag{3.45}$$

so that, up to irrelevant scalar multiplication,

$$Y^{j}_{j''\,j'}\left[\begin{smallmatrix} l\pm\frac{1}{2} & & \frac{1}{2}-\varepsilon \\ & l''' \; l' & \\ & \varepsilon''' \; \varepsilon' & \end{smallmatrix}\right] = Z^{j}_{j''\,j'}\left[\begin{smallmatrix} l\pm\frac{1}{2} & & \frac{1}{2}-\varepsilon \\ & l''' \; l' & \\ & \varepsilon''' \; \varepsilon' & \end{smallmatrix}\right] \Big/ F_{l\pm\frac{1}{2}\,l''}^{l''' \, \frac{1}{2} \, l'} \tag{3.46}$$

is independent of l''.

3.2.2 $h_{p,q} - h_{p,q\pm1}$

We consider next the operator product expansion of

$$\Psi^{\frac{1}{2}}\left[\begin{smallmatrix} 0 & & \frac{1}{2} \\ & l'' \; l'' & \\ & \varepsilon''' \; \varepsilon'' & \end{smallmatrix}\right](\,.\,,z)\,\Psi^{j}\left[\begin{smallmatrix} l & & \varepsilon \\ & l'' \; l' & \\ & \varepsilon'' \; \varepsilon' & \end{smallmatrix}\right](\,.\,,w). \tag{3.47}$$

Each coefficient of the expansion in powers of $z-w$ is a linear combination of $\hat{su}(2)_N \times \hat{su}(2)_1$ vertex operators, descendant from the primary

$$\Phi^{l}_{l'' \, l'}(\,.\,,w) \odot \Phi^{\frac{1}{2}-\varepsilon}_{\varepsilon''' \, \varepsilon'}(\,.\,,w) \tag{3.48}$$

and each term has behaviour $\sim (z-w)^{\mu}$, $\mu \in -\varepsilon + \mathbf{Z}$. On the other hand, in terms of their respective expansions in terms of $\hat{su}(2)_{N+1} \times \mathcal{L}_c$-primaries, we have

$$\sum_{j''',j'',j'} Y^{\frac{1}{2}}_{j'''\,j''}\left[\begin{smallmatrix} 0 & & \frac{1}{2} \\ & l'' \; l'' & \\ & \varepsilon''' \; \varepsilon'' & \end{smallmatrix}\right] Y^{j}_{j''\,j'}\left[\begin{smallmatrix} l & & \varepsilon \\ & l'' \; l' & \\ & \varepsilon'' \; \varepsilon' & \end{smallmatrix}\right]$$
$$\left[\Phi^{\frac{1}{2}}_{j'''\,j''}(\,.\,,z)\Phi^{j}_{j''\,j'}(\,.\,,w)\right] \odot \left[\Phi^{h_{1,2}}_{h_{p'',q'''}\,h_{p'',q''}}(z)\Phi^{h_{p,q}}_{h_{p'',q''}\,h_{p',q'}}(w)\right] \tag{3.49}$$

and retaining only the terms (in powers of $z-w$) that have coefficients that are also primaries, there are *a priori* and generically 4 terms:

$$\sum_{j_1,j_2=j\pm\frac{1}{2}} (z-w)^{s(l,j,j_1,j_2)} \sum_{j''',j'} Z^{j_1,\,j_2}_{j'''\,j'}\left[\begin{smallmatrix} l & & \frac{1}{2}-\varepsilon \\ & l'' \; l' & \\ & \varepsilon''' \; \varepsilon' & \end{smallmatrix}\right] \Phi^{j_1}_{j'''\,j'}(\,.\,,w) \odot \Phi^{h_{p,q_2}}_{h_{p'',q'''}\,h_{p',q'}}(w), \tag{3.50}$$

where $q_2 = 2j_2 + 1$ and

$$Z^{j_1,\,j_2}_{j'''\,j'}\left[\begin{smallmatrix} l & & \frac{1}{2}-\varepsilon \\ & l'' \; l' & \\ & \varepsilon''' \; \varepsilon' & \end{smallmatrix}\right] =$$
$$\sum_{j''} Y^{\frac{1}{2}}_{j'''\,j''}\left[\begin{smallmatrix} 0 & & \frac{1}{2} \\ & l'' \; l'' & \\ & \varepsilon''' \; \varepsilon'' & \end{smallmatrix}\right] Y^{j}_{j''\,j'}\left[\begin{smallmatrix} l & & \varepsilon \\ & l'' \; l' & \\ & \varepsilon'' \; \varepsilon' & \end{smallmatrix}\right] F^{j'''\,\frac{1}{2}\,j\,j'}_{j_1\,j''}\; F^{h_{p'',q''}\,h_{1,2}\,h_{p,q}\,h_{p',q'}}_{h_{p,q_2}\,h_{p'',q''}} \tag{3.51}$$

and

$$s(l,j,j_1,j_2) = \begin{cases} \pm(j-l) & \text{if } j_1 = j_2 = j \pm \frac{1}{2}; \\ \mp(j-l) \pm \dfrac{2j+1}{N+3} & \text{if } j_1 = j \pm \frac{1}{2},\ j_2 = j \mp \frac{1}{2}. \end{cases} \tag{3.52}$$

Again, the remaining terms in the expansion have coefficients that are descendents of the coefficients of one of these terms, and they come in integrally larger powers of $z - w$. The exceptions to the generic case occur when $j = 0$ or $N/2$, whence the fusion rules are more restrictive, and the four terms reduce to the one, corresponding to $j_1 = j_2 = j + 1/2$ and $j - 1/2$ respectively. Then (3.51) is a sum over a single value only of j'', given by j' and $N/2 - j'$ respectively.

But $j + l + \varepsilon \in \mathbf{Z}$, and $(2j+1)/(N+3) \notin \mathbf{Z}$, so that, for each pair of spins j''', j', we must have

$$Z^{j_1,\,j_2}_{j''' \; j'} \begin{bmatrix} l & \frac{1}{2}-\varepsilon \\ l'' \; l' & \varepsilon''' \; \varepsilon' \end{bmatrix} = 0 \quad \text{if } j_1 \neq j_2. \tag{3.53}$$

This is a notable property[2] for the branching coefficients. The fusion matrices in the sum (3.51) have no vanishing entries and are, of course, invertible, so that (3.53) implies that

$$Z^{j\pm\frac{1}{2},\,j\pm\frac{1}{2}}_{j''' \; j'} \begin{bmatrix} l & \frac{1}{2}-\varepsilon \\ l'' \; l' & \varepsilon''' \; \varepsilon' \end{bmatrix} \propto Y^{\frac{1}{2}}_{j''' \; j''} \begin{bmatrix} 0 & \frac{1}{2} \\ l'' \; l'' & \varepsilon''' \; \varepsilon'' \end{bmatrix} Y^{j}_{j'' \; j'} \begin{bmatrix} l & \varepsilon \\ l'' \; l' & \varepsilon'' \; \varepsilon' \end{bmatrix} \tag{3.54}$$

for either and both of $j'' = j''' \pm 1/2$.

So we have just the two terms whose coefficients are $\hat{su}(2)_{N+1} \times \mathcal{L}_c$-primaries. The more singular term (as $z - w \to 0$) is identified with its corresponding term on the left-hand-side and so can be written as a linear combination of $\hat{su}(2)_N \times \hat{su}(2)_1$ vertex operators, and it follows that the same is true for any of its $\hat{su}(2)_{N+1} \times \mathcal{L}_c$ descendents. Since $2|j - l| \in \mathbf{Z}$, we should also identify the less singular term with some linear combination of $\hat{su}(2)_N \times \hat{su}(2)_1$ vertex operators, after appropriately subtracting away some descendents of the more singular term. In both instances, once more using the 'state-field correspondence', the decomposition of the $\hat{su}(2)_N \times \hat{su}(2)_1$ vertex operators as a linear combination of $\hat{su}(2)_{N+1} \times \mathcal{L}_c$-primaries of particular spin and conformal dimension characterizes it up to scalar multiplication as

$$\Psi^{j\pm\frac{1}{2}} \begin{bmatrix} l & \frac{1}{2}-\varepsilon \\ l'' \; l' & \varepsilon''' \; \varepsilon' \end{bmatrix} (\,.\,,w) \tag{3.55}$$

so that, up to a scalar multiple,

$$Y^{j\pm\frac{1}{2}}_{j''' \; j'} \begin{bmatrix} l & \frac{1}{2}-\varepsilon \\ l'' \; l' & \varepsilon''' \; \varepsilon' \end{bmatrix} = Z^{j\pm\frac{1}{2},\,j\pm\frac{1}{2}}_{j''' \; j'} \begin{bmatrix} l & \frac{1}{2}-\varepsilon \\ l'' \; l' & \varepsilon''' \; \varepsilon' \end{bmatrix}. \tag{3.56}$$

[2] When we come to construct a full non-chiral theory, we will want to consider, for each pair of conformal dimensions $(h, \bar{h})$, only particular linear combinations of tensor products of chiral and antichiral primaries of the corresponding spins. Braiding and the operator product expansion of pairs of such primaries should not produce new linear combinations of chiral-antichiral pairs. Here, the situation is similar but with the antichiral primary replaced by a $\hat{su}(2)_{N+1}$ primary. Observe that the branching coefficients are highly constrained by this property, which is guaranteed by the fact that these particular combinations of $\hat{su}(2)_{N+1} \times \mathcal{L}_c$ primaries can be written as $\hat{su}(2)_N \times \hat{su}(2)_1$ vertex operators.

3.3 Remark on fusion rules

We have proved the analogue of the GKO coset construction for Virasoro vertex operators of the unitary discrete series. In particular, a $\mathcal{L}_c$-primary

$$\Phi^{h}_{h'' \; h'}(z)$$

that is allowed to exist by the fusion rules established in Appendix A can be found tensored with a $\hat{su}(2)_{N+1}$ primary by projecting a suitable $\hat{su}(2)_N \times \hat{su}(2)_1$ vertex operator onto the relevant GKO summands. It is natural to ask if we can turn our arguments into a proof of the fusion rules. In this paper, fusion rules have been defined to be the sufficient *and* necessary conditions for the existence of primary fields. We now argue that we can modify our arguments so that we do not need to assume any prior knowledge of the fusion rules. Then we have a proof of sufficiency. Proof of necessity will rely on some plausible assumptions. The modifications suggest that the fusion rules for other coset theories may be proved in a similar way and, by a similar approach, other properties of the vertex operators may also follow as they will for the case at hand here.

The generating Virasoro primaries were found without any recourse to the unitary discrete series fusion rules. Now, we used the operator product expansion for Virasoro primaries in order to find the remaining primaries. This is valid if we know the fusion rules or, more precisely, that the primary fields which appear when we braid pairs of primaries are already known to exist; and not valid if we do not already know this. This is the usual obstruction to proving the (sufficiency of) the fusion rules from knowledge of the generating primaries. Here, however, this problem can be circumvented. Consider (analytic continuations of) matrix elements of (3.37), (3.41) and their expansions in powers of $z - w$, corresponding to the operator product expansion for (3.37), but not necessarily so for (3.41), though each

$$\langle\eta|\Phi^{h_{2,1}}_{h_{p''',q''} \; h_{p'',q''}}(z)\Phi^{h_{p,q}}_{h_{p'',q''} \; h_{p',q'}}(w)|\xi\rangle \tag{3.57}$$

certainly has such an expansion. This expansion corresponds to the analytic continuation of this local solution of the corresponding hypergeometric equation at the singular point $x = w/z = 0$ to the singular point $x = 1$. Now the operator product expansion of (3.37) is a sum of descendents of the primaries (3.38), and the coefficient of each descendent is independent of $l''', l', \varepsilon''', \varepsilon'$, and dependent on l'', ε'' only by a scalar factor (an entry of the fusion matrix) that is the same for each family of descendents. In particular, we can look at the case $l' = \varepsilon' = 0$, so that $l'' = l$, $\varepsilon'' = \varepsilon$, $\varepsilon''' = 1/2 - \varepsilon$; and at the action of (3.37) on the $\hat{su}(2)_N \times \hat{su}(2)_1$-vacuum vector $|0\rangle \otimes |0\rangle$. Then (3.41) gives

$$\Phi^{j}_{j\;0}(\,.\,,w)|0\rangle \otimes \left[\Phi^{h_{2,1}}_{h_{p''',q} \; h_{p,q}}(z)\Phi^{h_{p,q}}_{h_{p,q} \; 0}(w)|0\rangle\right] \tag{3.58}$$

and the operator product expansion of this pair of Virasoro primaries *can* be used because the Virasoro primaries $\Phi^{h}_{h\;0}(z)$ are known to exist already, simply by the 'state-field correspondence' for the $\hat{su}(2)$ theory. So we have, in this special case, (3.42). As before, we can pick up the coefficient on the left-hand-side corresponding to the powers of $z - w$ appearing in these two terms. But by the 'state-field correspondence', there is a unique descendent of

$$\Phi^{l\pm\frac{1}{2}}_{l\pm\frac{1}{2}\;0}(\,.\,,w) \otimes \Phi^{\frac{1}{2}-\varepsilon}_{\frac{1}{2}-\varepsilon\;0}(w) \tag{3.59}$$

whose action on the vacuum is given by

$$\Phi^{j}_{j\;0}(\,.\,,w)|0\rangle \otimes \Phi^{h_{p\pm1,q}}_{h_{p\pm1,q}\;0}(w)|0\rangle \tag{3.60}$$

so we can identify it as

$$\Psi^{j}\begin{bmatrix} l\pm\frac{1}{2} & \frac{1}{2}-\epsilon \\ l\pm\frac{1}{2}\,0 & \frac{1}{2}-\epsilon\,0 \end{bmatrix}(\,.\,,w). \tag{3.61}$$

We go back to the general case. It follows that for arbitrary $l''', l'', l', \varepsilon''', \varepsilon'', \varepsilon'$,

$$\Psi^{j}\begin{bmatrix} l\pm\frac{1}{2} & \frac{1}{2}-\epsilon \\ l''\;l' & \varepsilon'''\;\varepsilon' \end{bmatrix}(\,.\,,w) \tag{3.62}$$

is the coefficient of the same power of $z-w$ in (3.57) as for (3.61), and we have obtained (3.42) and (3.43) again, with the fusion matrix in (3.43) now understood to be a connection matrix for a hypergeometric equation, which we have to show to have non-vanishing entries. This follows from Appendix B. The same applies to the arguments for the induction in the direction $h_{p,q} \to h_{p,q\pm1}$.

In the next section, we will show that correlation functions of Virasoro primaries converge to holomorphic functions and can be written in terms of correlation functions of $\hat{su}(2)$ vertex operators. Moreover, independent of a direct analysis of the 4-point BPZ equations, pairs of Virasoro primaries braid under analytic continuation, and in particular have operator product expansions. The set of Virasoro primaries found by the coset construction obviously has the following *completeness* property: it contains the Virasoro primaries

$$\Phi^{h}_{h\,0}(z) \tag{3.63}$$

for each unitary highest weight h; and it contains enough Virasoro primaries for the braiding behaviour

$$\Phi^{h_1}_{h_1\,0}(z)\Phi^{h_2}_{0\,h_2}(w) = \sum_{h} \Phi^{h_2}_{h_1\,h}(w)\Phi^{h_1}_{h\,h_2}(z)\, C^{h_1h_1h_2h_2}_{h\,0} \tag{3.64}$$

to make sense for all unitary highest weights h_1, h_2. In fact, if we can establish that none of the braiding matrix elements in (3.64) vanish, only the entire set of Virasoro primaries has this completeness property. We conjecture that such a completeness property characterizes the fusion rules for coset theories in general. The reason is that such a complete set of primaries is expected to be sufficient data to define Connes fusion, a commutative associative (up to equivalence) product on the category of unitary positive energy representations of the symmetry algebra of the coset theory. (The W-algebra case is still unclear.) The corresponding representation ring is given by the fusion ring which we can construct abstractly from the fusion rules, as it should be.

4. Correlation functions of Virasoro primaries

We now show that the convergence of correlation functions of Virasoro primaries follows from the result for $\hat{su}(2)$ primaries and the constructions of the previous section. In either case, convergence for general vertex operators follows from convergence for primary fields. The M-point function

$$\langle \Psi_M(\,.\,,z_M)\cdots\Psi_1(\,.\,,z_1)\rangle \tag{4.1}$$

of $\hat{su}(2)_N \times \hat{su}(2)_1$ vertex operators is known [2] to converge for $|z_M| > \ldots > |z_1|$, where

$$\Psi_i(\,.\,,z_i) \;=\; \Psi^{j_i}\begin{bmatrix} l_i & \epsilon_i \\ l'_{i+1}\;l'_i & \varepsilon'_{i+1}\;\varepsilon'_i \end{bmatrix}(\,.\,,z_i) \qquad (i=1,\ldots,M) \tag{4.2}$$

with $l'_1 = l'_{M+1} = \varepsilon'_1 = \varepsilon'_{M+1} = 0$. Now this is a linear combination of the terms

$$\langle \Phi^{j_M}_{j'_{M+1}\,j'_M}(\,.\,,z_M)\cdots\Phi^{j_1}_{j'_2\,j'_1}(\,.\,,z_1)\rangle\; \langle \Phi^{h_{p_M,q_M}}_{h_{p'_{M+1},q'_{M+1}}\;h_{p'_M,q'_M}}(z_M)\cdots\Phi^{h_{p_1,q_1}}_{h_{p'_2,q'_2}\;h_{p'_1,q'_1}}(z_1)\rangle \tag{4.3}$$

where $j'_1 = j'_{M+1} = 0$, and the j'_i range over the obvious values. The first factor is a M-point function of $\hat{su}(2)_{N+1}$ primaries and thus a convergent series. But by projecting (4.1) onto an appropriate irreducible $su(2)$-submodule of the tensor product

$$V_{j_M}\otimes\cdots\otimes V_{j_1}, \tag{4.4}$$

we see that each term in the sum converges separately. We do this by first projecting $V_{j_2}\otimes V_{j_1}$ onto the irreducible component isomorphic to $V_{j'_3}$ for a choice of j'_3, then sequentially projecting each $V_{j_i}\otimes V_{j'_i}$ onto $V_{j'_{i+1}}$ for a choice of j'_{i+1}, for $i=3,\ldots,M-1$. By inspection of the operator product expansion of a pair of $\hat{su}(2)$ primaries, we see that this obviously works, that only one term in the sum of terms (4.3) survives the projection. This is a notable property. It follows that the second factor, a M-point function of Virasoro primaries, is also convergent. As an immediate corollary of the proof, we have the expression of a correlation function of Virasoro primaries in terms of correlation functions of the corresponding $\hat{su}(2)$ theories. Indeed, each is a sum of quotients of $\hat{su}(2)_N \times \hat{su}(2)_1$ correlation functions by $\hat{su}(2)_{N+1}$ correlation functions, as befits a 'coset theory'.

Actually, it is sufficient to know that correlation functions of generating Virasoro primaries converge to obtain that those of arbitrary Virasoro primaries converge. For 4-point functions involving at least one generating primary have already been shown to be convergent, so that the operator product expansion of a generating primary with an arbitrary primary is understood. Beginning with a correlation function of generating primaries, we successively replace a pair of primaries, one a generating primary, with a primary field appearing in their operator product expansion. The result is also a convergent correlation function. But we easily show that we can obtain an arbitrary correlation function in this way, using the fact that the braiding matrices involving a generating primary have no vanishing entries. Thus is the term 'generating primary field' also justified.

It follows from Appendix A that pairs of Virasoro primaries braid under analytic continuation but it is better, particularly when considering other coset theories, that we do not have to analyze systems of partial differential equations to prove this. We now show that this property is inherited, via the coset construction, from the same for $\hat{su}(2)$ primaries. This is straightforward to establish by considering the braiding

$$\Psi^{j_1}\begin{bmatrix} l_1 & \epsilon_1 \\ l'''\;l'' & \varepsilon'''\;\varepsilon'' \end{bmatrix}(\,.\,,z)\;\Psi^{j_2}\begin{bmatrix} l_2 & \epsilon_2 \\ l''\;l' & \varepsilon''\;\varepsilon' \end{bmatrix}(\,.\,,w) =$$
$$\sum_{l,\varepsilon} C^{l'''l_1l_2l'}_{l''}\, C^{\varepsilon'''\epsilon_1\epsilon_2\varepsilon'}_{\varepsilon\varepsilon''}\;\Psi^{j_2}\begin{bmatrix} l_2 & \epsilon_2 \\ l'''\;l & \varepsilon'''\;\varepsilon \end{bmatrix}(\,.\,,w)\;\Psi^{j_1}\begin{bmatrix} l_1 & \epsilon_1 \\ l\;l' & \varepsilon\;\varepsilon' \end{bmatrix}(\,.\,,z) \tag{4.5}$$

in terms of the $\hat{su}(2)_{N+1}\times\mathcal{L}_c$ components. To see this, observe that if we *project* the $su(2)$-module $V_{j_1}\otimes V_{j_2}$ onto each of its irreducible submodules in turn, the (matrix elements of the) operators

$$\Phi^{j_1}_{j'''\,j''}(\,.\,,z)\Phi^{j_2}_{j''\,j'}(\,.\,,w) \tag{4.6}$$

as j'' varies, are pairwise linearly dependent, and analytically continue to an operator proportional to

$$\Phi^{j_2}_{j'''\,j}(\,.\,,w)\Phi^{j_1}_{j\,j'}(\,.\,,z) \tag{4.7}$$

for any of the values of j, these operators being likewise pairwise linearly dependent. The number of operators (4.6) indexed by j'' is equal to the number of operators (4.7) indexed by j, and this is again the number of different projections onto an irreducible submodule

of $V_{j_1} \otimes V_{j_2}$ such that (4.6) does not vanish identically. This last assertion follows from considering the operator product expansion of (4.6), i.e. the analytic continuation of (4.6) from $|x| < 1$ to $|(1-x)/x| < 1$, where $x = w/z$. We are, of course, just performing a change of basis for the solution space of a system of KZ equations.

The braiding (4.5) implies, for each j''', j', the equality of

$$\sum_{j''} Y^{j_1}_{j'''\,j''} \begin{bmatrix} l_1 & \epsilon_1 \\ l''' \; l'' & \epsilon''' \; \epsilon'' \end{bmatrix} Y^{j_2}_{j''\,j'} \begin{bmatrix} l_2 & \epsilon_2 \\ l'' \; l' & \epsilon'' \; \epsilon' \end{bmatrix}$$

$$\Phi^{j_1}_{j'''\,j''}(\,.\,,z)\Phi^{j_2}_{j''\,j'}(\,.\,,w) \otimes \Phi^{h_{p_1,q_1}}_{h_{p'''},q'''\;h_{p''},q''}(z)\Phi^{h_{p_2,q_2}}_{h_{p''},q''\;h_{p'},q'}(w) \qquad (4.8)$$

with

$$\sum_{l,\epsilon,j} C^{l'''l_1l_2l'}_{ll''} C^{\epsilon'''\epsilon_1\epsilon_2\epsilon'}_{\epsilon\epsilon''} Y^{j_2}_{j'''\,j} \begin{bmatrix} l_2 & \epsilon_2 \\ l'''\; l & \epsilon'''\; \epsilon \end{bmatrix} Y^{j_1}_{j\,j'} \begin{bmatrix} l_1 & \epsilon_1 \\ l\; l' & \epsilon\; \epsilon' \end{bmatrix}$$

$$\Phi^{j_2}_{j'''\,j}(\,.\,,w)\Phi^{j_1}_{j\,j'}(\,.\,,z) \otimes \Phi^{h_{p_2,q_2}}_{h_{p'''},q'''\;h_{p,q}}(w)\Phi^{h_{p_1,q_1}}_{h_{p,q}\;h_{p'},q'}(z) \qquad (4.9)$$

in the sense of analytic continuation. By performing each of the projections onto the irreducible components of $V_{j_1} \otimes V_{j_2}$ in (4.8) and (4.9), and factoring out the $\hat{su}(2)_{N+1}$ contribution, we obtain a system of equations for the braiding of particular linear combinations of the Virasoro primaries (4.6) which, by a change of basis, is of the form

$$\Phi^{h_{p_1,q_1}}_{h_{p'''},q'''\;h_{p''},q''}(z)\Phi^{h_{p_2,q_2}}_{h_{p''},q''\;h_{p'},q'}(w) = \sum_{h_{p,q}} \Phi^{h_{p_2,q_2}}_{h_{p'''},q'''\;h_{p,q}}(w)\Phi^{h_{p_1,q_1}}_{h_{p,q}\;h_{p'},q'}(z)\, C^{h_{p'''},q'''\,h_{p_1,q_1}\,h_{p_2,q_2}\,h_{p'},q'}_{h_{p,q}\,h_{p''},q''} \qquad (4.10)$$

for some braiding matrix, in principle computable from the braiding of $\hat{su}(2)$ primaries and the branching coefficients.

5. Comments

We have proved an analogue of the GKO coset construction for Virasoro primary fields. This lets us deduce properties of vertex operators in the unitary discrete series from the properties of vertex operators in the $su(2)$-WZW theories, which are easier to study. This is our guiding principle. Here, we have obtained as a corollary the fundamental property of convergence of the Virasoro correlation functions without the need to analyse differential equations that are more complex than a hypergeometric equation. The construction also gives a proof of the Virasoro fusion rules under the assumption that a completeness property characterizes them. Properties such as the braiding behaviour of primaries, and therefore also their operator product expansions, can likewise be obtained from this construction. We hope to give further applications in a future paper.

The considerations here also apply to the unitary discrete series for the super-Virasoro algebra. In particular, we can obtain the fusion rules (with the same caveat) in an elementary way. It also gives us a simple construction of the 'twist-fields' that intertwine the Neveu-Schwarz and Ramond sectors, which are difficult to define algebraically. We will address these questions in a forthcoming paper. More generally, we believe the approach taken here will be useful in studying analogues of primary fields in coset theories, say for which the extended algebra is a W-algebra. The simplest of such analogues are the 'twist-fields' of the super-Virasoro algebra. We expect that the utility of the approach taken here will increase as we proceed to study the less understood coset theories.

Acknowledgements

This work was supported by the Overseas Research Students Awards Scheme and a studentship from Trinity College, Cambridge. I am very grateful to P. Goddard, A. Kent and A. Wassermann for guidance, useful conversations and comments.

Appendix A. Basic algebraic theory for the unitary discrete series

We present a summary of the construction and basic algebraic properties of vertex operators of the unitary discrete series, following Tsuchiya-Kanie [2], followed by a short discussion of the solution space of a system of BPZ and projective invariance equations.

We will be concerned with unitary positive energy modules for the Virasoro algebra, where unitarity is with respect to a contravariant inner product [9]. An irreducible highest weight module $H(c,h)$ for the Virasoro algebra is the quotient of a Verma module $M(c,h)$ by its maximal submodule $J(c,h)$. Of these, the unitary ones are the irreducible components of the unitary positive energy modules. For central charge $c < 1$, they constitute a discrete series given by [1,4]:

$$\begin{aligned} c &= 1 - \frac{6}{m(m+1)}, \quad m = 3,4,\ldots \\ h &= \frac{[p(m+1) - qm]^2 - 1}{4m(m+1)}, \quad p = 1,\ldots,m-1; \;\; q = 1,\ldots,p. \end{aligned} \qquad (A.1)$$

We will actually let $1 \le p \le m-1$, $1 \le q \le m$, and take appropriate account of the double counting given by $h_{p,q}(c) = h_{m-p,m+1-q}(c)$. It is known [10,11] that the maximal submodule $J(c, h_{p,q}(c))$ is generated by a pair of singular vectors $O_{p,q}(c)|h_{p,q}(c)\rangle$, $O_{m-p,m+1-q}(c)|h_{p,q}(c)\rangle$, respectively at levels pq, $(m-p)(m+1-q)$.

Given some Virasoro modules, we can try to obtain new modules from them. We can take direct sums and tensor products, but the central charge is additive under the latter operation whilst it is natural to try to remain within a class of modules with the same central charge. We might hope to take the tensor product of a unitary positive energy module with another which has vanishing central charge, but there is only the trivial module [12]. On the other hand, we have the spaces of 'densities' $V_{\alpha,\beta}$ [9] (irreducible if $\alpha \notin \mathbf{Z}$ or $\beta \neq 0, 1$)

$$\begin{aligned} \nu_k &= z^{k+\alpha}(dz)^\beta \quad (k \in \mathbf{Z}) \\ L(n)\,\nu_k &= -(k + \alpha + \beta + \beta n)\,\nu_{n+k} \end{aligned} \qquad (A.2)$$

which are modules with vanishing central charge, for which $L(0)$ is diagonal with finite-dimensional integrally-graded eigenspaces, but whose spectrum is unbounded above and below. Then we can try to find a space of 'densities' whose tensor product with a unitary highest weight module possesses a unitary highest weight quotient, i.e. construct a module of linear operators from a unitary highest weight module to another which transforms as a 'density': a Virasoro primary field. These constructions are labelled by a triplet of numbers α, β and the highest weight (c, h) of the initial unitary highest weight module. With no loss of generality, α can be chosen such that the new unitary highest weight module has highest weight $(c, h - \alpha - \beta)$. We are led to the determination of the *fusion rules* of the unitary discrete series of the Virasoro algebra: for each c, the set of triplets of numbers

$$\begin{pmatrix} h_2 \\ h_3\; h_1 \end{pmatrix} \qquad (A.3)$$

called *allowed vertices*, for which

$$H(c,h_3) \cong V_{\alpha,\beta} \otimes H(c,h_1)/\sim, \quad \alpha = h_2 + h_1 - h_3 - 1,\ \beta = 1 - h_2 \tag{A.4}$$

as an isomorphism of highest weight modules, where $H(c,h_1)$, $H(c,h_3)$ are unitary highest weight modules and the quotient is by the maximal proper submodule containing the strictly negative eigenspaces of $L(0) - h_3$. That is, we have a module of operators

$$\begin{gathered}\Phi(k)\colon H(c,h_1) \to H(c,h_3) \quad (k \in \mathbb{Z}) \\ [L(n), \Phi(k)] = -(n + k + h_1 - h_3 - nh_2)\,\Phi(n+k)\end{gathered} \tag{A.5}$$

or, equivalently,

$$\begin{gathered}\Phi(z) = \sum_{k\in\mathbb{Z}} \Phi(k)\, z^{-k-(h_2+h_1-h_3)} \\ [L(n), \Phi(z)] = z^{n+1}\frac{d\Phi(z)}{dz} + h_2\,(n+1)\,z^n\,\Phi(z).\end{gathered} \tag{A.6}$$

It is easy to see that $\Phi(z)$ is uniquely determined by these relations and the normalization

$$\langle h_3|\,\Phi(z)\,|h_1\rangle = z^{-(h_2+h_1-h_3)} \tag{A.7}$$

where $|h_i\rangle \in H(c,h_i)$ $(i = 1,3)$ are highest weight vectors of unit norm. We call $\Phi(z)$ the *Virasoro primary field* corresponding to the *vertex* (A.3), and h_2 its *conformal dimension*, and write it as

$$\Phi^{h_2}_{h_3\ h_1}(z).$$

Its conjugate is defined to be

$$\Phi^*(z) = \left[\bar{z}^{-2h_2}\Phi(\frac{1}{\bar{z}})\right]^\dagger \tag{A.8}$$

and is easily seen to be $\Phi^{h_2}_{h_1\ h_3}(z)$. Equivalently, $\Phi(z)$ is a sesquilinear form on $H(c,h_3) \times H(c,h_1)$, and simultaneously a degenerate form $\phi(z)$ on $M(c,h_3) \times M(c,h_1)$ that factors to the quotient space:

$$\phi(\xi, \eta; z) = \langle\tilde{\xi}|\,\Phi(z)\,|\tilde{\eta}\rangle \tag{A.9}$$

where $\tilde{\xi} = \xi + J(c,h_3)$, $\tilde{\eta} = \eta + J(c,h_1)$.

Now, for any pair of unitary highest weights (c,h_1), (c,h_3) and $h_2 \in \mathbb{C}$, there is a unique sesquilinear form $\phi(z)$ on $M(c,h_3) \times M(c,h_1)$ such that

$$\begin{gathered}\phi(|h_3\rangle, |h_1\rangle; z) = z^{-(h_2+h_1-h_3)} \\ \phi(L(-n)\xi, \eta; z) - \phi(\xi, L(n)\eta; z) = \left[z^{n+1}\frac{d}{dz} + h_2\,(n+1)\,z^n\right]\phi(\xi, \eta; z)\end{gathered} \tag{A.10}$$

for all $\xi \in M(c,h_3)$, $\eta \in M(c,h_1)$, $n \in \mathbb{Z}$. Here, by abuse of notation, $|h_i\rangle \in M(c,h_i)$ $(i = 1,3)$ are highest weight vectors of unit norm. Then $\phi(z)$ defines a primary field if and only if it factors to the quotient $H(c,h_3) \times H(c,h_1)$. Let $h_i = h_{p_i,q_i}$ $(i = 1,3)$. We easily see that it factors if and only if

$$\begin{gathered}\phi(O_{p_3,q_3}|h_3\rangle, |h_1\rangle; z) = 0 \\ \phi(O_{m-p_3,m+1-q_3}|h_3\rangle, |h_1\rangle; z) = 0\end{gathered} \tag{A.11}$$

and

$$\begin{gathered}\phi(|h_3\rangle, O_{p_1,q_1}|h_1\rangle; z) = 0 \\ \phi(|h_3\rangle, O_{m-p_1,m+1-q_1}|h_1\rangle; z) = 0.\end{gathered} \tag{A.12}$$

Using (A.10), each of these reduce to a polynomial equation in h_2. The left-hand-side of the equation corresponding to $O_{p,q}|h_{p,q}\rangle$ is a differential operator of order pq acting on $z^{-(h_2+h_1-h_3)}$, giving a polynomial of degree pq in h_2.

We now appeal to a result [13–15] on the BPZ equations for a 3-point function. Let

$$O_{p_1,q_1} = \sum_{N=1}^{p_1q_1} \sum_{\substack{i_1,\ldots,i_N>0 \\ i_1+\cdots+i_N=p_1q_1}} a^{p_1,q_1}_{i_1\ldots i_N}\, L(-i_1)\cdots L(-i_N) \tag{A.13}$$

and $\hat{O}^{3;1}_{p_1,q_1}$ be the same with each $L(m)$ replaced by

$$\hat{L}^{3;1}(m) = -\sum_{j=2,3} z_{j1}^{m+1}\frac{\partial}{\partial z_j} + h_j\,(m+1)\,z_{j1}^m. \tag{A.14}$$

Then

$$\hat{O}^{3;1}_{p_1,q_1}\, z_{21}^{-\Delta_{21}} z_{32}^{-\Delta_{32}} z_{31}^{-\Delta_{31}} = \left(\frac{z_{32}}{z_{12}z_{31}}\right)^{p_1q_1} g_{p_1,q_1}(h_2,h_3)\, z_{21}^{-\Delta_{21}} z_{32}^{-\Delta_{32}} z_{31}^{-\Delta_{31}} \tag{A.15}$$

where $\Delta_{21} = h_2 + h_1 - h_3$ etc., and $g_{p_1,q_1}(h_2,h_3) = 0$ is a polynomial in h_2 of degree p_1q_1 and the roots are given by

$$\begin{gathered}\frac{[p(m+1) - qm]^2 - 1}{4m(m+1)}, \quad p = p_3 - p_1 + 1, p_3 - p_1 + 3, \ldots, p_3 + p_1 - 1 \\ q = q_3 - q_1 + 1, q_3 - q_1 + 3, \ldots, q_3 + q_1 - 1.\end{gathered} \tag{A.16}$$

It is noteworthy that *both* h_2 and h_{p_3,q_3} can be arbitrary complex numbers at this stage, but that the pair of equations

$$g_{p_1,q_1}(h_2,h_3) = 0, \quad g_{m-p_1,m+1-q_1}(h_2,h_3) = 0 \tag{A.17}$$

force h_{p_3,q_3} to be a unitary highest weight. It is now easy to see that the pairs of equations (A.11) and (A.12) are equivalent and that the solutions are given by

$$\begin{aligned}&h_2 = h_{p,q}, \\ &p = |p_3 - p_1| + 1, |p_3 - p_1| + 3, \ldots, \min(p_3 + p_1 - 1, 2m - p_3 - p_1 - 1) \\ &q = |q_3 - q_1| + 1, |q_3 - q_1| + 3, \ldots, \min(q_3 + q_1 - 1, 2(m+1) - q_3 - q_1 - 1)\end{aligned} \tag{A.18}$$

and we have the well-known fusion rules for the unitary discrete series. In particular, h_2 is a unitary highest weight. It is clear that the fusion rules are symmetric: a vertex (A.3) is allowed if and only if the vertices obtained by permuting h_1, h_2, h_3 are also allowed. We can define a commutative ring with identity, the *fusion ring*, which, as a $\mathbb{Z}$-module, has a basis given by the set of unitary highest weights (for some fixed central charge), and with multiplication given by

$$h_ih_j = \sum_k c_{ijk}h_k \tag{A.19}$$

where $c_{ijk} = 1$ if

$$\begin{pmatrix} & h_j & \\ h_i & & h_k \end{pmatrix}$$

is an allowed vertex, and vanishes otherwise. Commutativity is immediate and associativity can be checked directly but also follows from the fact that the 4-point functions span the solution space of a system of BPZ and projective invariance equations for a 4-point function (see below). The fusion ring will be the representation ring (for unitary positive energy representations at a fixed central charge) of the Virasoro algebra, when fusion is defined.

Let

$$T(z) = \sum_{n\in\mathbf{Z}} L(n)\, z^{-n-2}. \tag{A.20}$$

Recall that a unitary highest weight module $H(c,h)$ decomposes as a direct sum of finite-dimensional positive eigenspaces of $L(0)$:

$$H(c,h) = \bigoplus_{r\in h+\mathbf{N}_0} H(c,h;r) \tag{A.21}$$

and the inner product identifies its algebraic dual as

$$\hat{H}(c,h) = \prod_{r\in h+\mathbf{N}_0} H(c,h;r). \tag{A.22}$$

It is useful to think of primary fields and $T(z)$ as functions on $\mathbf{C}^*$ taking values in linear operators from the unitary modules to their duals, holomorphic in the sense that matrix elements are holomorphic (say, in $z \in \mathbf{C}^*$, $-\pi < \arg(z) < \pi$). The chief interest is the composition of these 'fields', by which we mean for a sequence $A_i(z_i) = \sum_{n_i} A_i(n_i) z_i^{-n_i-\Delta_i}$ $(i = 1,\ldots,N)$, if the formal series

$$\sum_{n_N,\ldots,n_1} \langle\eta|A_N(n_N)\cdots A_1(n_1)|\xi\rangle\, z_N^{-n_N-\Delta_N}\cdots z_1^{-n_1-\Delta_1} \tag{A.23}$$

is convergent on some domain in $|z_N| > \cdots > |z_1|$, for all η, ξ. A sequence of $T(z)$'s and a Virasoro primary field is composable, converging on $|z_N| > \ldots > |z_1|$, and can be analytically continued to a holomorphic function, single-valued, on $\{(z_N,\ldots,z_1) \in \mathbf{C}^{*N}\colon z_i \neq z_j, i \neq j\}$. Moreover, any permutation of the fields yields the same function. In particular, in the sense of analytic continuation

$$\begin{aligned} T(z)T(w) &= T(w)T(z) \\ T(z)\Phi(w) &= \Phi(w)T(z) \end{aligned} \tag{A.24}$$

and for $|w| > |z-w|$, we have convergent expansions in powers of $z-w$:

$$\begin{aligned} T(z)T(w) &= \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{1}{z-w}\frac{dT(w)}{dw} + \ldots \\ T(z)\Phi(w) &= \frac{h\,\Phi(w)}{(z-w)^2} + \frac{\Phi(w)}{z-w} + \ldots. \end{aligned} \tag{A.25}$$

For each primary field $\Phi(z)$ corresponding to a vertex, of conformal dimension h, we can define an irreducible highest weight module with highest weight h spanned by

$$\hat{L}(m_N)\cdots\hat{L}(m_1)\Phi(z), \tag{A.26}$$

which are, like the primary field they descend from, holomorphic functions on $\mathbf{C} - (-\infty, 0]$ taking values in sesquilinear forms on products of unitary highest weight modules; given by

$$\frac{1}{(2\pi i)^N}\oint_{C_N}\cdots\oint_{C_1} dw_N\cdots dw_1\,(w_N - z)^{m_N+1}\cdots(w_1 - z)^{m_1+1}\,T(w_N)\cdots T(w_1)\Phi(z) \tag{A.27}$$

where each contour encircles z and C_i goes around C_{i-1}. The action of the Virasoro algebra on this space is the obvious one which we can check by the usual 'contour trick' or directly using

$$\hat{L}(m)\Psi(z) = \left[\sum_{r=0}^{\infty} a_r(-z)^r L(m-r)\right]\Psi(z) - \Psi(z)\left[\sum_{r=0}^{\infty} a_r(-z)^{m+1-r}L(r-1)\right] \tag{A.28}$$

where $(1+x)^{m+1} = \sum_{r=0}^{\infty} a_r x^r$, $|x| < 1$. Irreducibility follows from (A.15). That is, for $h = h_{p,q}$, we have

$$\begin{aligned} \hat{O}_{p,q}\Phi(z) &= 0 \\ \hat{O}_{m-p,m+1-q}\Phi(z) &= 0 \end{aligned} \tag{A.29}$$

where, of course, each $\hat{O}_{p,q}$ is $O_{p,q}$ with the $L(m)$'s replaced by $\hat{L}(m)$'s. Elements of these modules of 'field operators' are the *chiral vertex operators*. For each $\psi \in H(c,h)$, we write the corresponding descendent of $\Phi^{h}_{h'',h'}(z)$ as

$$\Phi^{h}_{h'',h'}(\psi, z).$$

An N-point function of Virasoro primaries

$$\langle\Phi_N(z_N)\cdots\Phi_1(z_1)\rangle \tag{A.30}$$

where $\Phi_i(z_i) = \Phi^{h_i}_{k_{i+1},k_i}(z_i)$, $h_i = h_{p_i,q_i}$ $(i = 1,\ldots,N)$ is *a priori* a formal series:

$$\sum_{n_N,\ldots,n_1} \langle 0|\Phi_N(n_N)\cdots\Phi_1(n_1)|0\rangle\, z_N^{-n_N-\Delta_N}\cdots z_1^{-n_1-\Delta_1}. \tag{A.31}$$

From (A.29), we obtain for $|z_N| > \cdots > |z_1|$ the $2N$ BPZ equations [5]:

$$\begin{aligned} \hat{O}^{N;i}_{p_i,q_i}\,\langle\Phi_N(z_N)\cdots\Phi_1(z_1)\rangle &= 0 \\ \hat{O}^{N;i}_{m-p_i,m+1-q_i}\,\langle\Phi_N(z_N)\cdots\Phi_1(z_1)\rangle &= 0 \qquad (i = 1,\ldots,N) \end{aligned} \tag{A.32}$$

where $\hat{O}^{N;i}_{p,q}$ is likewise $O_{p,q}$ with each $L(m)$ replaced by

$$\hat{L}^{N;i}(m) = -\sum_{\substack{j=1\\ j\neq i}}^{N} z_{ji}^{m+1}\frac{\partial}{\partial z_j} + h_j(m+1)\,z_{ji}^{m}. \tag{A.33}$$

The coefficients in the differential operators have convergent series expansions and the equations are understood to be those of formal series. In addition, we have the 3 projective invariance equations

$$\sum_{j=1}^{N}\left[z_j^{m+1}\frac{\partial}{\partial z_j} + h_j(m+1)\,z_j^{m}\right]\langle\Phi_N(z_N)\cdots\Phi_1(z_1)\rangle = 0 \quad (m = 0, \pm 1). \tag{A.34}$$

Thus, corresponding to a sequence $h_N, \ldots, h_1$ of unitary highest weights is a system of $2N+3$ differential equations. A corollary of the results in this paper is that the corresponding N-point functions converge. We would like to know that they also span the solution space of this system of equations (see also [16]). It will emerge that any 2 of the N pairs of BPZ equations can be disregarded, as is already apparent for $N = 2, 3$.

In the rest of this section, we will suppose that we have already determined that the N-point functions are convergent. We now show that there are no other regular non-logarithmic solutions. The change to the variables of Tsuchiya-Kanie [2]

$$\begin{aligned} w_i &= z_i/z_{i+1} \quad (i = 1, \ldots, N-1) \\ w_N &= z_N \end{aligned} \tag{A.35}$$

is an analytic isomorphism from $\{(z_N, \ldots, z_1) \in \mathbf{C}^{*N}\colon |z_N| > \cdots > |z_1|\}$ to $\{(w_N, \ldots, w_1) \in \mathbf{C}^{*N}\colon |w_i| < 1, i = 1, \ldots, N-1\}$. We consider the system of $2N+3$ equations corresponding to the sequence of unitary highest weights $h_N, \ldots, h_1$, $h_i = h_{p_i, q_i}$. A regular non-logarithmic solution at the origin is a convergent series solution:

$$w_N^{\alpha_N} \cdots w_1^{\alpha_1} \sum_{n_{N-1}, \ldots, n_1 \geq 0} c_{n_{N-1} \cdots n_1} \, w_{N-1}^{n_{N-1}} \cdots w_1^{n_1} \tag{A.36}$$

for some $\alpha_i \in \mathbf{C}$, with $\alpha_N = -\sum_{i=1}^N h_i$, $\alpha_{N-1} = \alpha_N + 2h_N$, and $\alpha_1 = 0$. In terms of the z-variables,

$$z_N^{-2h_N} \prod_{i=2}^{N-1} z_i^{-(h_i + k_i - k_{i+1})} \sum_{n_{N-1}, \ldots, n_1 \geq 0} c_{n_{N-1} \cdots n_1} \, z_N^{-n_N} \cdots z_i^{n_i - n_{i-1}} \cdots z_1^{n_1} \tag{A.37}$$

where $k_N = h_N$, $k_2 = h_1$. Denote this solution by $f(z_N, \ldots, z_1)$, analytically continuing it when necessary, and observe that its leading term as $z_2 - z_1 \to 0$ has behaviour $\sim (z_2 - z_1)^{-(h_2 + h_1 - k_3)}$. Let

$$g(z_N, \ldots, z_3, z_1) = \frac{1}{2\pi i} \oint_C dz_2 \, (z_2 - z_1)^{h_2 + h_1 - k_3 - 1} f(z_N, \ldots, z_1) \tag{A.38}$$

be the coefficient of this term. Observe that (A.15) implies that k_3 is a unitary highest weight such that

$$\begin{pmatrix} h_2 \\ k_3 \; h_1 \end{pmatrix} \tag{A.39}$$

is an allowed vertex. We now show that $g(z_N, \ldots, z_3, z_1)$ is a solution of the $2(N-1)+3$ equations corresponding to $h_N, \ldots, h_3, k_3$. The projective equations are easily seen to be satisfied. It is sufficient to show that it also satisfies the $N-3$ pairs of BPZ equations corresponding to $h_N, \ldots, h_4$. Then, by induction on N, we are done, since the case $N = 4$ is clear.

Observe that, for $|z_N| > \cdots > |z_1|$,

$$f(z_N, \ldots, z_1) = \langle 0 | \Phi_{0\,h_N}^{h_N}(z_N) F_N(z_{N-1}, \ldots, z_2) \Phi_{h_1\,0}^{h_1}(z_1) | 0 \rangle \tag{A.40}$$

where $F_N(z_{N-1}, \ldots, z_2)$ is a holomorphic function in its variables taking values in sesquilinear forms on $H(c, h_N) \times H(c, h_1)$; defined by

$$\langle h_N | F_N(z_{N-1}, \ldots, z_2) | h_1 \rangle = \lim_{z_1 \to 0} \lim_{z_N \to \infty} z_N^{2h_N} f(z_N, \ldots, z_1) \tag{A.41}$$

and

$$[L(m), F_N(z_{N-1}, \ldots, z_2)] = \left[\sum_{i=2}^{N-1} z_i^{m+1} \frac{\partial}{\partial z_i} + h_i (m+1) z_i^m \right] F_N(z_{N-1}, \ldots, z_2) \tag{A.42}$$

for all $m \in \mathbf{Z}$. This is well-defined by the BPZ equations corresponding to h_N and h_1. From

$$\frac{1}{2\pi i} \oint_C dz_2 \, z_2^{h_2 + h_1 - k_3 - 1} \langle h_N | O_{p_N, q_N}^\dagger F_N(z_{N-1}, \ldots, z_2) | h_1 \rangle = 0 \tag{A.43}$$

we have

$$\hat{O}_{p_N, q_N}^\dagger \left[\lim_{z_1 \to 0} \lim_{z_N \to \infty} z_N^{2h_N} \, g(z_N, \ldots, z_3, z_1) \right] = 0 \tag{A.44}$$

where $\hat{O}_{p,q}^\dagger$ is $O_{p,q}^\dagger$ with each $L(m)$ replaced by

$$\sum_{i=3}^{N-1} z_i^{m+1} \frac{\partial}{\partial z_i} + h_i (m+1) z_i^m. \tag{A.45}$$

But, by the projective equations, this is the statement that $g(z_N, \ldots, z_3, z_1)$ satifies the BPZ equation corresponding to $O_{p_N, q_N} | h_N \rangle$. The same arguments lead to the BPZ equation corresponding to $O_{m - p_N, m+1-q_N} | h_N \rangle$.

Analytically continuing the functions $f(z_N, \ldots, z_1)$ and $g(z_N, \ldots, z_3, z_1)$ to the domain $|z_j| > |z_N| > \ldots > |z_1|$, for $j = 4, \ldots, N-1$, we can repeat the construction above to obtain $F_j(z_N, \ldots, \hat{z}_j, \ldots, z_2)$, holomorphic and taking values in forms on $H(c, h_j) \times H(c, h_1)$. The precise paths for analytic continuation are unimportant, and the leading behaviour in $z_2 - z_1 \to 0$ is unchanged. We thus obtain that $g(z_N, \ldots, z_1)$ satisfies the BPZ equations corresponding to h_j in the corresponding domain and thus, by analytic continuation along the reverse path, in the original domain as well. So we have shown that the N-point functions span the regular non-logarithmic solutions of the BPZ and projective invariance equations. To show that all solutions are regular at the origin $(w_N, \ldots, w_1) = 0$ and are non-logarithmic is a little more involved, unless $N \leq 4$.

For $N = 4$, the projective invariance equations plus one of the BPZ equations can be shown to be an integrable Pfaffian system with regular singularities of the type considered in Appendix C. Then all the solutions are regular at the origin and it remains to show that none are logarithmic. Since $N = 4$, this is the problem of an ordinary differential equation with a regular singularity at $x = 0$. Recall that logarithmic solutions occur only if the indicial equation has repeated solutions, whence they must occur; or if it has solutions differing by integers, whence they can but need not appear. In the latter case, we can keep track of the possibly logarithmic solutions by keeping track of the solutions to the indicial equations, since they are all distinct. To find the indicial equation, consider a formal solution

$$\prod_{\substack{i,j=1 \\ i<j}}^{4} z_{ij}^{-\gamma_{ij}} \, x^s (\log x)^r \, [1 + \ldots], \quad x = \frac{z_{12} z_{34}}{z_{13} z_{24}} \tag{A.46}$$

where $\sum_{j, i<j} \gamma_{ij} + \sum_{j, j<i} \gamma_{ji} = 2h_i$ $(i = 1, \ldots, 4)$. Let $|z_4| > \cdots > |z_1|$, and the leading behavior as $z_{12} \to 0$ be $\sim (\log z_{12})^r z_{12}^{-(h_1 + h_2 - k_3)}$. Then the leading behaviour as $z_{34} \to$

0 is $\sim (\log z_{34})^r z_{34}^{-(h_3+h_4-k_3)}$. It follows easily from (A.15) that the indicial equations corresponding to the O_{p_1,q_1} and $O_{m-p_1,m+1-q_1}$ equations are respectively

$$\begin{aligned} g_{p_1,q_1}(h_2,k_3) &= 0 \\ g_{m-p_1,m+1-q_1}(h_2,k_3) &= 0 \end{aligned} \tag{A.47}$$

and corresponding to the O_{p_4,q_4} and $O_{m-p_4,m+1-q_4}$ equations are respectively

$$\begin{aligned} g_{p_4,q_4}(h_3,k_3) &= 0 \\ g_{m-p_4,m+1-q_4}(h_3,k_3) &= 0. \end{aligned} \tag{A.48}$$

The common solutions of the indicial equations correspond precisely to the 4-point function solutions. Hence there are no logarithmic solutions. We do not have a proof for $N > 4$, but the general case is less important for the following reason.

The fundamental corollary is the braiding relation:

$$\langle\eta|\Phi^{h_3}_{h_4\,h}(z)\Phi^{h_2}_{h\,h_1}(w)|\xi\rangle = \sum_{h'} \langle\eta|\Phi^{h_2}_{h_4\,h'}(w)\Phi^{h_3}_{h'\,h_1}(z)|\xi\rangle\, C^{h_4h_3h_2h_1}_{h'h} \tag{A.49}$$

in the sense of analytic continuation, for all η, ξ. The braiding matrix $(C^{h_4h_3h_2h_1}_{h'h})_{h'h}$, which depends on the class of path chosen, is necessarily square, with h, h' ranging over values such that the respective vertices appearing above are allowed. It is easy to see that (A.49) holds for arbitrary η, ξ if it holds for highest weight vectors. We can also see that the same relation still holds if each of the primaries above is replaced with one of its descendents. It follows from the braiding relations that N-point functions, defined in one domain, are always analytically continued to other N-point functions, defined in another domain. If the BPZ equations did have further solutions, the space of solutions is nevertheless 'reducible' in this sense.

It is easy to show that for each vertex operator $\Psi(z)$,

$$[L(-1), \Psi(z)] = \frac{d\Psi(z)}{dz} \tag{A.50}$$

from which follows for $|z| > |w|$:

$$\Psi(z)e^{wL(-1)} = e^{wL(-1)}\Psi(z-w) \tag{A.51}$$

and

$$\Phi^{h}_{h\,0}(\psi, z)|0\rangle = e^{zL(-1)}|\psi\rangle. \tag{A.52}$$

Then it is straightforward to show that we have for $|w| > |z-w|$:

$$\Phi^{h_3}_{h_4\,h}(z)\Phi^{h_2}_{h\,h_1}(w) = \sum_{h'} F^{h_4h_3h_2h_1}_{h'h} \sum_{a_{h'}} \langle a_{h'}|\Phi^{h_3}_{h'\,h_2}(z-w)|h_2\rangle\, \Phi^{h'}_{h_4\,h_1}(a_{h'}, w), \tag{A.53}$$

where $\{a_{h'}\}$ is a homogeneous orthonormal basis of $H(c, h')$, and

$$F^{h_4h_3h_2h_1}_{h'h} = C^{h_4h_1h'0}_{h_1h'}\, C^{h_4h_3h_1h_2}_{h'h}\, C^{hh_2h_10}_{h_2h_1}. \tag{A.54}$$

gives the fusion matrix in terms of braiding matrices. For similar arguments in a meromorphic context, see [17].

Appendix B. Computing braiding matrices

Let h_4, h_3, h_2, h_1 be a sequence of unitary highest weights, $h_i = h_{p_i,q_i}$ $(i = 1,\ldots,4)$, with $h_2 = h_{1,2}$ or $h_{2,1}$. We want to calculate the braiding matrices $(C^{h_4h_3h_2h_1}_{h'h})_{h'h}$. Recall that 4-point functions satisfy a system of (BPZ and projective invariance) differential equations which reduces to a system of o.d.e.'s of Fuchsian type with 3 singular points, which we can and do choose to be 0, 1 and ∞. It will be apparent that the braiding matrix is essentially the connection matrix for a Fuchsian equation (for some path), relating the local solutions at the singular points 0 and ∞. Since $h_2 = h_{1,2}$ or $h_{2,1}$, one of these Fuchsian equations is of order 2, and thus after suitable transformations a hypergeometric equation.

Corresponding to $h_2 = h_{1,2}$ or $h_{2,1}$, we have the BPZ equation:

$$\left[\frac{3}{4h_2+2}\frac{\partial^2}{\partial z_2^2} + \sum_{j=1,3,4} z_{j2}^{-1}\frac{\partial}{\partial z_j} - h_j\, z_{j2}^{-2}\right] f = 0 \tag{B.1}$$

in addition to the projective invariance equations:

$$\left[\sum_{j=1}^{4} z_j^{m+1}\frac{\partial}{\partial z_j} + h_j\,(m+1)\,z_j^m\right] f = 0 \quad (m = -1, 0, 1). \tag{B.2}$$

We will consider this system of equations, which we reduce to a hypergeometric equation $H(\alpha,\beta,\gamma)$, unique up to Kummer's 24 generically distinct transformations that take one hypergeometric equation into another:

$$x(1-x)\frac{d^2f}{dx^2} + \{\gamma - (\alpha+\beta+1)x\}\frac{df}{dx} - \alpha\beta f = 0 \tag{B.3}$$

with Riemann scheme

$$\begin{pmatrix} 0 & 1 & \infty \\ 0 & 0 & \alpha \\ 1-\gamma & \gamma-\alpha-\beta & \beta \end{pmatrix}. \tag{B.4}$$

Our problem then reduces to the classical connection problem for the hypergeometric equation, whose solution is known [18,19].

The projective invariance equations have the general solution

$$\prod_{\substack{i,j=1\\ i<j}}^{4} z_{ij}^{-\gamma_{ij}}\, f(x), \quad x = \frac{z_{12}z_{34}}{z_{13}z_{24}}, \tag{B.5}$$

where the γ_{ij} are arbitrary solutions to

$$2h_i = \sum_{\substack{j=1\\ i<j}}^{4} \gamma_{ij} + \sum_{\substack{j=1\\ j<i}}^{4} \gamma_{ji} \quad (i = 1,\ldots,4). \tag{B.6}$$

We can choose, say, γ_{12} and γ_{23} arbitrarily, and obtain

$$\begin{aligned} \gamma_{13} &= h_1 + h_2 + h_3 - h_4 - \gamma_{12} - \gamma_{23} \\ \gamma_{14} &= h_1 + h_4 - h_2 - h_3 + \gamma_{23} \\ \gamma_{24} &= 2h_2 - \gamma_{12} - \gamma_{23} \\ \gamma_{34} &= h_3 + h_4 - h_1 - h_2 + \gamma_{12}. \end{aligned} \tag{B.7}$$

With this Ansatz, the BPZ equation above reduces to

$$\frac{3}{4h_2+2}\frac{d^2f}{dx^2}+\left(\left[1-\frac{3\gamma_{12}}{2h_2+1}\right]\frac{1}{x}-\left[1-\frac{3\gamma_{23}}{2h_2+1}\right]\frac{1}{1-x}\right)\frac{df}{dx} \tag{B.8}$$
$$\left(\frac{-\left[\frac{3\gamma_{12}\gamma_{23}}{2h_2+1}+\gamma_{13}\right]}{x(1-x)}+\frac{\left[\frac{3\gamma_{12}(\gamma_{12}+1)}{4h_2+2}-\gamma_{12}-h_1\right]}{x^2}+\frac{\left[\frac{3\gamma_{23}(\gamma_{23}+1)}{4h_2+2}-\gamma_{23}-h_3\right]}{(1-x)^2}\right)f=0.$$

Now let γ_{12},γ_{23} be respectively solutions to

$$\begin{aligned}3\gamma_{12}(\gamma_{12}+1)-(4h_2+2)(\gamma_{12}+h_1)&=0\\ 3\gamma_{23}(\gamma_{23}+1)-(4h_2+2)(\gamma_{23}+h_3)&=0\end{aligned} \tag{B.9}$$

so that we get the hypergeometric equation $H(\alpha,\beta,\gamma)$, with

$$\begin{aligned}\gamma&=\frac{4h_2+2}{3}-2\gamma_{12}\\ \alpha+\beta&=\frac{8h_2+1}{3}-2(\gamma_{12}+\gamma_{23})\\ \alpha\beta&=2\gamma_{12}\gamma_{23}+\frac{4h_2+2}{3}\gamma_{13}.\end{aligned} \tag{B.10}$$

Further, we easily check that

$$\begin{aligned}\{0,1-\gamma\}&=\begin{cases}\{h_{p_1,q_1\pm1}-h_1-h_2+\gamma_{12}\} & \text{if } h_2=h_{1,2}\\ \{h_{p_1\pm1,q_1}-h_1-h_2+\gamma_{12}\} & \text{if } h_2=h_{2,1}\end{cases}\\ \{\alpha,\beta\}&=\begin{cases}\{h_{p_4,q_4\pm1}+h_2-h_4-\gamma_{12}-\gamma_{23}\} & \text{if } h_2=h_{1,2}\\ \{h_{p_4\pm1,q_4}+h_2-h_4-\gamma_{12}-\gamma_{23}\} & \text{if } h_2=h_{2,1}.\end{cases}\end{aligned} \tag{B.11}$$

Of course, $h_{p,q}$ need not be a unitary highest weight for p, q not in the usual ranges.

The 4 generically distinct choices for $(\gamma_{12},\gamma_{23})$ correspond to the Kummer transformations that act trivially on the variable x. For definiteness, we take:

$$\begin{aligned}h_2=h_{1,2}:\quad \gamma_{12}&=\frac{-1-[p_1(m+1)-q_1m]}{2(m+1)},\quad \gamma_{23}=\frac{-1-[p_3(m+1)-q_3m]}{2(m+1)},\\ \left.\begin{matrix}\alpha\\ \beta\end{matrix}\right\}&=h_{p_4,q_4\pm1}-h_{p_4,q_4}+h_2-(\gamma_{12}+\gamma_{23})\\ &=\frac{1}{2}[1+(p_1+p_3\mp p_4)-(q_1+q_3\mp q_4)]+\frac{q_1+q_3\mp q_4}{2(m+1)}\\ \gamma&=1+h_1+h_2-\gamma_{12}-h_{p_1,q_1+1}\\ &=1+p_1-q_1+\frac{q_1}{m+1}\end{aligned} \tag{B.12}$$

and

$$\begin{aligned}h_2=h_{2,1}:\quad \gamma_{12}&=\frac{1+[p_1(m+1)-q_1m]}{2m},\quad \gamma_{23}=\frac{1+[p_3(m+1)-q_3m]}{2m},\\ \left.\begin{matrix}\alpha\\ \beta\end{matrix}\right\}&=h_{p_4\pm1,q_4}-h_{p_4,q_4}+h_2-(\gamma_{12}+\gamma_{23})\\ &=\frac{1}{2}[1-(p_1+p_3\mp p_4)+(q_1+q_3\mp q_4)]-\frac{p_1+p_3\mp p_4}{2m}\\ \gamma&=1+h_1+h_2-\gamma_{12}-h_{p_1+1,q_1}\\ &=1-p_1+q_1-\frac{p_1}{m}.\end{aligned} \tag{B.13}$$

Replacing (p_4,q_4) by $(m-p_4,m+1-q_4)$ interchanges α and β. Replacing (p_3,q_3) by $(m-p_3,m+1-q_3)$, (p_1,q_1) by $(m-p_1,m+1-q_1)$, or both, implements a Kummer transformation, given equivalently by $(\alpha,\beta,\gamma)\mapsto(\gamma-\beta,\gamma-\alpha,\gamma)$, $(\alpha,\beta,\gamma)\mapsto(1+\alpha-\gamma,1+\beta-\gamma,2-\gamma)$, $(\alpha,\beta,\gamma)\mapsto(1-\beta,1-\alpha,2-\gamma)$ respectively. Together with the identity, these transformations form a subgroup of Kummer transformations ($\cong \mathbf{Z}_2\times\mathbf{Z}_2$).

We look at the case: $h_2=h_{1,2}$. Observe:

$$\begin{aligned}\gamma-1&=p_1-q_1+\frac{q_1}{m+1},\\ \alpha+\beta-\gamma&=p_3-q_3+\frac{q_3}{m+1},\\ \beta-\alpha&=p_4-q_4+\frac{q_4}{m+1}\qquad\notin\mathbf{Z}.\end{aligned} \tag{B.14}$$

The conditions γ, $\alpha+\beta-\gamma$, $\beta-\alpha\notin\mathbf{Z}$ respectively guarantee that the local solutions at the singular points 0, 1, ∞ are non-logarithmic. Because of the other 7 BPZ equations, which we have ignored so far, we are interested only in the case when at least one of

$$\begin{pmatrix} & h_{p_3,q_3} & \\ h_{p_4,q_4} & & h_{p_1,q_1\pm1}\end{pmatrix}$$

is an allowed vertex. Then we have, exclusively, either that $p_1+p_3+p_4\in2\mathbf{Z}+1$, $q_1+q_3+q_4\in2\mathbf{Z}$; or that $p_1+p_3+p_4$, $q_1+q_3+q_4\in m+1+2\mathbf{Z}$. Equivalently, $\sum_{i=1}^4(p_i+q_i)\in2\mathbf{Z}$ or $2\mathbf{Z}+1$, respectively. These two cases are exchanged by any one of the involutions $(p_i,q_i)\mapsto(m-p_i,m+1-q_i)$, $i=1,3,4$, so there is no loss of generality in restricting attention to one case only. We consider the former case. Then:

$$\begin{aligned}\alpha\in\mathbf{Z}&\Leftrightarrow q_1+q_3-q_4=0\\ \beta\in\mathbf{Z}&\Leftrightarrow q_1+q_3+q_4=2(m+1)\\ \alpha-\gamma\in\mathbf{Z}&\Leftrightarrow q_1+q_4-q_3=0\\ \beta-\gamma\in\mathbf{Z}&\Leftrightarrow q_3+q_4-q_1=0.\end{aligned} \tag{B.15}$$

Observe that these are mutually exclusive conditions. Each of the equivalent left- and right-hand-side conditions has a simple interpretation:

Let $f_a(x,\nu)$, $a=0,\infty$, denote the local solution of $H(\alpha,\beta,\gamma)$ at the singular point $x=a$, with characteristic exponent ν (we won't need to consider the singular point $x=1$).

More precisely,

$$\begin{aligned} f_0(x,0) &= F(\alpha,\beta,\gamma;x) \\ f_0(x,1-\gamma) &= x^{1-\gamma}F(\alpha-\gamma+1,\beta-\gamma+1,2-\gamma;x) \\ f_\infty(x,\alpha) &= x^{-\alpha}F(\alpha,\alpha-\gamma+1,\alpha-\beta+1;x^{-1}) \\ f_\infty(x,\beta) &= x^{-\beta}F(\beta,\beta-\gamma+1,\beta-\alpha+1;x^{-1}), \end{aligned} \tag{B.16}$$

where

$$\begin{aligned} F(\alpha,\beta,\gamma;x) &= \sum_{m=0}^{\infty} \frac{(\alpha)_m(\beta)_m}{(\gamma)_m(1)_m}\, x^m, \\ (\mu)_m &= \mu(\mu+1)\cdots(\mu+m-1) \end{aligned} \tag{B.17}$$

is the hypergeometric series. We take $-\pi < \arg x < \pi$. By analytic continuation, we can compare the 2 pairs of solutions; we have the classical result [18,19]:

$$(f_0(x;0),\, f_0(x;1-\gamma)) = (f_\infty(x;\alpha),\, f_\infty(x;\beta))\, P(\alpha,\beta,\gamma) \tag{B.18}$$

where, taking the path for analytic continuation along the negative imaginary axis, the connection matrix is given by

$$P(\alpha,\beta,\gamma) = \begin{pmatrix} e^{-\pi i\alpha}\frac{\Gamma(\gamma)\Gamma(\beta-\alpha)}{\Gamma(\beta)\Gamma(\gamma-\alpha)} & e^{-\pi i(\alpha-\gamma+1)}\frac{\Gamma(2-\gamma)\Gamma(\beta-\alpha)}{\Gamma(\beta-\gamma+1)\Gamma(1-\alpha)} \\ e^{-\pi i\beta}\frac{\Gamma(\gamma)\Gamma(\alpha-\beta)}{\Gamma(\alpha)\Gamma(\gamma-\beta)} & e^{-\pi i(\beta-\gamma+1)}\frac{\Gamma(2-\gamma)\Gamma(\alpha-\beta)}{\Gamma(\alpha-\gamma+1)\Gamma(1-\beta)} \end{pmatrix} \tag{B.19}$$

for $\alpha,\beta,\gamma-\alpha,\gamma-\beta \notin \mathbf{Z}$. Both sides of the equation are analytic in x,α,β,γ, and the points corresponding to $\alpha,\beta,\gamma-\alpha$, or $\gamma-\beta \in \mathbf{Z}$ are removable singularities. Since $\Gamma(z)\Gamma(1-z) = \pi/\sin(\pi z)$, and $\sin(z)$ is an entire function, $\Gamma(z)$ has no zeros. Thus all the entries of the connection matrix $P(\alpha,\beta,\gamma)$ are non-vanishing unless one of $\alpha,\beta,\gamma-\alpha,\gamma-\beta$ is an integer.

For a sequence of unitary highest weights $(h_i = h_{p_i,q_i})_{i=1,2,3,4}$, such that $h_2 = h_{1,2}$, there corresponds a vector space of 4-point functions of dimension 1 or 2 (or, trivially, 0) spanned by

$$\langle h_4|\Phi^{h_3}_{h_4\ h}(z)\Phi^{h_2}_{h\ h_1}(w)|h_1\rangle, \quad h = h_{p_1,q_1\pm 1} \tag{B.20}$$

with the convention that a vertex operator is just zero if the vertex is not an allowed one. We assume again without loss of generality that $\sum_{i=1}^4 (p_i+q_i) \in 2\mathbf{Z}$. By the fusion rules, the dimension is 2 if and only if

$$q_1 \pm 1 \in \{|q_3-q_4|+1, |q_3-q_4|+3, \ldots, \min(q_3+q_4-1, 2(m+1)-q_3-q_4-1)\} \tag{B.21}$$

or, symmetrically, if and only if:

$$\begin{aligned} &2 \le q_i \le m-1 \quad (i=1,3,4). \\ &q_1+q_3-q_4,\ q_1+q_4-q_3,\ q_3+q_4-q_1 \ge 2, \quad \text{and} \\ &q_1+q_3+q_4 \le 2m. \end{aligned} \tag{B.22}$$

The dimension is 1 if and only if one of the following holds:

$$\begin{aligned} q_1+1 &= |q_3-q_4|+1 \\ q_1-1 &= \min(q_3+q_4-1,\ 2(m+1)-q_3-q_4-1) \end{aligned} \tag{B.23}$$

or, symmetrically, if and only if one of the following holds:

$$\begin{aligned} q_1+q_3-q_4 &= 0 \\ q_1+q_4-q_3 &= 0 \\ q_3+q_4-q_1 &= 0 \\ q_1+q_3+q_4 &= 2(m+1). \end{aligned} \tag{B.24}$$

Thus the dimension of the space of 4-point functions is 2 if and only if the connection matrix $P(\alpha,\beta,\gamma)$ has no vanishing entries; the dimension is 1 if and only if it has precisely one vanishing entry, whence the pair of solutions $(f_0(x,\nu_1), f_\infty(x,\nu_2))$ that are multiples of each other correspond to the 4-point function.

Finally, we compute the braiding matrix. Our normalization is:

$$\begin{aligned} \langle h_i|h_i\rangle &= 1 \\ \langle h_k|\Phi^{h_i}_{h_k\ h_j}(z)|h_j\rangle &= z^{-(h_i+h_j-h_k)}. \end{aligned} \tag{B.25}$$

Observe that, for $|z_3| > |z_2|$,

$$\begin{aligned} \langle h_4|\Phi^{h_3}_{h_4\ h}(z_3)\Phi^{h_{1,2}}_{h\ h_1}(z_2)|h_1\rangle &= z_3^{-(h_3+h-h_4)} z_2^{-(h_2+h_1-h)}[1+\ldots] \\ &= z_{32}^{-\gamma_{23}} z_3^{-\gamma_{13}} z_2^{-\gamma_{12}} f_0(x;\nu) \\ &= z_3^{-\gamma_{13}-\gamma_{23}} z_2^{-\gamma_{12}} (1-x)^{-\gamma_{23}} f_0(x;\nu) \end{aligned} \tag{B.26}$$

where $x = z_2/z_3$, and

$$\nu = \begin{cases} 1-\gamma & \text{if } h = h_{p_1,q_1+1} \\ 0 & \text{if } h = h_{p_1,q_1-1}. \end{cases} \tag{B.27}$$

And that, for $|z_2| > |z_3|$,

$$\begin{aligned} \langle h_4|\Phi^{h_2}_{h_4\ h'}(z_2)\Phi^{h_3}_{h'\ h_1}(z_3)|h_1\rangle &= z_2^{-(h_2+h'-h_4)} z_3^{-(h_3+h_1-h')}[1+\ldots] \\ &= z_{23}^{-\gamma_{23}} z_3^{-\gamma_{13}} z_2^{-\gamma_{12}} f_\infty(x;\nu') \\ &= z_3^{-\gamma_{13}-\gamma_{23}} z_2^{-\gamma_{12}} (x-1)^{-\gamma_{23}} f_\infty(x;\nu'), \end{aligned} \tag{B.28}$$

where

$$\nu' = \begin{cases} \alpha & \text{if } h' = h_{p_4,q_4+1} \\ \beta & \text{if } h' = h_{p_4,q_4-1}. \end{cases} \tag{B.29}$$

We choose a homotopy class of path in $\{(z_2,z_3) \in \mathbf{C}^{*2}\colon z_2 \ne z_3\}$ from $|z_3| > |z_2|$ to $|z_2| > |z_3|$, such that the class of path in the cut plane $\mathbf{C} - (-\infty,0]$ is along the negative imaginary axis. Then we have by analytic continuation along this path,

$$(1-x)^{-\gamma_{23}} = e^{-\pi i\gamma_{23}}(x-1)^{-\gamma_{23}}. \tag{B.30}$$

Hence, denoting

$$C^{h_4 h_3 h_{1,2} h_1}_{h_{p_4,q_4\pm 1} h_{p_1,q_1\pm 1}} = C_{\pm\pm},$$

we have

$$\begin{pmatrix} C_{+-} & C_{++} \\ C_{--} & C_{-+} \end{pmatrix} = e^{-\pi i\gamma_{23}} P(\alpha,\beta,\gamma). \tag{B.31}$$

Observe that the braiding relations are invariant under $(p_i, q_i) \mapsto (m - p_i, m + 1 - q_i)$, each $i = 1, 3, 4$, as needs be.

The case $h_2 = h_{2,1}$ proceeds in exactly the same way, with corresponding results on the non-vanishing of entries in the braiding matrix. Likewise writing

$$C^{h_4 h_3 h_{2,1} h_1}_{h_{p_4 \pm 1, q_4} h_{p_1 \pm 1, q_1}} = C_{\pm\pm},$$

we have the same formula for the braiding matrix.

Moreover, by inspection of the connection matrices for the analytic continuation of solutions between other pairs of singular points, we obtain that the same results on the non-vanishing of the entries in the braiding matrix applies if any one of the h_i $(i = 1, 3, 4)$ is chosen instead of h_2 to be $h_{1,2}$ or $h_{2,1}$.

Appendix C. Pfaffian system with regular singular points

We consider a system of equations

$$\left[\sum_{j=1}^{4} z_j^{m+1} \frac{\partial}{\partial z_j} + h_j\,(m+1)\, z_j^m\right] f = 0 \quad (m = 0, \pm 1)$$
$$O(\vec{z}, \frac{\partial}{\partial \vec{z}})\, f = 0 \tag{C.1}$$

where coefficients in $O(\vec{z}, \frac{\partial}{\partial \vec{z}})$ have convergent expansions when $|z_4| > \cdots > |z_1|$; and the Ansatz:

$$f = \prod_{\substack{i,j=1 \\ i<j}}^{4} z_{ij}^{-\gamma_{ij}}\, F(x), \quad x = \frac{z_{12} z_{34}}{z_{13} z_{24}} \tag{C.2}$$

$$\sum_{\substack{i=1 \\ i<j}}^{4} \gamma_{ij} + \sum_{\substack{i=1 \\ i>j}}^{4} \gamma_{ji} = 2h_j \quad (j = 1, \ldots, 4) \tag{C.3}$$

reduces this latter equation to a Fuchsian equation in the variable x with 3 singular points, at 0, 1, ∞:

$$\left[\frac{d^n}{dx^n} + a_1(x)\frac{d^{n-1}}{dx^{n-1}} + \ldots + a_n(x)\right] F = 0 \tag{C.4}$$

where each

$$a_k(x) = \frac{p_k(x)}{x^k(1-x)^k}, \quad p_k \text{ a polynomial of degree} \leq k. \tag{C.5}$$

It should be clear that all the equations above can be interpreted as equations between formal series in the variables z_i $(i = 1, \ldots, 4)$. By $F(x)$ is meant a formal series F that satisfies the equations

$$\sum_{j=1}^{4} z_j^{m+1} \frac{\partial F}{\partial z_j} = 0 \quad (m = 0, \pm 1). \tag{C.6}$$

The equation involving derivatives in the variable x is meant in the following sense: use the projective invariance equations to replace the equation $O(\vec{z}, \frac{\partial}{\partial \vec{z}})\, f = 0$ with one involving only derivatives with respect to z_1, say, and then change variable from z_1 to x. There

is an obvious canonical correspondence between formal series in the variables (z_i) and formal series in the (w_i), where $w_i = z_i/z_{i+1}$ $(i = 1, 2, 3)$, $w_4 = z_4$. This change of variables gives an analytic isomorphism from $\{(z_4, \ldots, z_1) \in \mathbf{C}^{*4}\colon |z_4| > \cdots > |z_1|\}$ to $D = \{(w_4, \ldots, w_1) \in \mathbf{C}^{*4}\colon 1 > |w_i| > 0,\ i = 1, 2, 3\}$.

We observe that

$$w_i \frac{\partial F}{\partial w_i} = \alpha_i(w) \frac{\partial F}{\partial w_1} \quad (i = 2, 3, 4) \tag{C.7}$$

where each $\alpha_i(w)$ is holomorphic on D. Explicitly,

$$\alpha_2(w) = \frac{-(w_1 - 1)(w_1 w_2^2 w_3 - 1)}{(w_2 - 1)(w_2 w_3 - 1)}, \quad \alpha_3(w) = \frac{w_3(w_1 - 1)(w_1 w_2 - 1)}{(w_3 - 1)(w_2 w_3 - 1)}, \quad \alpha_4(w) = 0. \tag{C.8}$$

Then for each $r \in \mathbf{N}_0$,

$$w_i \frac{\partial}{\partial w_i}\left(\frac{\partial^r F}{\partial w_1^r}\right) = \sum_{s=1}^{r} A_{i,rs}(w) \frac{\partial^s F}{\partial w_1^s} \quad (i = 2, 3, 4). \tag{C.9}$$

where each $A_{i,rs}(w)$ is holomorphic on D. Observe also that with the change of variables from x to z_1 to w_1,

$$\frac{\partial}{\partial w_1} = w_2\, \alpha(w) \frac{d}{dx} \tag{C.10}$$

where $\alpha(w), \alpha(w)^{-1}$ are holomorphic on D with the property that $d^r\alpha(w)/dx^r$ is holomorphic in D for each $r \in \mathbf{N}_0$. Explicitly,

$$\alpha(w) = \frac{(w_2 - 1)(w_3 - 1)}{(w_2 w_3 - 1)(w_1 w_2 - 1)^2}. \tag{C.11}$$

Writing

$$\left(\alpha(w)\frac{d}{dx}\right)^{n-1} = \sum_{k=0}^{n-1} \beta_k(w) \frac{d^r}{dx^r}$$
$$\left(\alpha(w)^{-1}\frac{\partial}{\partial w_1}\right)^{i} = \sum_{j=0}^{i} \gamma_j^i(w) \frac{\partial^j}{\partial w_1^j} \quad (i \in \mathbf{N}_0). \tag{C.12}$$

where the $\beta_i(w)$, $\gamma_j^i(w)$ are holomorphic on D, we obtain

$$\frac{\partial^n F}{\partial w_1^n} = \sum_{j=0}^{n-1} \sum_{i=0}^{j} \left[w_2^{n-i}\, \beta_i(w) - \alpha(w)^n\, w_2^{n-i}\, a_{n-i}(x)\right] \gamma_j^i(w) \frac{\partial^j F}{\partial w_1^j}. \tag{C.13}$$

By the Fuchsian property of (C.4), each $w_2^k\, a_k(x)$ is holomorphic on D, so that each of the coefficients above is holomorphic on D. We conclude that

$$\frac{\partial \vec{F}}{\partial w_i} = \frac{A_i(w)\vec{F}}{w_i} \quad (i = 1, \ldots, 4), \qquad \vec{F} = \left(\frac{\partial^{r-1} F}{\partial w_1^{r-1}}\right)_{r=1,\ldots,n} \tag{C.14}$$

where each $A_i(w)$ is an $n \times n$ matrix-valued function, holomorphic on D. Unsurprisingly, we readily verify that $\vec{f} = (\partial^{r-1} f/\partial {w_1}^{r-1})_{r=1,\ldots,n}$ satisfies a similar system of equations.

To show that the projective invariance equations plus a BPZ equation for a 4-point function is equivalent to an integrable Pfaffian system of the form (C.14), we have to check 2 points. First is the compatibility of the BPZ equation with the projective invariance equations, whence integrability follows from that of an ordinary differential equation. This involves checking its invariance under Möbius transformations, which follows from its construction. Second, to verify the form (C.14), we consider the BPZ equation given by $\hat{O}^{4;1}_{p,q}$, which is $O_{p,q}$ with each $L(m)$ replaced by

$$\hat{L}^{4;1}(m) = -\sum_{i=2}^{4} z_{i1}^{m+1}\frac{\partial}{\partial z_i} + h_i\,(m+1)\,z_{i1}^{m}. \qquad (C.15)$$

The equation is of the form

$$\left[\frac{\partial^{pq}}{\partial z_1^{pq}} + \sum_{\substack{i_1,\dots,i_N>0\\ i_1+\dots+i_M=pq}} a^{p,q}_{i_1\cdots i_M}\,\hat{L}^{4;1}(-i_1)\cdots\hat{L}^{4;1}(-i_M)\right] f = 0 \qquad (C.16)$$

and it follows in a straightforward way, by noting the form (C.15) of the $\hat{L}^{4;1}(-i)$'s and by inspection of the projective invariance equations, which give all the other derivatives in terms of

$$\frac{\partial}{\partial w_1} = z_2\frac{\partial}{\partial z_1}, \qquad (C.17)$$

that

$$\tilde{f} = \left(\frac{\partial^{r-1} f}{\partial w_1^{r-1}}\right)_{r=1,\dots,pq} \qquad (C.18)$$

satisfies a system of equations of the form (C.14). Without considering the other BPZ equations, this is sufficient to show that all the solutions to the full set of equations are regular at the origin $(w_4,\dots,w_1)=0$.

Appendix D. Decomposing vertex operators

Here, we show that a $\hat{su}(2)_N \times \hat{su}(2)_1$ vertex operator descendant from the primary

$$\Phi^{l}_{l''\,l'}(\,.\,,z)\otimes\Phi^{\epsilon}_{\epsilon''\,\epsilon'}(\,.\,,z) \qquad (D.1)$$

can be decomposed as a sum of $\hat{su}(2)_{N+1}\times\mathcal{L}_c$ vertex operators descendant from the primaries

$$\Phi^{j}_{j''\,j'}(\,.\,,z)\otimes\Phi^{h_{p,q}}_{h_{p'',q''}\,h_{p',q'}}(z) \qquad (D.2)$$

where $p=2l+1$, $q=2j+1$ etc., for values of j, j' and j'' given by the GKO decompositions of $(N,l)\otimes(1,\varepsilon)$, $(N,l')\otimes(1,\varepsilon')$ and $(N,l'')\otimes(1,\varepsilon'')$ respectively. It is sufficient to show that the $\hat{su}(2)_N\times\hat{su}(2)_1$ vertex operator

$$\Psi(\,.\,,z) = \Psi^{j}\left[\begin{matrix} l & & \epsilon \\ l'' & l' & \epsilon'' \;\; \epsilon' \end{matrix}\right](\,.\,,z) \qquad (D.3)$$

corresponding to the lowest energy subspace $\langle\psi^{j,l}_{a}\rangle$ of a summand of the GKO decomposition

$$(N+1,j)\otimes(c,h_{p,q})\subset(N,l)\otimes(1,\varepsilon) \qquad (D.4)$$

decomposes as the appropriate $\hat{su}(2)_{N+1}\times\mathcal{L}_c$ primaries. The general case follows from the observation that the $\hat{su}(2)_{N+1}\times\mathcal{L}_c$ descendents of such vertex operators span the entire space of $\hat{su}(2)_N\times\hat{su}(2)_1$ vertex operators descendant from (D.1).

By definition, we have the relations

$$\begin{aligned}\hat{X}_{N+1}(m)\Psi(\,.\,,z) &= 0 \quad (m\in\mathbb{N})\\ \hat{X}_{N+1}(0)\Psi(\,.\,,z) &= \Psi(X\,.\,,z)\\ \hat{L}_{N+1}(0)\Psi(\,.\,,z) &= \Delta(j;N+1)\,\Psi(\,.\,,z)\end{aligned} \qquad (D.5)$$

for all $X\in su(2)$, from which follows

$$\hat{L}_{N+1}(m)\Psi(\,.\,,z) = 0 \quad (m\in\mathbb{N}); \qquad (D.6)$$

and also the relations

$$\begin{aligned}\hat{L}_c(m)\Psi(\,.\,,z) &= 0 \quad (m\in\mathbb{N})\\ \hat{L}_c(0)\Psi(\,.\,,z) &= h_{p,q}\,\Psi(\,.\,,z)\end{aligned} \qquad (D.7)$$

where the subscripts stand for the algebras $\hat{su}(2)_{N+1}$ and $\mathcal{L}_c$.

We check directly that its conjugate

$$\Psi^{*}(\,.\,,z) = \left[\Psi(c_j(\,.\,),\frac{1}{\bar{z}})\,\bar{z}^{-2h_{p,q}-2\Delta(j;N+1)}\right]^{\dagger} \qquad (D.8)$$

also satisfies the equations (D.5)–(D.7), so that we must have

$$\Psi^{j}\left[\begin{matrix} l & & \epsilon \\ l'' & l' & \epsilon'' \;\; \epsilon' \end{matrix}\right]^{*}(\,.\,,z) \propto \Psi^{j}\left[\begin{matrix} l & & \epsilon \\ l' & l'' & \epsilon' \;\; \epsilon'' \end{matrix}\right](\,.\,,z) \qquad (D.9)$$

where $c_j\colon V_j\to V_j^{*}$ is the antilinear anti-isomorphism from the $su(2)$-module of spin j to its conjugate module, defined by the inner product on the $\hat{su}(2)$-module of spin j by restriction to its lowest energy subspace. It has the property

$$X(c_j(\xi)) = -c_j(X^{\dagger}\xi) \qquad (D.10)$$

where the sign comes from regarding V_j^{*} as a left $su(2)$-module.

It follows from the equations (D.5)–(D.7) for $\Psi(\,.\,,z)$ *and* its conjugate $\Psi^{*}(\,.\,,z)$ that

$$\begin{aligned}[X_{N+1}(m),\,\Psi(\,.\,,z)] &= z^{m}\,\Psi(X\,.\,,z)\\ [L_{N+1}(m),\,\Psi(\,.\,,z)] &= z^{m+1}\,[L_{N+1}(-1),\,\Psi(\,.\,,z)] + \Delta(j;N+1)\,(m+1)\,z^{m}\,\Psi(\,.\,,z)\\ [L_c(m),\,\Psi(\,.\,,z)] &= z^{m+1}\,[L_c(-1),\,\Psi(\,.\,,z)] + h_{p,q}\,(m+1)\,z^{m}\,\Psi(\,.\,,z)\end{aligned} \qquad (D.11)$$

for *all* $m\in\mathbb{Z}$. We of course have, with $T(z)=T_N(z)+T_1(z)=T_{N+1}(z)+T_c(z)$,

$$[L(-1),\,\Psi(\,.\,,z)] = \frac{d\Psi(\,.\,,z)}{dz}. \qquad (D.12)$$

From

$$[L(0),\,\Psi(\,.\,,z)] = z\,\frac{d\Psi(\,.\,,z)}{dz} + [h_{p,q}+\Delta(j;N+1)]\,\Psi(\,.\,,z), \qquad (D.13)$$

we have a decomposition into homogeneous components

$$\Psi(\,.\,,z) = \sum_{n\in\mathbf{Z}} \Psi(\,.\,;n)\, z^{-n-\Delta} \tag{D.14}$$

where

$$\Delta = h_{p,q} + \Delta(j;N+1) + \Delta(l';N) + \Delta(\varepsilon';1) - \Delta(l'';N) - \Delta(\varepsilon'';1) \tag{D.15}$$

and each $\Psi(\,.\,;n)$ $(n \in \mathbf{Z})$ is homogeneous of degree $-n$. As a corollary, for each j', j'' and

$$\begin{aligned} \psi^{j',l'} &= |j'\rangle \otimes |h_{p',q'}\rangle \in (N+1,j') \otimes (c,h_{p',q'}) \subset (N,l') \otimes (1,\varepsilon') \\ \psi^{j'',l''} &= |j''\rangle \otimes |h_{p'',q''}\rangle \in (N+1,j'') \otimes (c,h_{p'',q''}) \subset (N,l'') \otimes (1,\varepsilon'') \end{aligned} \tag{D.16}$$

we have

$$\langle \psi^{j'',l''} | \Psi(\,.\,,z) | \psi^{j',l'} \rangle = Y_{j''\,j'}\, \langle j'' | \Phi^{j}_{j''\,j'}(\,.\,,z) | j' \rangle\, \langle h_{p'',q''} | \Phi^{h_{p,q}}_{h_{p'',q''}\,h_{p',q'}}(z) | h_{p',q'} \rangle \tag{D.17}$$

for some $Y_{j''\,j'} \in \mathbf{C}$, possibly vanishing. The equations (D.11) and (D.17) characterize $\Psi(\,.\,,z)$ (as an operator-valued function); but they are satisfied by

$$\sum_{j',j''} Y_{j''\,j'}\, \Phi^{j}_{j''\,j'}(\,.\,,z) \otimes \Phi^{h_{p,q}}_{h_{p'',q''}\,h_{p',q'}}(z) \tag{D.18}$$

so we are done. We will write

$$Y_{j''\,j'} = Y^{j}_{j''\,j'} \begin{bmatrix} l & \varepsilon \\ l'' & l' \quad \varepsilon'' \quad \varepsilon' \end{bmatrix}. \tag{D.19}$$

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M.S.
I.D. No.
NAME