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BRAIDED GEOMETRY: A NEW APPROACH TO q-DEFORMATIONS

SHAHN MAJID*
Department of Applied Mathematics & Theoretical Physics
University of Cambridge, Cambridge CB3 9EW, U.K.

Abstract. Many constructions in mathematics can be generalised by replacing transpositions of symbols by a braiding. Algebraic information now flows along braids. Physically this extends the philosophy of supersymmetry to the case of braid statistics and provides a new approach to systematically q-deforming everything, by q-deforming the notion of tensor product itself. Such deformations offer the possibility of regularising infinities in physics. We report on this approach as developed over the last three years by the author. We describe the braided Lie algebra gl_2 in detail.

Key words: supersymmetry – braided groups – quantum groups – non-commutative geometry – q-deformations – braided Lie algebras

O time, thou must untangle this, not I.
It is too hard a knot for me t'untie.
Viola, in Twelfth Night by W. Shakespeare

1. Introduction

In recent years there has been much interest in developing some kind of quantum or non-commutative geometry. This should be more general than usual notions of non-Euclidean geometry, namely general enough to be useful in the quantum domain. Usually, when we enter quantum physics we are forced to throw away the large part of our intuition concerning classical geometry, because of Heisenberg's uncertainty principle. Probably it is commonly accepted by now that there is a clear need for developing new tools to recover this geometrical picture in the quantum world, and this is the long term goal of quantum geometry. This point of view is somewhat looser than the concrete models and predictions coming out of operator algebras in the work of A. Connes and others[1][2] as we have seen at this conference. On the other hand, if quantum geometry can be developed as systematically as Riemannian geometry then we will have a powerful new language in which many questions that involve both quantum physics and macroscopic physics can be resolved. In the author's view this includes is-

* SERC Fellow and Fellow of Pembroke College, Cambridge. This paper is in final form and no version of it will be submitted for publication elsewhere.

sues about quantum measurement as well as interactions between gravity, information theory and quantum statistical mechanics.

From this point of view, which is that of developing a language, we expect quantum geometry to be useful for the formulation of questions (rather than the making of predictions) well below the Planck scale. One might also use the geometry to ask questions and make predictions about physics at or beyond the Planck scale[3][4].

There is also another point of view in which the parameter q controlling the degree of non-commutativity of the quantum geometry is *not* Planck's constant, but rather is viewed as a formal deformation parameter used to regularise infinities in quantum theory[5]. The case of interest would then be $q = 1$ and singularities in physics could appear as poles at $q = 1$. In this line of development the parameter q is considered orthogonal to physics as we know it. For this makes it an ideal regulator since all usual constructions should be preserved (albeit q-deformed), in contrast to more conventional regularisation schemes (such as momentum cut-off) which seriously disrupt the mathematical structure, making it hard to renormalise in anything other than a formal power-series.

This is a slightly ambitious programme because from this point of view one should not expect serious predictions until all structures that we use in quantum theory (not only geometrical ones, but quantum mechanical ones also) have been q-deformed. It is not generally consistent to q-deform one aspect of our theory and leave another alone. This provides then a mathematical challenge: how to q-deform everything in a systematic way ?

We also recover our previous more restrictive point of view if q is made a function of $\hbar$ but this is not necessary. Conceptually speaking, these two points of view on q are different and we propose that they should not be confused. In this note we want to argue that they correspond in fact to two different kinds of non-commutativity in physics. Non-commutativity of the first kind is the usual one proportional to $\hbar$ and corresponding to quantisation within a given system. Non-commutativity of the second kind is the outer or statistical non-commutativity encountered between independent systems with fermionic or other statistics. Associating q whole-heartedly with this second kind of non-commutativity turns out to be very powerful, and we shall see that many questions that physicists have asked in recent years concerning q-deformation are solved at once using the techniques that we explain here.

To understand the nature of these two different kinds of non-commutativity it is necessary to appreciate that in the algebraic description of geometry there are two different structures. One is the algebra A which we think of as like $C(X)$ the functions on some 'quantum space'. This is well known by now. The second structure is how we think about two independent copies of the space. This is expressed algebraically in the notion of tensor product of

algebras. So $A \otimes A$ is to be thought of as like $C(X \times X)$. This is relevant to the product and also to the coproduct $A \to A \otimes A$ if our quantum space is some kind of group. Here the coproduct directly allows one to do random walks etc so our statistical point of view is a very precise one[6]. Thus, orthogonal to the idea of making the algebra A non-commutative is the idea of making the tensor product $\otimes$ non-commutative. We have

1. *Non-commutative geometry: A is allowed to be non-commutative, but $\otimes$ is the usual one.*
2. *Braided geometry: A remains in some sense commutative but $\otimes$ becomes non-commutative.*

In non-commutative geometry the non-commutativity is often determined as quantising a Poisson bracket etc. Likewise in this second conception of quantum geometry, which we call *braided geometry*, the non-commutativity of $\otimes$ is controlled by a structure Ψ, the braiding. We will see that the R-matrix revolution which in recent years has provided us with a rich supply of non-commutative geometries in the form of quantum groups $A(R)$ etc [7][8], provides equally well a rich supply of braided geometries in the form of braided groups $B(R)$ etc [9]. Moreover, these two directions for quantum geometry are orthogonal in the sense that they are perfectly compatible with each other. There are also functors of transmutation and bosonization that broadly connect these approaches,

$$\boxed{\text{Non-commutative Geometry}} \underset{\text{bosonization}}{\overset{\text{transmutation}}{\rightleftarrows}} \boxed{\text{Braided Geometry}}$$

Some quantum groups are more naturally braided groups and vice versa via these functors. For example, one setting where the braided versions should be more natural is as the generalised internal symmetries in low dimensional quantum theory, where the generators of the symmetry should be built from fields with braid statistics. In other situations the braided geometry is not only natural but seems to be essential.

Not only geometry but other structures too can be systematically braided. Mathematically, this process of *braidification* is a generalisation of the more familiar process of superisation that was popular some years ago. Recall that in supersymmetry the idea is to replace the transposition operator implicit in all constructions by the super-transposition $\Psi = \pm 1$ corresponding to Bose-Fermi statistics. This forms a representation of the symmetric group. Other representations of the symmetric group give essentially the same results. Now we go further and replace the symmetric group by the Artin braid group. The possible representations of the braid group are much richer than just one parameter q. But in particular our braidification process also solves the problem of q-deforming everything in a completely systematic way.

This systematic notion of braided geometry has been introduced over the last three years by the author. Just as the simplest geometries are groups,

so the first main notions were those of braided groups[10][9]. But by now, braided lines, planes, matrices, differential operators and also a proposal for braided-Lie algebras have all been found[10][11][9][12][13][14][15]. There are about thirty papers, some with collaborators, and also two review articles [16] and [17] for physicists and mathematicians respectively. The present note is intended as an invitation to these works.

Preliminaries on braided wiring diagrams

Our first task is to formalise our notion of tensor product. Thus we need a collection of objects V, W, Z etc, which the reader can keep in mind as vector spaces. Between any two we need a tensor product $V \otimes W$ again in our collection. There should be an identity object $\underline{1}$ and isomorphisms $V \otimes \underline{1} \cong V \cong \underline{1} \otimes V$ and $V \otimes (W \otimes Z) \cong (V \otimes W) \otimes Z$ expressing the identity and associativity. These isomorphisms obey all the usual coherence properties and will be suppressed. The fancy name for all this is that of a monoidal category[18]. The reader can keep in mind $\underline{1} = \mathbf{C}$ or other field, and the usual vector space tensor product.

A *braiding* in this context is then a collection of isomorphisms $\Psi_{V,W} : V \otimes W \to W \otimes V$. It expresses a degree of commutativity of $\otimes$, and has the usual natural properties that we take for granted for transposition or super-transposition. These are

$$\Psi_{V \otimes W, Z} = \Psi_{V,Z}\Psi_{W,Z}, \qquad \Psi_{V, W \otimes Z} = \Psi_{V,Z}\Psi_{V,W} \tag{1}$$

$$\Psi_{W,Z}(\phi \otimes \mathrm{id}) = (\mathrm{id} \otimes \phi)\Psi_{V,Z}, \quad \Psi_{Z,W}(\mathrm{id} \otimes \phi) = (\phi \otimes \mathrm{id})\Psi_{Z,V} \ \forall \phi \begin{array}{c} V \\ \downarrow \\ W \end{array}. \tag{2}$$

Such a Ψ gives what is called a braided-monoidal or quasitensor category. These have been studied in category theory[19] and also arise in the representation theory of quantum groups[20, Sec 7] for example.

Our goal is to start doing algebra and geometry while working entirely in such a category. The key to this is a good notation, which is just the notation used by electronics engineers when writing wiring diagrams. Each operation is a box or node with inputs and outputs. These are joined up with wires. Obviously a lot of mathematics can be expressed in this way (and it has become fashionable in recent times in other contexts). We adopt the convention to write all maps with information generally flowing from top to bottom. As every engineer knows, it will generally be necessary at some points for wires to cross on our diagram. It means lexicographically that symbols are being transposed. The idea of braided mathematics is to

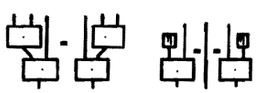
put at all such points the braiding Ψ for an over crossing and Ψ^{-1} for an undercrossing

Note that we did *not* suppose that these two isomorphisms $V \otimes W \to W \otimes V$ coincide. This is the main difference between braided mathematics and usual or super mathematics. The functoriality condition (2) now says exactly that we can take morphisms as boxes through braid crossings just as an engineer might pick up a wire and move it over or under to the other side of a chip. The coherence theorem for braided categories combined with this functoriality says that we can rearrange our operations or wiring diagrams in this way and always get the same answer if we do not cut any wires. We will use this in all our calculations below.

One can generalise these ideas to higher dimensions easily enough, but we shall content ourselves here with the above three-dimensional calculus. One more ingredient which is familiar for vector spaces and which we also need to generalise is that of duals. One can have left duals, right duals or both. They come with evaluation and coevalutation maps represented by cups or caps in the diagrammatic notation since $\underline{1}$ is represented by omission. Inclusion of these duals turns our braided calculus into a knotted one.

2. Braided algebras and groups

We work in a braided category as described informally in the preliminaries. Now a braided-algebra means B in our collection of objects and morphisms $\underline{\eta} : \underline{1} \to B$ and $\cdot : B \otimes B \to B$ obeying the usual properties of an associative algebra with unit, which we write as diagrams

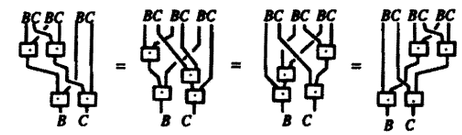


(3)

The fundamental construction which we need is

LEMMA 2.1. *[10] Let B, C be algebras in a braided category. Then there is an algebra $B \underline{\otimes} C$ also in the category, the* braided tensor product algebra. *It is built on $B \otimes C$ with product $(\cdot_B \otimes \cdot_C)\Psi_{C,B}$. Here B, C are subalgebras.*

The proof that this is associative is



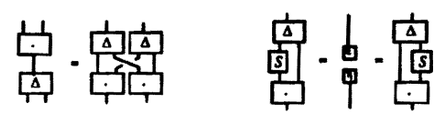
where the two blocks on the left are each the product in $B \underline{\otimes} C$ and we have to show that the combination is the same as the products in $B \underline{\otimes} C$ taken in the other order. The middle equality is associativity (3) in each of B, C. Note also that in the concrete case the product is explicitly

$$(a \otimes c)(b \otimes d) = a\Psi_{C,B}(c \otimes b)d, \qquad \forall a, b \in B, c, d \in C$$

in analogy with the super-tensor product of super-algebras.

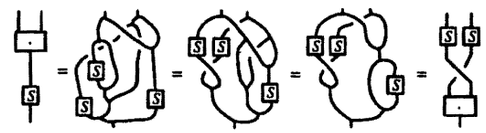
Just as easy is the notion of braided-coalgebra. This is an object B with counit $\underline{\varepsilon} : B \to \underline{1}$ and coproduct $\underline{\Delta} B \to B \underline{\otimes} B$ morphisms obeying just the axioms above turned up-side-down (turn the pages of this book upside down to see them).

DEFINITION 2.2. *[10] A braided-Hopf algebra or braided group is $(B, \underline{\varepsilon}, \underline{\Delta}, \underline{S})$ where B is a braided algebra and coalgebra with $\underline{\varepsilon} : B \to \underline{1}$, $\underline{\Delta} : B \to B \underline{\otimes} B$ algebra homomorphisms. The homomorphism property of $\underline{\Delta}$ and the axioms for the braided-antipode $\underline{S} : B \to B$ are*



We shall use the term 'braided group' fairly loosely to include the case without antipode also. There are also more restrictive notions of commutativity etc which can be studied under this general heading[10][11]. Our task now is to convince the reader that these braided groups naturally behave like groups or algebras of functions on groups. The latter is our motivation, but the axioms are self-dual and one can just as easily think of B as modelled on the group algebra $\mathbb{C}G$ generated by elements of a group G with $\underline{\Delta} g = g \otimes g$, $\underline{\varepsilon} g = 1$ and $\underline{S} g = g^{-1}$, and trivial braiding. One finds indeed that many (but not all) constructions familiar for groups or Hopf algebras work just as well in this braided setting. For example, $\underline{S}$ is like an 'inverse'. Usual inverses are

anti-multiplicative and likewise we have



Here we used the axioms and the braided wiring diagram techniques above with the additional shorthand of collapsing the boxes representing products or coproducts to nodes. See [16] for a full explanation of the proof. Another important construction for groups is that of the adjoint action. This and its action property are now

(4)

The definition here is given by breaking down how we think of the usual adjoint action for groups into steps, and writing the steps as boxes. One can have a lot of fun proving this and other main results of braided group theory in this way[16][17]. One has of course to be careful to choose consistently between under and over crossings and check every time that the constructions do not get tangled up. Suffice it to say that the braided-representation theory, braided actions and coactions and cross products, braided-invariant differentials, braided-invariant integrals etc all go through. One notion that takes care is that of braided-commutativity or cocommutativity which is introduced in [10][11] in a weak (but useful) form as relative to an action. Another problem where one can get tangled up is the notion of tensor product of braided groups. This can be done so far only in a small class of braided groups[21].

3. Braided lines, planes and matrices

Here we would like to introduce the reader right away to a whole bunch of examples. Many recent puzzles in the theory of quantum groups are resolved once one realises that the relevant structure to look for is not that of a quantum group but a braided one. Throughout, we let q be a generic parameter. We can work over any field. I would like to stress that the above abstract braided group theory led to the following examples of braided coproducts of interest to physicists, and not the other way around.

3.1. *Braided line.*

We take $B = \mathbf{C}[x]$ the polynomials in one variable. The braiding, counit and coproduct are

$$\Psi(x^m \otimes x^n) = q^{mn} x^n \otimes x^m, \quad \underline{\Delta} x = x \otimes 1 + 1 \otimes x, \quad \underline{\varepsilon} x = 0.$$

This innocent example is the structure underlying the familiar q-analysis[22]. For example, let $[m;q] = \frac{1-q^m}{1-q}$ then formally

$$e_q(x) = \sum_{m=0}^{\infty} \frac{x^m}{[m;q]!}, \qquad \underline{\Delta} e_q(x) = e_q(x) \otimes e_q(x), \quad \underline{\varepsilon} e_q(x) = 1.$$

This provides an abstract characterisation of the well-known q-exponential $e_q(x)$. It takes a Lie-algebra like element x to a group-like one.

3.2. *Free braided planes.*

We take $B = \mathbf{C}\langle x_i \rangle$ the free non-commutative algebra on generators $\mathbf{x} = (x_i)$ which we regard as a row vector. We let $R \in M_n \otimes M_n$ be an invertible matrix solution of the QYBE so $R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}$. The braiding, counit and coproduct are

$$\Psi(\mathbf{x}_1 \otimes \mathbf{x}_2) = \mathbf{x}_2 \otimes \mathbf{x}_1 R, \quad \underline{\Delta} \mathbf{x} = \mathbf{x} \otimes 1 + 1 \otimes \mathbf{x}, \quad \underline{\varepsilon} \mathbf{x} = 0.$$

Group-like elements are formally provided by the R-exponentials $\exp_R(\mathbf{x}|\mathbf{v})$ constructed later in (5) with $\mathbf{v} = (v^i)$ an ordinary vector. This is the natural generalisation to n-dimensions of the braided line and its plane-waves.

3.3. *Quantum planes.*

We can quotient the previous braided group by introducing relations of the form $\mathbf{x}_1\mathbf{x}_2 = \mathbf{x}_2\mathbf{x}_1 R'$ for certain matrices R'[13]. This is the general *braided covector algebra*. For example, in the Hecke case one can take $R' \propto R$ so we have a braided group structure on the usual quantum exchange algebras for $GL_q(n)$ etc.

For example, consider the celebrated quantum plane algebra $B = \mathbf{C}\langle x, y \rangle$ with the relations $yx = qxy$. As usual we can write the generators $\mathbf{x} = (x, y)$ as a row vector. They are like the co-ordinate functions on $\mathbf{C}^2$ as is well known. But how about adding such quantum row vectors? Our general construction provides us precisely such an addition law in the form of a braided group with linear coproduct as above. The braiding Ψ is given by the $GL_q(2)$ R-matrix.

Remember that the braided coproduct $\underline{\Delta}$ maps into the braided tensor product algebra $B \underline{\otimes} B$. Writing the generators of the second copy of B with

a prime, the algebra $B\underline{\otimes}B$ has the two copies of B as subalgebras and the cross relations or *braid statistics*

$$x'x = q^2xx',\ x'y = qyx',\ y'y = q^2yy',\ y'x = qxy' + (q^2-1)yx'.$$

The fact that $\underline{\Delta}$ is a homomorphism then says precisely that the vector sum

$$(x'', y'') = (x, y) + (x', y')$$

obeys the same quantum plane relations. The same construction works for the fermionic quantum plane where the statistics tend as $q \to 1$ to fermionic ones. There are also examples where the limit $q \to 1$ is a mixture of bosonic and fermionic co-ordinates. Any R-matrix normalised so that the corresponding braiding operator has an eigenvalue -1 leads to associated covector algebras like this[13].

3.4. *Quantum matrices.*

Now we come to the celebrated quantum matrices $A(R)$[7][8]. They have a matrix $\mathbf{t} = (t^i{}_j)$ of generators with the relations $R\mathbf{t}_1\mathbf{t}_2 = \mathbf{t}_2\mathbf{t}_1R$. Everybody knows now that they form a usual quantum group or bialgebra with coproduct $\Delta\mathbf{t} = \mathbf{t}\otimes\mathbf{t}$ corresponding to matrix multiplication. But how about the addition of quantum matrices? We show in [23] that one has

$$\Psi(\mathbf{t}_1\otimes\mathbf{t}_2) = R_{21}\mathbf{t}_2\otimes\mathbf{t}_1R,\quad \underline{\Delta}\mathbf{t} = \mathbf{t}\otimes 1 + 1\otimes\mathbf{t},\quad \underline{\varepsilon}\mathbf{t} = 0.$$

So we have both addition and multiplication of matrices on this algebra. The two are distributive in the form

$$(\mathrm{id}\otimes\cdot)\tau_{23}(\Delta\otimes\Delta)\underline{\Delta} = (\underline{\Delta}\otimes\mathrm{id})\Delta$$

$$(\cdot\otimes\mathrm{id})\tau_{23}(\Delta\otimes\Delta)\underline{\Delta} = (\mathrm{id}\otimes\underline{\Delta})\Delta$$

where τ_{23} is usual transposition of the middle two factors. This construction works for R of Hecke type, which includes the $GL_q(n)$ series. So at least in the Hecke case we have a *quantum ring*. It is a hybrid object part quantum group and part braided group. The same coaddition holds for the rectangular $m\times n$ quantum matrices[21] associated to a pair of Hecke R-matrices. This reduces to the quantum row or column vectors in the $1\times n$ or $m\times 1$ cases.

The usual 2×2 quantum matrices $M_q(2)$ are an obvious example and one can compute their explicit braid statistics as we did for the quantum plane. Then

$$\begin{pmatrix} a'' & b'' \\ c'' & d'' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$$

obeys the $M_q(2)$ relations when the a, a' etc have the right statistics. By contrast, their matrix product obeys the same algebra when the two copies commute. See [23] for further details and an application.

3.5. *Braided matrices and q-Minkowski space.*

Finally we come to the braided matrices $B(R)$ introduced by the author in [9] by transmutation of the quantum matrices. They have a matrix $\mathbf{u} = (u^i{}_j)$ of generators with the relations $R_{21}\mathbf{u}_1R\mathbf{u}_2 = \mathbf{u}_2R_{21}\mathbf{u}_1R$. There is a braided comultiplication

$$\Psi(R^{-1}\mathbf{u}_1\otimes R\mathbf{u}_2) = \mathbf{u}_2R^{-1}\otimes\mathbf{u}_1R,\quad \underline{\Delta}\mathbf{u} = \mathbf{u}\otimes\mathbf{u},\quad \underline{\varepsilon}\mathbf{u} = \mathrm{id}.$$

For example, for the 2×2 case we have the algebra $BM_q(2)$ in [9]. We can multiply them as

$$\begin{pmatrix} a'' & b'' \\ c'' & d'' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$$

provided we remember the corresponding braid statistics between two copies computed from Ψ. There is also a natural $*$-structure $\begin{pmatrix} a^* & b^* \\ c^* & d^* \end{pmatrix} = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$ appropriate to *hermitian matrices* and a bosonic and central braided determinant $\mathrm{BDET}(\mathbf{u}) = ad - q^2cb$. All of this makes this particular algebra ideally suited to serve as q-Minkowski space. The generators

$$x_0 = qd + q^{-1}a,\quad x_1 = \frac{b+c}{2},\quad x_2 = \frac{b-c}{2i},\quad x_3 = d - a$$

are some natural self-adjoint space-time co-ordinates while BDET comes out as

$$\frac{q^2}{(q^2+1)^2}x_0^2 - q^2x_1^2 - q^2x_2^2 - \frac{(q^4+1)q^2}{2(q^2+1)^2}x_3^2 + \left(\frac{q^2-1}{q^2+1}\right)^2\frac{q}{2}x_0x_3$$

and provides a q-deformed Lorentz metric. See [24] for details and an application. Conceptually, the transmutation process mentioned in the introduction naturally turns the usual Euclidean quantum 3-sphere $SU_q(2)$ into a Lorentzian one.

But what about the addition of these braided-Hermitian matrices, i.e. addition on q-Minkowski space? This has recently been found by U. Meyer in [25] in the braided form

$$\Psi(\mathbf{u}_1\otimes\mathbf{u}_2) = \mathbf{u}_2\otimes\mathbf{u}_1R_L,\quad \underline{\Delta}\mathbf{u} = \mathbf{u}\otimes 1 + 1\otimes\mathbf{u},\quad \underline{\varepsilon}\mathbf{u} = 0$$

where R_L is a new 16×16 solution of the QYBE and defines also a q-Lorentz group with respect to which the q-Minkowski space is covariant. The general scheme of [13] then ensures that there is also a q-Poincaré group (with dilatation) given by the semidirect product of the braided-addition by extended q-Lorentz transformations. See [25] for the details for this example.

4. Braided differential operators

We can now begin to do some braided geometry. The most basic notion is that of differential operator. We do not mean here one forms etc, which in our approach come later. Rather, we define as differentiation the operators corresponding to infinitesimal actions of braided groups on braided spaces. The simplest case is the braided covector algebras above. The coaddition $\underline{\Delta}$ corresponds to translation. Writing the shorthand $\mathbf{a} \equiv \mathbf{x} \otimes 1$ and $\mathbf{x} \equiv 1 \otimes \mathbf{x}$ we know that the braided tensor product $B \underline{\otimes} B$ is generated by $\mathbf{a}, \mathbf{x}$ with the cross relations $\mathbf{x}_1\mathbf{a}_2 = \mathbf{a}_2\mathbf{x}_1 R = \mathbf{a}_1\mathbf{x}_2(PR)$ in the compact notation. For the second form we used P the permutation matrix. Then we define $\partial^i : B \to B$ as the operator[14]

$$\partial^i f(\mathbf{x}) = \left(a_i^{-1}(f(\mathbf{a}+\mathbf{x}) - f(\mathbf{x}))\right)_{\mathbf{a}=0} \equiv \text{coeff of } a_i \text{ in } f(\mathbf{a}+\mathbf{x}).$$

This is just the definition due to Leibniz or Newton of differentiation as infinitesimal translation! We have

$$\begin{aligned}\partial^i(\mathbf{x}_1 \cdots \mathbf{x}_m) &= \text{coeff}_{a_i}((\mathbf{a}_1+\mathbf{x}_1)(\mathbf{a}_2+\mathbf{x}_2)\cdots(\mathbf{a}_m+\mathbf{x}_m)) \\ &= \text{coeff}_{a_i}(\mathbf{a}_1\mathbf{x}_2\cdots\mathbf{x}_m + \cdots + \mathbf{x}_1\cdots\mathbf{x}_{m-1}\mathbf{a}_m) \\ &= \text{coeff}_{a_i}\mathbf{a}_1\mathbf{x}_2\cdots\mathbf{x}_m[m;R]_{1\cdots m} \\ &= \mathbf{e}^i{}_1\mathbf{x}_2\cdots\mathbf{x}_m[m;R]_{1\cdots m}\end{aligned}$$

where $\mathbf{e}^i$ is a basis covector $(\mathbf{e}^i)_j = \delta^i{}_j$, i.e.

$$\partial^i x_{i_1}\cdots x_{i_m} = \delta^i{}_{j_1}x_{j_2}\cdots x_{j_m}[m;R]^{j_1\cdots j_m}_{i_1\cdots i_m}$$

and

$$[m;R] = 1 + (PR)_{12} + (PR)_{12}(PR)_{23} + \cdots + (PR)_{12}\cdots(PR)_{m-1,m}$$

is a certain matrix living in the m-fold matrix tensor product. We call these matrices *braided integers*.

In the one-dimensional case $R = (q)$ we recover the usual q-differential $\partial f(x) = \frac{f(x)-f(qx)}{(1-q)x}$. So, the celebrated q-differential is derived from something deeper, namely as an infinitesimal translation on the braided line. Its q-derivation property is interpreted now as a braided-Leibniz rule where ∂ has degree -1. The general form is

$$\partial^i(ab) = (\partial^i a)b + \Psi^{-1}(\partial^i \otimes a)b$$

$$\Psi^{-1}(\partial^i \otimes \mathbf{x}_1\cdots\mathbf{x}_r) = \mathbf{e}^i{}_1\mathbf{x}_2\cdots\mathbf{x}_r\mathbf{x}_{r+1}(PR)_{12}\cdots(PR)_{r,r+1} \otimes \partial_{r+1}.$$

Moreover, the ∂^i obey the relations of the associated quantum vector algebra of the form $\partial_1\partial_2 = R'\partial_2\partial_1$. The vector algebras here are defined like the covector algebras in Section 3.3 and have a linear braided coproduct.

In the free case of Section 3.2 we can assume that the braided integers are all invertible. Then[14]

$$\exp_R(\mathbf{x}|\mathbf{v}) = \sum_{m=0}^{\infty} \mathbf{x}_1\cdots\mathbf{x}_m[m;R]^{-1}_{1\cdots m}\cdots[2;R]^{-1}_{m-1,m}\mathbf{v}_m\cdots\mathbf{v}_1. \tag{5}$$

is an eigenfunction $\partial^i \exp_R(\mathbf{x}|\mathbf{v}) = \exp_R(\mathbf{x}|\mathbf{v})v^i$. The eigenvalues v^i can be usual numbers. With the ordering shown, they could also be elements of a non-commutative algebra, such as the vector algebra. Using this, one can prove a braided Taylor's theorem

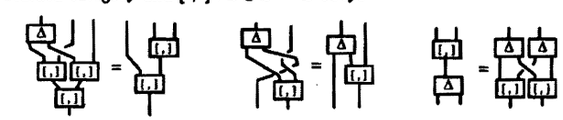
$$\exp_R(\mathbf{a}|\partial)f(\mathbf{x}) = f(\mathbf{a}+\mathbf{x}).$$

A general vector field is of the form $f_i(\mathbf{x})\partial^i$ i.e. an element of the vector space $B \otimes B^*$ where B^* denotes the quantum vector algebra. Since B^* acts on B one may make a braided semidirect product $B \rtimes B^*$ which is the braided Heisenberg-Weyl algebra. This is the basis of R-quantum mechanics and recovers the usual q-quantum mechanics algebra $px - qxp = 1$ in the one-dimensional case. On the other hand, all these constructions work quite generally. One may define left-invariant vector fields etc in a similar way on any braided group.

5. Braided Lie algebras

Many authors have sought some form of Lie-algebra like structure for the quantum groups $U_q(g)$. One even knows good generators for this purpose[8] but what axioms should they obey? The right ones appear to be the following, coming out of braided geometry.

We begin by studying the formal properties of the braided-adjoint action (4) of a braided group on itself. We suppose also that this action is cocommutative in the sense of [10] since of course a usual enveloping algebra would be cocommutative. This is the key assumption and one finds various properties of the map $[\ ,\ ] = \text{Ad} : B \otimes B \to B$. If $\mathcal{L} \subset B$ is stable under Ad and $\underline{\Delta}$ then these same properties apply to $\mathcal{L}$ without knowing all of B. This gives the following abstract notion of a braided-Lie algebra[15]. Namely, a braided-Lie algebra is $(\mathcal{L}, \underline{\Delta}, \underline{\varepsilon}, [\ ,\ ])$ where the first three are a coalgebra in a braided category and $[\ ,\ ] : \mathcal{L} \otimes \mathcal{L} \to \mathcal{L}$ obeys



(6)

Note that for ordinary Lie algebras there is an implicit additive coalgebra $\Delta\xi = \xi \otimes 1 + 1 \otimes \xi$ in the Lie algebra with 1 adjoined. If one puts this

form into (6) one has something a bit more classical-looking. But we are not tied to this form. Thus, there are many forms or flavours of braided-Lie algebra depending on what form of $\underline{\Delta}$ we assume. A convenient one is the multiplicative form $\underline{\Delta}\mathbf{u} = \mathbf{u}\otimes\mathbf{u}$. The braiding as in Section 3.5 and a suitable bracket[26] fulfill the above axioms in this form.

The main result about our notion of braided-Lie algebra, not possible in other attempts so far, is that we have also an enveloping algebra functor U that associates to a braided-Lie algebra $\mathcal{L}$ a braided group $U(\mathcal{L})$. See [15] for details. Again, in general this is done diagrammatically. In the example above we have $U(\mathcal{L}) = B(R)$ the braided matrices again!. So these braided groups are both like the algebras of functions on a braided space and like the enveloping algebra of a braided-Lie algebra. If we change variables to $\chi = \mathbf{u} - \mathrm{id}$ then the coalgebra and relations of $U(\mathcal{L})$ are

$$\underline{\Delta}\chi = \chi\otimes 1 + 1\otimes\chi + \chi\otimes\chi, \qquad R_{21}\chi_1 R\chi_2 - \chi_2 R_{21}\chi_1 R = Q\chi_2 - \chi_2 Q$$

where $Q = R_{21}R$. Typically as $q\to 1$ the matrix Q becomes close to 1, and it is the rescaled generators $\bar{\chi} = \hbar^{-1}\chi$ that tend to a usual Lie algebra. This is exactly how a braided-Lie algebra deforms a usual Lie algebra. Moreover, for general q one has the algebra of $U_q(g)$ as isomorphic to $U(\mathcal{L})$ modulo further determinant-like relations[15].

Without dwelling on this general theory we give an example, namely the braided Lie algebra $\underline{gl_2}$. This is the four-dimensional vector space $\mathcal{L} = \mathrm{span}\{a,b,c,d\}$ with matrix coalgebra and braiding as in Section 3.5. This is an example of the general matrix-Lie algebras described above. For comparison with the classical limit it is also convenient to change variables to

$$\begin{pmatrix} H \\ X \\ Y \\ \gamma \end{pmatrix} = (q^2-1)^{-1}\begin{pmatrix} d-a \\ b \\ c \\ q^{-2}(a-1)+d-1 \end{pmatrix} \tag{7}$$

which all lie in the kernel of $\underline{\epsilon}$. In terms of these, the general braided-Lie bracket in [15] becomes

$$\begin{aligned}
&[H,X] = (q^{-2}+1)X = -q^2[X,H]\\
&[H,Y] = -(q^{-2}+1)q^{-2}Y = -q^{-2}[Y,H]\\
&[X,Y] = q^{-2}H = -[Y,X]\\
&[H,H] = (1-q^{-4})H, \quad [\gamma,\begin{Bmatrix} H \\ X \\ Y \end{Bmatrix}] = (1-q^{-4})\begin{Bmatrix} H \\ X \\ Y \end{Bmatrix}
\end{aligned}$$

with zero for the others. We see that for $q\to 1$ the γ mode decouples and we have the Lie algebra $su_2\oplus u(1)$, but for $q\neq 1$ these are unified. There is also

a braided-Killing form provided by the braided-trace of the braided-adjoint representation[15] which comes out in the above basis as

$$q^{-4}\begin{pmatrix} [4;q^{-2}][2;q^{-2}] & 0 & 0 & 0 \\ 0 & 0 & q^{-2}[4;q^{-2}] & 0 \\ 0 & [4;q^{-2}] & 0 & 0 \\ 0 & 0 & 0 & q^2(1-q^{-4})^2[3;q^{-2}] \end{pmatrix}$$

Thus the natural braided-Killing form on $\underline{gl_2}$ is non-degenerate for generic $q\neq 1$, i.e. the su_2 and $u(1)$ are truly unified. Moreover, as $q\to 1$ we recover the usual degenerate Killing form for $su_2\oplus u(1)$. This means that in a q-deformed electroweak gauge theory with bare Lagrangian based on the norm of a $\underline{gl_2}$ curvature tensor via this Killing form, the $u(1)$ part would decouple as $q\to 1$ and enter with infinite bare coupling constant into the Lagrangian. This suggests the possibility indeed of q-regularising electroweak theory. This is a direction for further development.

Finally, the braided-enveloping algebra comes out as our old 2×2 braided-matrices $BM_q(2)$ in Section 3.5. So

$$U(\underline{gl_2}) \cong q - \text{Minkowski Space}, \qquad q\neq 1.$$

This braided group isomorphism is an example of a ***purely quantum phenomenon*** in the sense that it holds only for generic $q\neq 1$. As $q\to 1$ the left hand side tends as we saw to $U(su_2\oplus u(1))$ a classical enveloping algebra. The right hand side tends by contrast to the algebra of co-ordinate functions on Minkowski spacetime! The H,X,Y,γ generators are needed for the first interpretation, and the a,b,c,d generators for the second. Their connection in (7) is singular as $q\to 1$.

This kind of unification of two completely different classical concepts is attributable to a remarkable self-duality property of a number of braided-groups. This in turn is made possible by the self-duality (or input-output symmetry) of the very axioms of a braided-Hopf algebra or indeed of an ordinary Hopf algebra. If $q\neq 1$ due to quantum corrections to the classical geometry, then we might expect this self-duality to be physically manifest at the Planck scale. This is a genuinely new phenomenon in physics which might in principle be experimentally verified. The same self-duality phenomenon is also known for the Hopf algebras in [3] arising as quantisations of certain homogeneous spaces, where it was interpreted physically as a symmetry between observables and states. Moreover, the requirement of self-duality there forced one into a singular dynamics roughly resembling motion in a black hole background.

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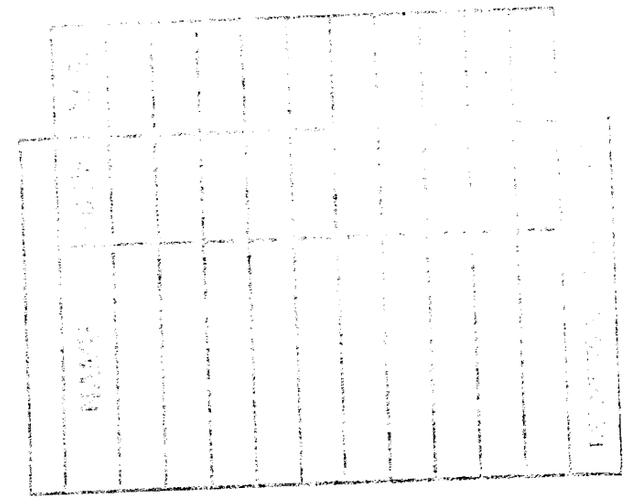
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