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The Dressed Sphere: An Exactly Solved Model of Resonance in Potential Scattering

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Abstract

A dressed boundary arises from a rigid one by placing on the physical region side of it a suitably parametrised narrow deep potential well. Some examples of quantum mechanics in one dimension in the presence of dressed boundaries are followed by an exact treatment of scattering by a dressed sphere. Results for s- and p-wave amplitudes are used to illustrate a variety of issues regarding the occurrence of and role of bound state and resonance poles of scattering amplitudes in scattering. A careful derivation of a Breit-Wigner resonance formula is described.

1. Introduction

The phenomenon of resonance is a topic in the quantum mechanics of scattering by a potential that is often found hard to expound effectively and yet at the same time is of considerable importance. The scarcity of examples that can be treated explicitly in sufficient detail accounts for much of the difficulty. Thus it is hoped that the provision of a new example in which resonance features explicitly and simply is helpful.

The purpose of the present paper is to examine an example, here called the *dressed sphere*, in which we not only can determine conditions under which a resonance occurs but also give a good description of it when these conditions are satisfied.

We begin with an explanation of what we mean by the term dressed sphere. We set out from the central potential

$$\begin{aligned} V(r) &= \infty, \quad 0 < r < a, \\ &= -\hbar^2\mu^2/(2m), \quad a < r < b, \\ &= 0, \quad b < r. \end{aligned} \tag{1}$$

For fixed a, we specify the quantities $\mu(>0)$ and b by writing

$$b - a = \frac{\pi}{2}a\gamma\epsilon, \quad \mu(b-a) = \frac{\pi}{2} + \eta\epsilon, \tag{2}$$

where η is allowed only to take either the value $+1$ or -1, and γ is a real positive *dressing* parameter. The dressed sphere is realised now by passing to the limit $\epsilon \to 0$. Since (2) now implies

$$\mu = \frac{1}{a\gamma}\left(\frac{1}{\epsilon} + \frac{2\eta}{\pi}\right), \tag{3}$$

we see that the dressed sphere consists of a hard sphere of radius a, with a suitably parametrised narrow deep well just outside it. It involves the dressing parameter γ, together with the parameter η, which enters because our analysis of scattering brings in the expression $\tan\mu b$ and hence depends critically, via (2), on the sign of η. Our work below determines, amongst other things the conditions on η and γ under which a resonance appears in p-wave scattering by a dressed sphere.

Motivation for the present study came originally from the use [1] of a similar dressing procedure at the boundary of an interval of the real axis to which are confined quantum mechanical particles moving in one dimension. The context of this use is in the theory of an integrable many body system made famous in [2]; this is soluble by Bethe ansatz methods, and the aim was to see whether or not changing boundary conditions destroyed integrability properties. In [1] it is indicated that, whereas a plane wave is reflected at a rigid boundary with only a change of phase, a more general reflection coefficient, given explicitly, is possible if the rigid boundary is preceded by a suitably parametrised narrow, deep well. In fact, an equivalent but more abstract characterisation of a dressed boundary is available. This states that the wave function $\psi(x)$ at a dressed boundary $x = 0$ is subject to a boundary condition of the form

$$\psi(0) = \nu\psi'(0), \tag{4}$$

in which ν is a real parameter. Of course, if a dressing procedure is specified in detail, then ν is determined in terms of its parameters.

It is of interest to note that the boundary condition (4) has recently occurred in a totally different context of current theoretical interest, the theory of systems of anyons [3]. There the relative motion of two anyons moving in one dimension is seen to be governed by a Schrödinger equation on the half line $x > 0$ with a boundary condition like (4), except that now ν is a parameter that depends on the anyon statistics

The contents of the paper are as follows. Section two describes the dressing procedure by reference to simple examples in one dimension. In section three we introduce the dressed sphere and calculate exactly the s- and p-wave phase-shifts for scattering by it. In section four, there is discussion of the poles of the corresponding s- and p-wave scattering amplitudes, and their implications for bound states and resonances, correlation of model parameters with their occurrence being described in detail. Section five turns to the careful derivation of a Breit-Wigner form of the p-wave scattering cross-section when this is dominated by a single low energy resonance.

Standard textbook background to this paper is available in many excellent books, for example [4] or [5]. We refer to [6] for additional discussion of several aspects of potential scattering related to the pole properties of scattering amplitudes in the complex plane $\mathbf{C}_k$ of the wave-number k of the scattering process.

2. Examples in One Dimension

We consider first a quantum mechanical particle strictly confined to the interval $-a \leq x \leq b$, where $a, b \geq 0$, and subject there to the potential

$$V(x) = 0, \quad -a \leq x \leq 0,$$
$$V(x) = -\hbar^2\mu^2/(2m), \quad 0 \leq x \leq b,$$

where μ is real and positive. We now take a limit like that specified above in which $\mu \to \infty$ and $b \to 0$. More precisely we set

$$b = \frac{\pi}{2}a\gamma\epsilon, \quad \mu b = \frac{\pi}{2} + \eta\epsilon, \tag{5}$$

so that also

$$\mu = \frac{1}{a\gamma}(\frac{1}{\epsilon} + \frac{2\eta}{\pi}). \tag{6}$$

Herein η and γ are as defined in section one, and we take the limit in which $\epsilon \to 0$ through positive values. The reasons why we have defined our dressing procedure in the way just described soon emerge clearly when we solve our quantum mechanical problem for its bound state solutions. Looking first for solutions of finite positive energy $E = \hbar^2k^2/(2m)$, we define $f^2 = \mu^2 + k^2$, and chose wave functions of the form

$$\begin{aligned} \psi_I &= A\sin k(x+a), \quad -a \leq x < 0, \\ \psi_{II} &= B\sin f(x-b), \quad 0 < x \leq b. \end{aligned} \tag{7}$$

The continuity of ψ and ψ' at $x = 0$ then yields

$$\frac{\tan ka}{k} = -\frac{\tan fb}{f}. \tag{8}$$

As we seek solutions that are of finite k, and hence E, in the limit $\epsilon \to 0$, we may directly replace f by μ in (8). Hence, using also (5), we find

$$\tan fb \approx \tan\mu b = -\cot\eta\epsilon = -\frac{1}{\eta\epsilon} + O(\epsilon). \tag{9}$$

From (9) and (6), we see that the right side of (8) reduces effectively to

$$\left(\frac{1}{\eta\epsilon}\right) / \left(\frac{1}{a\gamma\epsilon}\right),$$

which is finite in the limit as $\epsilon \to 0$, as required. Thus the equation which determines the discretely distributed allowed values of k and hence the positive energy part of the spectrum of E is

$$\frac{\tan ka}{ka} = \eta\gamma. \tag{10}$$

One seeks information about the solutions of (10) by well-known graphical procedures involving intersections of the functions $y = \tan x$, $y = \eta\gamma x$, where $x = ka$. For $\eta = \pm 1$, it is thereby seen that (10) has roots which approach $(2r \pm 1)\frac{\pi}{2}$ as r goes to ∞ through positive integral values. Further, there is a root in $0 < ka < \frac{\pi}{2}$ if $\eta = 1$ and $\gamma > 1$, and no solution in the interval in question otherwise.

Turning to negative energy solutions of our problem, we use

$$\begin{aligned} \psi_I &= A\sinh\alpha(x+a), \\ \psi_{II} &= B\sinh\beta(x-b), \end{aligned} \tag{11}$$

where $E = -\hbar^2\alpha^2/(2m)$, and $\beta^2 = \mu^2 - \alpha^2$. By following routine steps and taking the limit $\epsilon \to 0$ in the same way as above, we are led to the condition

$$\frac{\tanh\alpha a}{\alpha a} = \eta\gamma,$$

for discrete solutions. This time the nature of the tanh function tells us there is one solution if $\eta = 1$ and $\gamma \leq 1$ and no solutions otherwise.

We can now give a more abstract specification of our problem. We see that (7) and (10) imply

$$\psi'(0_-) = Ak\cos ka = k\cot ka\,\psi(0_-) = (a\gamma\eta)^{-1}\psi(0_-),$$

where 0_- indicates a value reached by approaching 0 through negative values. In other words a boundary condition of the form (4) is satisfied. Similar considerations apply to a solution of the type (11) whenever one such exists. Thus our model can also be viewed as a physical realisation of the quantum mechanical problem of a particle confined to the interval $-a \leq x \leq 0$, in which a boundary condition (4) applies at $x = 0_-$.

An interesting view of the orthogonality of the solutions of the problem follows by doing two integrations by parts on the quantity

$$I_{12} = \int_{-a}^{0} \sin k_1(x+a)\sin k_2(x+a)\,dx,$$

where k_1 and k_2 are distinct roots of (11). One finds

$$(k_1^2 - k_2^2)I_{12} = k_2\sin k_1a\cos k_2a - k_1\cos k_1a\sin k_2a.$$

Then $I_{12} = 0$ follows from

$$\frac{\tan k_1a}{k_1a} = \frac{\tan k_2a}{k_2a}.$$

A similar proof applies to other cases. One can also normalise the solutions found explicitly and consider the classical limit of large r, where r indexes the roots k_r of (11). It is no surprise to find that the dressing procedure goes unnoticed in the classical limit.

There are of course many other problems of (class-room?) interest that arise by replacing a rigid boundary by a dressed one. For example [1], a wave e^{ikx} incident form $x < 0$ upon $x = 0$ will, when we describe the dressed boundary by the boundary condition

$\psi(0) = \nu\psi'(0)$, no longer be accompanied by the simple reflected wave $-e^{-ikx}$. Rather the solution in $x \leq 0$ will be

$$\psi = e^{ikx} + Re^{-ikx},$$

where

$$R = -\frac{1 - i\nu k}{1 + i\nu k}.$$

The reflection coefficient R has a pole at $k = i/\nu$ which lies on the positive half of the imaginary axis of the complex k-plane if and only if ν is positive. It should therefore follow that our problem admits a bound state of energy $E = -\hbar^2/(2m\nu^2)$ under exactly these conditions. We leave it to the reader to verify that it does so follow. For valuable background information on scattering and bound state problems with special emphasis on one dimension, see the section entitled Scattering theory for Pedestrians in [7].

3. The Dressed Sphere

The dressed sphere has been described in detail in section one. Here we study the scattering of a particle of energy $E = \hbar^2k^2/(2m)$ by a dressed sphere, intending to evaluate s- and p-waves amplitudes explicitly and investigate bound state and resonance properties of these in detail. The radial Schrödinger equation of the potential (1) is well-known [4,5] to admit solutions of the type

$$\begin{aligned} R_{l1} &= c_l\{j_l(Kr)\cos\phi_l - n_l(Kr)\sin\phi_l\}, \quad a \leq r \leq b, \\ R_{l2} &= d_l\{j_l(kr)\cos\delta_l - n_l(kr)\sin\delta_l\}, \quad b \leq r, \end{aligned} \tag{12}$$

where the δ_l are the phase shifts, as usually [4,5] defined, c_l, d_l and ϕ_l are constants and $K^2 = k^2 + \mu^2$. We use also the boundary condition

$$R_{l1}(a) = 0, \tag{13}$$

and, from continuity at $r = b$, the equation

$$Q_{l1}(b) = Q_{l2}(b), \tag{14}$$

where, for $i = 1, 2$, we write

$$Q_{li}(r) = \frac{1}{R_{li}(r)}\frac{\partial R_{li}(r)}{\partial r}. \tag{15}$$

To proceed further it is perhaps most convenient to treat s- and p-waves separately in turn. The explicit forms of the spherical Bessel functions that we need are given, for example, on p197 of [4].

For s-waves, $l = 0$, (12) and (13) give

$$R_{01}(a) = c_0\sin(Ka + \phi_0)/(Ka) = 0, \tag{16}$$

and (16) is satisfied by

$$\phi_0 = -Ka. \tag{17}$$

Then, using (12) to evaluate $Q_{01}(b)$, and (17) to eliminate ϕ_0 from the answer,one finds

$$Q_{01}(b) = K\cot K(b - a) - \frac{1}{b}. \tag{18}$$

As we wish to deal with solutions that are of finite k in the dressing limit, we may replace K by μ in evaluating $Q_{01}(b)$. Thus, we note first, using (2) and (3)

$$\mu b = \mu(b - a) + \mu a = \frac{1}{\gamma\epsilon} + O(1). \tag{19}$$

Next we obtain

$$\frac{\mu b}{\tan\mu(b - a)} = \{\frac{1}{\gamma\epsilon} + O(1)\}/\{\frac{-1}{\eta\epsilon} + O(\epsilon)\} = \frac{-1}{\gamma\eta} + O(\epsilon), \tag{20}$$

by using (19) on the numerator, and using (2) to treat the denominator in the manner shown in (9). Eqs. (18) and (20) thereby show us that $Q_{01}(b)$ approaches the finite limiting value

$$-(1 + \gamma\eta)/(a\gamma\eta)$$

in the dressing limit. Turning to the right side of (14), we see that we may let b approach a in it directly. Then (14) in the dressing limit reads

$$-(1 + 1/(\eta\gamma)) = ka\cot(ka + \delta_0) - 1,$$

or

$$\tan(ka + \delta_0) = -ka\eta\gamma. \tag{21}$$

Our analysis thus shows that, in the dressing limit, the interval $a \leq r \leq b$ can be disregarded, provided that for the physical region $r > b \approx a$ we use the boundary condition

$$Q_{02}(a) = -(1 + \gamma\eta)/(a\gamma\eta). \tag{22}$$

This shows that here, as in the one dimensional examples, we have an alternative, equivalent formulation of the problem just solved. It consists in discussing the (s-wave) scattering by a sphere of radius a that differs only from a hard sphere of radius a in that the boundary condition imposed on the radial function is not $R_{02}(a) = 0$ but instead is obtained from (15) and (22).

We come at last to the topic of greatest interest, p-wave scattering. It is of course harder to treat than was the s-wave case.

For p-waves, $l = 1$, (12) becomes

$$R_{11}(a) = c_1\{\frac{\sin(Ka + \phi_1)}{K^2a^2} - \frac{\cos(Ka + \phi_1)}{Ka}\},$$

so that (13) yields

$$\tan(Ka + \phi_1) = Ka. \tag{23}$$

Next, we calculate

$$Q_{11}(b) = \frac{2Kb + T(K^2b^2 - 2)}{b(T - Kb)}, \tag{24}$$

where an abbreviation $T = \tan(Kb + \phi_1)$ has been used. To eliminate ϕ_1, we write

$$T = \tan\{K(b-a) + (Ka + \phi_1)\} = \frac{\tan K(b-a) + Ka}{1 - Ka\tan K(b-a)},$$

when obvious use of (23) is made. To pass towards the dressing limit, we replace K by μ as before, and use (2) and (3) to obtain

$$T = \{\frac{-1}{\eta\epsilon} + \frac{1}{\gamma\epsilon} + O(1)\}/\{\frac{1}{\eta\gamma\epsilon^2} + O(1/\epsilon)\} = (\eta - \gamma)\epsilon + O(\epsilon^2).$$

Hence, from (24), we obtain

$$Q_{11}(b) = \{\frac{2}{\gamma\epsilon} + (\eta - \gamma)\epsilon\frac{1}{\gamma^2\epsilon^2} + O(1)\}/\{\frac{-a}{\gamma\epsilon} + O(1)\},$$

so that finally the limiting value of Q_{11} is

$$Q_{11} = -(1 + \gamma\eta)/(a\gamma\eta). \tag{25}$$

Thus to write (14) for $l = 1$ in the dressing limit, we may let b approach a on the right side. This leads to

$$\frac{-(1+\gamma\eta)}{a\gamma\eta} = \frac{2ka + t(k^2a^2 - 2)}{a(t - ka)}, \tag{26}$$

where the abbreviation $t = \tan(ka + \delta_1)$ has been used. We remark that the same steps that were used in the evaluation of (24) above have been used here to produce a right-side for (26) of very similar appearance. Rearrangement of (26) now produces the final result

$$\tan(ka + \delta_1) = \frac{ka\{1 - (\gamma\eta)^{-1}\}}{1 - (\gamma\eta)^{-1} - k^2a^2}. \tag{27}$$

We note that the right sides of (22) and (25) are equal. This is no coincidence. One can actually show, by a fairly delicate calculation, involving the small x behaviours of $j_l(x)$ and $n_l(x)$, that all $Q_l(a)$ have the same value in the dressing limit.

Discussion of (21) and (27) is given in section four.

4. Scattering Amplitudes and Poles

The scattering amplitude S_l for the l-th partial wave of scattering by a given potential is defined by

$$S_l = e^{2i\delta_l} = \frac{1 + i\tan\delta_l}{1 - i\tan\delta_l}. \tag{28}$$

Also useful is the equivalent result

$$e^{2ika}S_l = \frac{1 + i\tan(ka + \delta_l)}{1 - i\tan(ka + \delta_l)}, \tag{29}$$

because of the form in which the actual results of calculations, such as (21) and (27), emerge. Poles of S_l occur whenever $\tan\delta_l = -i$. When this is satisfied we have also $\tan(ka + \delta_l) = -i$, in agreement with (29). It is well-known, [6], p62 and p79, that S_l is a meromorphic function in the complex plane $\mathbf{C}_k$ of k, with simple poles of two distinct types. One type of pole lies on the imaginary axis of $\mathbf{C}_k$. This corresponds to a bound state in the l-th partial wave of the given potential only, [6], p78 and p204, if it lies on the positive half of the axis. The other type of pole is a complex pole in the lower half of $\mathbf{C}_k$. A complex pole with $\mathrm{Im}k < 0$ and $\mathrm{Re}k > 0$ corresponds, [6], p156, to a resonance in S_l. If one such exists it is accompanied by a pole obtained from it by reflection in the imaginary axis of $\mathbf{C}_k$.

Our results from section three for the dressed sphere give rise to a good number of illustrations of the general facts just summarised. We turn first to S_0. It follows from (27) and (21) that the equation for the poles of S_0 for the dressed sphere are given by the simple condition

$$ka\gamma\eta = i, \tag{30}$$

So there is one pole of the type mentioned first above, and it corresponds to an s-wave bound state for the dressed sphere only if $\eta = 1$. We leave it to the reader to follow standard methods to solve explicitly the Schrödinger equation in question for its s-wave bound states. The required treatment of the dressing limit is essentially the same for this case as we have already described for the positive energy or scattering case. One finds that if $\eta = 1$ then there is exactly one bound state whose energy follow by substituting the value of k given by (30) in $E = \hbar^2k^2/(2m)$, and no bound state if $\eta = -1$.

We note that $e^{2ika}S_0$ is completely determined by its pole. It can be written in Blaschke form, or canonical product form, [6], p94, using (29) and (24), as

$$e^{2ika}S_0 = \frac{1 + k/(i\kappa_P)}{1 - k/(i\kappa_P)}, \tag{31}$$

where we have rewritten (30) as $k = i\kappa_P$, $\kappa_P = 1/(a\gamma\eta)$.

Next we turn to the p-wave amplitude S_1. From (29) and (27), we see that the poles of S_1 are obtained by solution of the equation

$$x^2 + 2isx - 2s = 0, \tag{32}$$

Here we have made the definitions $2s = 1 - 1/(\gamma\eta)$, and $x = ka$. We will also deal sometimes with $\mathbf{C}_x$, the complex plane of x, instead of with $\mathbf{C}_k$. Eq. (32) has roots

$$x_\pm = -is \pm (2s - s^2)^{\frac{1}{2}}. \tag{33}$$

Two cases, which correspond to the two types of pole mentioned in the general discussion above, arise.

Case I: $x_\pm$ pure imaginary.

This case arises when $s(2-s) \leq 0$ and gives two sub-cases

Case I(a): $s \leq 0$, or $\eta = 1$ and $\gamma \leq 1$.

Case I(b): $s \geq 2$, or $\eta = -1$ and $\gamma \leq \frac{1}{3}$.

If we write $x_\pm = ia\kappa_{P\pm}$, then the poles in Case I correspond to bound states only if $\kappa_{P\pm}$ is positive. In fact this can be seen to obtain only in the case of κ_{P+} in Case I(a).

For both subcases of Case I, we can use (27) to evaluate (29) obtaining

$$e^{2ika}S_1 = \frac{2s - x^2 + 2ixs}{2s - x^2 - 2ixs} = \left(\frac{1 + k/(i\kappa_{P+})}{1 - k/(i\kappa_{P+})}\right)\left(\frac{1 + k/(i\kappa_{P-})}{1 - k/(i\kappa_{P-})},\right) \tag{34}$$

in Blaschke form.

Case II: $x_\pm$ complex.

In Case II the word complex implicitly excludes pure imaginary, which is covered by Case I, for it inconvenient to have to keep mentioning this.

This case arises when $0 \leq s \leq 2$, that is, when the square root in (33) yields the real part of $x_\pm$. Thus (33) implies that $\mathrm{Im}x_\pm$ is negative throughout case II. This illustrates an important point from the general discussion: that any complex poles that are not on the imaginary axis of $\mathbf{C}_x$ must be in the lower half of $\mathbf{C}_x$. Moreover, we may expect to interpret x_+ as the resonance pole, [6], p156, in the fourth quadrant of $\mathbf{C}_x$, with x_- its partner in the third. In terms of η and γ, case II yields two sub-cases.

Case II(a): $\eta = 1$ and $\gamma > 1$.

Case II(b): $\eta = -1$ and $\gamma > \frac{1}{3}$.

In Case II we can describe the poles $x_\pm$ geometrically in $\mathbf{C}_x$. They lie on the parts of a certain circle that lie in the fourth and third quadrants of $\mathbf{C}_x$. If we write $x_\pm = p_\pm + iq_\pm$, then (33) leads easily to

$$p_\pm^2 + (q_\pm + 1)^2 = 1. \tag{35}$$

Thus, as s varies over its range for Case II, the resonance pole x_+ traverses that half of the circle (35) of unit radius, centred at $x = -i$, which lies in the fourth quadrant of the $\mathbf{C}_x$. This picture will help in section five in the discussion of when a resonance pole causes its typically marked effect on the p-wave contribution to the cross-section for scattering by a dressed sphere.

To conclude, we write S_1 in Blaschke form, [6], p94. Set $x_+ = k_0 a$, so that k_0 gives the position of the resonance pole in $\mathbf{C}_k$. Then $x_- = -x_+^*$, and hence we obtain

$$e^{2ika}S_1 = \left(\frac{1 - k/k_0^*}{1 - k/k_0}\right)\left(\frac{1 + k/k_0}{1 + k/k_0^*}\right). \tag{36}$$

So again, apart from the phase factor that we have chosen to place on the left of such equations as (31), (34) and (36), the amplitude is completely determined in terms of its pole structure. For higher partial waves, a similar pattern is expected on general grounds. While we can calculate amplitudes of higher l explicitly, we cannot solve analytically the polynomial equations, of degree l, that determine their poles. But we expect to find one quotient of the type shown in (31) for each pole of the first type and a contribution like the entire right side of (36) for each pole k_0 of the second type.

5. Breit-Wigner Resonance Formula

We shall be content to examine one physical context in which the p-wave amplitude S_1 for scattering by a dressed sphere is dominated by a single resonance. In this context, we shall discuss the representation of the p-wave contribution σ_1 to the total scattering cross-section in Breit-Wigner form. One expects, [6], p156, a resonance pole of S_1 in $\mathbf{C}_k$, which is both sufficiently close to the real axis of $\mathbf{C}_k$ and well isolated from any other resonance pole, to give rise to a Breit-Wigner formula for σ_1 over a suitable interval of k-values. In the dressed sphere example, there can at most be one resonance pole. Chosing a model, within case II of section four, for which there is one such pole, it is therefore necessary only to ensure that it does indeed lie close to the real axis of k.

From (33), it can be seen that S_1 has a pole at k_0, where

$$k_0 a = x_0 = -is + \Delta, \quad \Delta = (2s - s^2)^{\frac{1}{2}}. \tag{37}$$

This pole is a resonance pole, in the fourth quadrant of $\mathbf{C}_k$, if $0 < s \leq 2$. It will lie close to the real axis of $\mathbf{C}_k$ if we chose s so that

$$0 < s \ll 1, \tag{38}$$

in which case we deal with a model belonging to Case II(a) of section four. In this model, $\eta = 1$ and $2s = 1 - \gamma^{-1}$, i.e.

$$\eta = 1, \quad \gamma \approx 1 + 2s \quad > 1. \tag{39}$$

For a dressed sphere subject to (39), we shall calculate σ_1 for incident wave-numbers k lying in the interval J in which

$$-\xi s + \Delta < x = ka < \xi s + \Delta, \tag{40}$$

where ξ is a constant of $O(1)$. Of course, we make these choices because we know, [6], p157, that the resonance pole (37) will produce a Breit-Wigner shape for σ_1, peaked at $x = \Delta$ and of half width $\Gamma = s$.

It turns out that Δ rather than s is the natural quantity to use in the description of the approximations involved in our calculations. Therefore, we invert the definition of Δ in (37) to obtain

$$s = 1 - (1 - \Delta^2)^{\frac{1}{2}}, \tag{41}$$

chosing a sign so that (38) is satisfied for Δ sufficiently small. It is good enough for our purposes to use the lowest order approximation

$$s = \frac{1}{2}\Delta^2 \tag{42}$$

to (41). We shall show that a Breit-Wigner type formula for σ_1 follows if we work to lowest non-trivial order in powers of Δ. But more can done quite easily; we give here also details of the calculation for σ_1 accurate in next-to-leading order.

Using (42), we write $x \in J$ in the form

$$x = \Delta + fs = \Delta(1 + \frac{1}{2}f\Delta), \tag{43}$$

where, from (40), we see that the variable f obeys $-\xi < f < \xi$. Also, f is independent of Δ; it is to be regarded as being of $O(\Delta^0)$.

To calculate σ_1, we employ

$$\sigma_1 = \frac{12\pi a^2}{x^2}\sin^2\delta_1. \tag{44}$$

Thus we must compute $\sin\delta_1$ from (36), written as

$$e^{2ika}\, e^{2i\delta_1} = e^{2i\phi_1}\, e^{2i\phi_2},$$

where

$$e^{2i\phi_1} = \frac{k_0^* - k}{k_0 - k}, \quad e^{2i\phi_2} = \frac{k_0 + k}{k_0^* + k}, \tag{45}$$

or, more simply, as

$$\delta_1 = \phi_1 + \phi_2 - x. \tag{46}$$

We look first at the factor $e^{2i\phi_1}$ associated with the resonance pole because we expect it to produce for us a peaked Breit-Wigner factor in (44).

Since (45) shows that $e^{-i\phi_1}$ is proportional to

$$\begin{aligned}(k_0 - k)a = x_0 - x &= -is - (x - \Delta)\\ &= -\frac{1}{2}i\Delta^2 - \frac{1}{2}f\Delta^2,\end{aligned}$$

we obtain

$$\sin\phi_1 = \frac{1}{(1+f^2)^{\frac{1}{2}}}, \quad \cos\phi_1 = \frac{-f}{(1+f^2)^{\frac{1}{2}}}.$$

If we can show that $(\phi_2 - x)$ produces $O(\Delta)$ corrections to the $O(1)$ evaluation of δ_1 that comes from using the version $\delta_1 \approx \phi_1$ of (46), we get the leading order value of σ_1 by using this and $x \approx \Delta$. It is

$$\sigma_1 = \frac{12\pi a^2}{\Delta^2}\frac{1}{(1+f^2)^{\frac{1}{2}}} = \frac{12\pi a^2}{\Delta^2}\frac{s^2}{(x-\Delta)^2 + s^2}, \tag{47}$$

where we use (43) to obtain the second form of (47). This is of Breit-Wigner form, resonance shaped with a peak at $x = \Delta$, and half-width $\Gamma = s$ at half-height.

Now we must return to (46) to complete the justification of our lowest order result (47) and to see how to improve upon it. For the factor $e^{2i\phi_2}$ belonging to the third quadrant partner of the resonance pole k_0, we note that $e^{i\phi_2}$ is proportional to

$$(k_0 + k)a = x_0 + x = 2\Delta - \frac{1}{2}i\Delta^2 + \frac{1}{2}f\Delta^2.$$

Hence, to sufficient accuracy, we have

$$\tan\phi_2 = \frac{-\Delta^2}{4\Delta + f\Delta^2} = -\frac{1}{4}\Delta(1 - \frac{1}{4}f\Delta),$$

and $\tan\phi_2 \approx \phi_2 \approx \sin\phi_2$, $\cos\phi_2 \approx 1$. Then, using (43) again for x, we find

$$\sin(\phi_2 - x) = -\frac{1}{4}\Delta(1 - \frac{1}{4}f\Delta).1 - 1.(\Delta + \frac{1}{2}\Delta^2) = -\frac{5}{4}\Delta + O(\Delta^2),$$

and similarly $\cos(\phi_2 - x) = 1 + O(\Delta^2)$. Hence, keeping here the $O(\Delta)$ corrections to the lowest order result, we reach

$$\begin{aligned}\sin\delta_1 &= \sin\phi_1\cos(\phi_2 - x) + \cos\phi_1\sin(\phi_2 - x),\\ &= \frac{1}{(1+f^2)^{\frac{1}{2}}}\{(1.1 + (-f)(-\frac{5}{4}\Delta)\}.\end{aligned}$$

This shows that a careful treatment of $(\phi_2 - x)$ justifies our previous lowest order approach, and also furnishes the correction to it in $O(\Delta)$. Again using (43), but this time to higher accuracy than was required in passing to (47), we achieve our final result

$$\sigma_1 = \frac{12\pi a^2}{\Delta^2}\frac{1}{(1+f^2)^{\frac{1}{2}}}\left(1 + \frac{3}{2}f\Delta\right). \tag{48}$$

The correction factor in (48) clearly does not mask the resonant nature of the leading order result (47). It just gives a modification in $O(\Delta)$ of its shape.

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