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THE EFFECTIVE POTENTIAL for gauge theories and at finite temperature+

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ABSTRACT

In two-dimensional, scalar field theories at zero temperature, the effective potential V is only one-fold differentiable at points of pure quantum corrected ground states. This non-differentiability can be established in the framework of a rigorous theory of phase transitions, based on the notion of Gibbs potentials. It justifies the approximation of the ground states by minima in a naive loop expansion. After reviewing this approach briefly, we discuss its validity for gauge field theories and at finite temperature.

1. Introduction

For quantum field theories with spontaneously broken symmetries, the perturbative loop expansion leads to a non-convex expression for the effective potential, despite a formal convexity argument, a fact which has caused great confusion in the past.[1-3] At the root of this failure lies the use of an ill-defined Legendre transform in the perturbative calculation:[3-5]

In perturbative calculations, starting from a scalar field theory with generating functional $Z[J]$, the effective action $\Gamma[\phi_c]$ is taken to be the Legendre transform of $W[J] = \ln Z[J]$,

$$\Gamma[\phi_c(x)] = W[J] - \int J(x)\phi_c(x)dx; \tag{1}$$

$$\phi_c(x) = \frac{\delta}{\delta J(x)} W[J]. \tag{2}$$

Usually, the effective potential $\hat{V}(\hat{\phi}_c)$ is then defined by setting ϕ_c in the finite volume effective action Γ_Ω to be a constant $\hat{\phi}_c$ and dividing the total space-time volume Ω:

$$\lim_{\Omega\to\infty} \frac{1}{\Omega}\Gamma_\Omega[\phi_c]_{\phi_c=\hat{\phi}_c} = -\hat{V}(\hat{\phi}_c). \tag{3}$$

Clearly, $\hat{V}$ is only well-defined for all field values $\hat{\phi}_c$ if the functional derivative of $W[J]$ in Eq. (2) exists for all J and if the infinite volume limit in Eq. (3) exists. In the case of spontaneously broken symmetries however, the derivative in Eq. (2) does not exist, because the left and right derivatives differ: they define the degenerate vacuum expectation values.

For scalar, Z_2-symmetric scalar two-dimensional field theories, the situation is completely under control:[4-7] there, the infinite volume density of $W[J]$ for constant source term, $w(J)$, is known to exist and is stricly convex.[6] Furthermore, Glimm, Jaffe and Spencer have shown that for appropriate interaction terms, this class of models shows symmetry breaking,[7]

$$\frac{d^\pm}{dJ} w(0) = \phi_\pm \neq 0, \tag{4}$$

where ϕ_+, $\phi_- = -\phi_+$ denote the degenerate vacua. Then, a well-defined effective potential is provided by the Fenchel transform of $w(J)$,

$$V(\hat{\phi}_c) = \sup_J [J\hat{\phi}_c - w(J)], \tag{5}$$

and the following holds:[4,5,8]

i) $V(\hat{\phi}_c)$ is defined for all $\hat{\phi}_c$, $\hat{V}(\hat{\phi}_c)$ is defined for $|\hat{\phi}_c| > \phi_+$ only. By construction, $V(\hat{\phi}_c) = \hat{V}(\hat{\phi}_c)$ in the domain of definition of $\hat{V}$.
ii) From (4) and (5), $V(\hat{\phi}_c)$ can be seen to be strictly convex for $|\hat{\phi}_c| > \phi_+$, containing a straight line between ϕ_- and ϕ_+.
iii) $V(\hat{\phi}_c)$ is exactly one-fold differentiable for $\hat{\phi}_c = \phi_\pm$, but infinitely often differentiable elsewhere.
iv) $V(\hat{\phi}_c)$ has an interpretation as a free Helmholtz energy density, if $w(J)$ is considered to be the (negative) energy density in the presence of an external field, i.e. the (negative) Gibbs potential.
v) The loop expansion of $\hat{V}(\hat{\phi}_c)$ in powers of $\hbar$ is asymptotic to $\hat{V}$ for $|\hat{\phi}_c| > \phi_+$, but not for $|\hat{\phi}_c| < \phi_+$.

The crucial point is that according to (iii), $V(\hat{\phi}_c)$ is not an analytic function, but consists of three analytic branches for $\hat{\phi}_c < \phi_-$, $|\hat{\phi}_c| < \phi_+$ and $\hat{\phi}_c > \phi_+$, connected at points of non-differentiability. Hence, a perturbative polynomial expansion of $V(\hat{\phi}_c)$ does not necessarily approximate all analytic branches at once. According to (v), only the two strictly convex branches are approximated. Furthermore, since $V(\hat{\phi}_c)$ is one-fold differentiable everywhere, it can be shown that

vi) ϕ_+ and ϕ_- are asymptotically approximated by the minima of the loop expansion.

This situation may be compared to the Landau-Ginzburg approach to phase transitions, where a polynomial (non-convex) ansatz for the non-analytic (convex) free energy density proves phenomenologically successful, apparently for very similar reasons.

2. The effective potential for gauge theories

As first observed by Jackiw,[9] the perturbative loop expansion of the effective potential $\hat{V}(\hat{\phi}_c)$ in gauge field theories leads to gauge-dependent expressions. In the framework of perturbation theory, several heuristic arguments have been put forward of how to obtain physical information from $\hat{V}$ despite its gauge-dependence.[10-13] Here, we aim at extending the non-perturbative discussion of the last section to gauge theories. In principle, the obvious strategy would be to prove

a) the existence of a strictly convex $w(J)$ for the gauge theory under consideration and
b) to establish analogously to Eq. (4) that different values are obtained for different directional derivatives of $w(J)$ at some J.

Then, the equivalent of Eq. (5) would be well-defined and the corresponding analysis would go through automatically. However, in practice it is very difficult to follow this strategy.

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Problem a) amounts to a construction of gauge field theories which for realistic models is far from being achieved. Still, there seems no harm in adopting the working assumption that a strictly convex $w(J)$ exists.

Problem b) on the other hand cannot be by-passed by a similar assumption. As pointed out by Wightman,[14] the vacuum expectation value is generically found to be zero in latticised versions of gauge theories without Faddeev-Popov (FP) gauge fixing.[15-17] Furthermore, in path integral formulations with FP gauge fixing, the exponential decay of gauge-invariant two-point functions has been established in the radiation gauge and in all Gupta-Bleuler gauges.[18,19]* This failure of the path integral formalism to lead to the spontaneous symmetry breakdown of gauge theories has its roots in the insufficient gauge fixing of the usual FP procedure:[15]

Let us sketch this argument for the lattice theory of a scalar matter field $\phi(x)$, defined by associating with each lattice point coordinates $[R(x), g_x]$. Here, $R(x) \in [0,\infty)$ and $g_x \in \mathbf{G}$, the gauge group. By a slight abuse of notation, we write for an n-dimensional irreducible representation $\phi(x) = g_x R(x)$ where $R(x) = (R(x), 0, ..., 0)$ is a n-dimensional vector and g_x a $n \times n$-matrix. The gauge field is given by link variables $g_{x,x+\mu}$ between neighbouring sites x and $x+\mu$, $g_{x,y} \in \mathbf{G}$. Then, the path integral (for finite volume Λ) is given by the Gibbs measure

$$d\mu_\Lambda = Z_\Lambda^{-1} e^{S_\Lambda} \prod_{xy} dh_{xy} \prod_x R^k(x) dR(x) \prod_x dg_x, \tag{6}$$

where k denotes the dimension of the Lie algebra of $\mathbf{G}$, Z_Λ the partition function and S_Λ the finite volume action. $h_{xy} = g_x^{-1} g_{xy} g_y$ is the new link variable, gauge invariant under the joint gauge transformation γ of gauge field and matter field,

$$g_x \longrightarrow g_x^\gamma = \gamma_x g_x, \tag{7a}$$

$$g_{xy} \longrightarrow g_{xy}^\gamma = \gamma_x g_{xy} \gamma_y^{-1}. \tag{7b}$$

The path integral for the gauge-invariant measure $d\mu_\Lambda$ then assigns an expectation value $\langle f(x)\rangle_\Lambda = \int f(x) d\mu_\Lambda$ to any function $f(x) = f(R(x), g_x, g_{xy})$ of the fields. Any gauge fixing F is a modification of Eq. (6),

$$d\mu_\Lambda \longrightarrow d\mu_\Lambda F, \tag{8}$$

which leaves the corresponding expectation value $\langle f(x)\rangle_\Lambda^F$ unchanged for gauge-invariant $f(x)$. This requirement is equivalent to

$$\int \prod_x dg_x F(R(x), h_{xy}, g_x) = 1. \tag{9}$$

At this point, it is useful to distinguish two different gauge fixings, F_1 and F_2, defined by

$$\int \prod_x dg_x F_1(R(x), h_{xy}, g_x) f(R(x), h_{xy}, g_x) = \int dg f(R(x), h_{xy}, g\bar{g}_x), \tag{10a}$$

$$\int \prod_x dg_x F_2(R(x), h_{xy}, g_x) f(R(x), h_{xy}, g_x) = f(R(x), h_{xy}, \bar{g}_x) \tag{10b}$$

* The abelian Higgs model in the Landau gauge is a remarkable exception.[19] There, an asymptotically non-vanishing two-point function can be shown to be a consequence of the failure of the cluster decomposition property in this gauge rather than a consequence of a non-zero vacuum expectation value.[15]

for any function f of the fields and a set of fixed gauge field variables $\bar{g}_x$. Then, the following can be shown:[15]

i) F_1 is the usual FP gauge fixing term which removes the local, but not the global gauge invariance from the path integral measure. $\langle f(x)\rangle_\Lambda^{F_1}$ corresponds to the scalar product on a Hilbert space $\mathcal{H} = \int_G dg \mathcal{H}_g$ associated with an integral representation of the field algebra, where $\mathcal{H}_g$ is an irreducible space obtained via the GNS-construction from a cyclic vacuum $|0\rangle_g$ in which ${}_g\langle 0|\phi|0\rangle_g = gR$. Represented on $\mathcal{H}$, all gauge-dependent elements of the field algebra have zero vacuum expectation value, particularly, $\langle\phi(x)\rangle_\Lambda^{F_1} = 0$.

ii) F_2 fixes both local and global gauge invariance of the path integral. $\langle f(x)\rangle_\Lambda^{F_2}$ corresponds to the scalar product on the irreducible space $\mathcal{H}_g$. Especially, there exist representations of the field algebra on $\mathcal{H}_g$, such that $\langle\phi(x)\rangle_\Lambda^{F_2} = \bar{g}_x R(x) \neq 0$.

iii) Subject to some technical caveats, i) and ii) hold in the infinite volume limit and the continuum limit, too.

We expect that these results have the following consequences for defining an effective potential for gauge theories: Depending on the gauge fixing chosen, one obtains gauge theories defined by $w_1(J)$, $w_2(J)$ respectively. Formally, the $\langle f(x)\rangle_\Lambda^{F_i}$ then result from the functional derivatives of $w_i(J)$, $i = 1, 2$. Eq. (5) defines the effective potential for the scalar field of the theory. V depends on the representation of the field algebra, i.e., on the measure in the path integral, and the following holds:

iv) No spontaneous symmetry breaking occurs in integral representations of the gauge field algebra. Hence, V is stricly convex for theories defined by $w_1(J)$.

v) In irreducible representations of the field algebra, spontaneous symmetry breaking can occur. However, since the gauge has been fixed completely in $w_2(J)$, $V(\hat{\phi}_c)$ does not take values for all $\hat{\phi}_c = gR$, but for $\hat{\phi}_c = \bar{g}R$ only, where $\bar{g} \in \mathbf{G}$ is fixed, i.e., V is a function of $R \in [0,\infty)$ only. Again, the symmetry is broken if $V(\bar{g}R)$ contains points of non-differentiability. Remark however that unlike the scalar case discussed in section 1, spontaneous symmetry breaking is not a completely dynamical problem any more. Whether symmetry breaking is possible is decided by the path integral $w_2(J)$, i.e. by the dynamics, which determines the form of $V(\bar{g}R)$, whether it takes place, however, depends on the choice of a particular representation of the field algebra.

The effective potential of a field theory with global continuous symmetries can be treated according to iv) and v), too.

These remarks seem to indicate why it is very difficult to extract physical information about spontaneous symmetry breaking from effective potentials calculated perturbatively in the path integral formalism with the usual FP gauge fixing.

3. The effective potential at finite temperature

The perturbative loop expansion of scalar field theories at finite temperature leads to non-convex expressions which may even become complex.[20,21] In principle, the discussion of section 1 can be applied: if $w(J)$ satisfies the equivalent of Eq. (4) at finite temperature, then V defined by Eq. (5) contains points of non-differentiability, i.e., the symmetry is broken. The form of V is stricly convex in the two outer branches which are connected by a straight line at their minima. This straight line has a free energy density interpretation.

To see this, one starts from the Helmholtz free energy functional[4,8]

$$\hat{F}(\rho) = Tr\Big(\rho\hat{H}(\phi) - T\hat{S}(\rho)\Big), \tag{11}$$

where ρ is an arbitrary state with entropy $\hat{S}(\rho) = -k\rho \ln \rho$ and $\hat{H}$ denotes the hamiltonian. The field ϕ is a linear functional of the state, $\phi(\rho)$ and the Helmholtz free energy density $\hat{f}(\rho)$ of $\hat{F}$ gives rise to the constraint Helmholtz free energy density

$$f_c(\hat{\phi}_c, T) = \inf_\rho \{\hat{f}(\rho) \mid \phi(\rho) = \hat{\phi}_c\}, \tag{12}$$

and the Gibbs potential

$$g(J, T) = \inf_\rho \{\hat{f}(\rho) - J\phi(\rho)\}, \tag{13}$$

where the intensive variable J is the conjugate of ϕ. In this thermodynamic setting, it can be shown that for values of $\hat{\phi}_c$ varying between the directional derivatives of $g(J.T)$ at a point of non-differentiability (i.e., in the case of phase transition),[4]

$$f_c(\hat{\phi}_c, T) = \sup_J [J\hat{\phi}_c + g(J, T)]. \tag{14}$$

In comparison with Eq. (5), one concludes that the straight line of V has a free energy density interpretation if $w(J)$ can be understood as a (negative) Gibbs potential. This however is the usual working assumption of every thermodynamic interpretation of the path integral formalism.

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