DAMTP 93-45

DAMTP/93-45

A new q-Lorentz group and a q-Minkowski space with both braided coaddition and q-spinor decomposition

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July 1993

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Abstract

We present a new version of q-Minkowski space. As opposed to earlier attempts, our new q-Minkowski space has both an $SL_q(2,\mathbf{C})$-spinor decomposition and a coaddition law. The additive structure forms a braided group rather than a quantum one. In the process, we obtain a new q-Lorentz group which coacts coariantly on this q-Minkowski space.

Keywords: quantum groups, braided groups, non-commutative geometry, q-regularisation.

1 Introduction

In recent years, there has been some speculation whether it could be possible to regularise singularities in quantum field theories by making spacetime slightly non-commutative. As well as the programme of A. Connes [3] based on the theory of operator algebras, there is also a more naive approach based on the idea of q-deformation. In this approach, which is the one we shall follow, non-commutativity is controlled by a parameter q such that one recovers the commutative case for $q = 1$. There are examples [7] of integrals over two-dimensional q-deformed planes which are of the form $\int(\ldots) = \frac{1}{q^2-1}(finite)$, i.e. are divergent only in the commutative case. Moreover, one hopes in such a q-regularisation scheme to preserve all symmetries as q-symmetries, using the standard techniques for q-deforming Lie algebras, etc. One would then set $q = 1$ after intelligent renormalisation, although, to take account of Planck scale corrections to the geometry, one might even keep $q \neq 1$.

As an important element of such a q-regularisation scheme, many q-Lorentz groups and q-Minkowski spaces have been recently proposed [17, 2, 16, 11]. One of the points of view in these works, which will be our point of view also, is that q-Minkowski space should have a q-spinor decomposition. Mathematically, q-Minkowski space should be a q-deformed version of 2×2 Hermitean matrices and the q-Lorentz group should act on it by conjugation by two q-deformed $SL(2,\mathbf{C})$ transformations The rôle of such a q-deformed $SL(2,\mathbf{C})$ can be provided by the quantum double [17] but q-Minkowski space and the q-Lorentz group itself are less well understood so far.

*Supported by an SERC Studentship and Studienstiftung des deutschen Volkes

Naively, one might try to construct q-Minkowski space as quantum 2×2 matrices, but this algebra is not covariant under the coaction of the q-deformed $SL(2, \mathbf{C})$ [15]. The solution to this problem is to consider braided rather than quantum Hermitean matrices as q-Minkowski space [10, 11]. Braided matrices are an example of so-called braided bialgebras introduced by S. Majid [9] as a generalisation of bialgebras, for which the ordinary tensor product in the bialgebra axioms is replaced by a ***braided tensor product.*** Braided tensor products are like the super tensor products encountered in the the theory of superspaces, but with ± 1 replaced by braid statistics. There is a general construction [13], called ***transmutation*** by means of which one can convert a suitable bialgebra, such as a usual quantum matrix algebra, into a braided bialgebra with better covariance properties. The algebra and coalgebra structure of such a braided bialgebra are covariant under the coaction of the quantum group. Thus the braided 2×2 matrices as the transmutation of the well-known 2×2 quantum matrices are a natural candidate for the algebra of q-Minkowski space. It is covariantly coacted upon by the q-deformed $SL(2, \mathbf{C})$.

Braided 2×2 matrices have the same matrix coalgebra structure as quantum matrices, but a different multiplication [10]. Similar as for 2×2 quantum matrices, one can find a braided determinant which central and grouplike with respect to the braided coproduct [10], to play the rôle of a q-Minkowski metric. Furthermore, these braided matrices allow for a q-spinor decomposition [15] and can also be equipped with a $*$-structure appropriate for Hermitean matrices [11].

Considering braided Hermitean matrices seems to lead in the right direction, but a fundamental structure is still missing: so far there is no q-deformed analogue of the additive group structure of Minkowski space. In this paper we solve this problem and generalize the group structure on Minkowski space as a 'braided coaddition' in the form of a new (braided) coalgebra structure on the algebra of braided matrices. The required braiding for the coaddition is a new one. We also give a new q-Lorentz group which acts covariantly on this new version of q-Minkowski space. In fact, one can construct many different q-Lorentz groups, all of which have the algebra of braided Hermitean matrices as a 'fundamental representation', but only the q-Lorentz group presented in this paper appears to be consistent with the braided coaddition law on them.

An outline of the paper is as follows. In section 2, we reformulate some classical considerations about the Lorentz group and Minkowski space suitable for later q-deformation. The q-Lorentz group (of function algebra type) is presented in section 3. In section 4 we present our braided coaddition law on q-Minkowski space and we also give a q-spinor decomposition. Our q-Lorentz group coacts on q-Minkowski space in a way which respects the braided coaddition law. These facts suggest that our proposal for q-Lorentz group may be a definitive one. Section 5 is devoted to the discussion of the q-deformation of the universal enveloping algebra of the Lorentz group, which is dual to the q-Lorentz group of function algebra type from section 3.

Preliminaries.

When working with matrices, we use lower-case letters for indices which run from 1 to 2 or n, and upper-case letters for multi-indices, e.g. $A = (a_0 a_1) = (11), (12), (21), (22)$.

For Hopf algebras, we use the notation and results from the standard textbooks [1, 20]. Recall that a (complex) coalgebra is a $\mathbf{C}$-vector space A equipped with an $\mathbf{C}$-linear coassociative comultiplication $\Delta : A \to A \otimes A$ and a $\mathbf{C}$-linear counit $\varepsilon : A \to \mathbf{C}$ satisfying certain axioms. Elements g in A obeying $\Delta g = g \otimes g$ are called grouplike. We use the notation $\Delta a = a_{(1)} \otimes a_{(2)}$ and omit summation signs for brevity. A complex bialgebra is an algebra and a coalgebra in a compatible way, such that both comultiplication and counit are algebra maps. If a bialgebra H also allows for a $\mathbf{C}$-linear antipode $S : H \to H$ obeying $(S \otimes \varepsilon) \circ \Delta = id = (\varepsilon \otimes S) \circ \Delta$, then H is called a Hopf algebra. We also use M. Sweedler's shorthand notation [20], where a suffix indicates the position in a matrix tensor product, e.g. $A_{12}B_{23}$ means $A^{ij}_{kl}B^{lm}_{xy}$, etc.

A $*$-Hopf algebra [21] is a Hopf algebra equipped with an antilinear involution $*$ such that $(S \circ *)^2 = id$, $\Delta \circ * = (* \otimes *) \circ \Delta$, and $\varepsilon \circ * = \bar{} \circ \epsilon$, where the bar denotes complex conjugation in the field, $\mathbf{C}$. Two $*$-Hopf algebras H and H' are called dually paired if there exists a bilinear pairing $< , >: H \otimes H' \to \mathbf{C}$ such that $< \alpha\beta, x >=< \alpha \otimes \beta, \Delta x >$, $< \alpha, xy >=< \Delta\alpha, x \otimes y >$, $< 1, x >= \varepsilon(x)$, $< \alpha, 1 >= \varepsilon(\alpha)$, $< S\alpha, x >=< \alpha, Sx >$, $< \alpha^*, x >= \overline{< \alpha, (Sx)^* >}$ $\forall \alpha, \beta \in H; x, y \in H'$.

We shall also need the notion of a right comodule, which is dual to the definition of a left module: a right comodule of a coalgebra A is a pair (C, β), where C is a vector space and β a linear map $\beta : C \to C \otimes A$ obeying $(id \otimes \Delta) \circ \beta = (\beta \otimes id) \circ \beta$ and $id = (id \otimes \varepsilon) \circ \beta$.

Of particular interest to us are non-commutative bialgebras, for which the non-commutativity is controlled by a so-called dual quasitriangular structure [12], which is defined as a convolution invertible map $\Re : A \otimes A \to k$ such that $b_{(1)}a_{(1)}\Re(a_{(2)} \otimes b_{(2)}) = \Re(a_{(1)} \otimes b_{(1)})a_{(2)}b_{(2)}$, $\Re(ab \otimes c) = \Re(a \otimes c_{(1)})\Re(b \otimes c_{(2)})$, and $\Re(a \otimes bc) = \Re(a_{(1)} \otimes c)\Re(a_{(2)} \otimes b)$ for all $a, b, c \in A$. In other words, $\Re$ is a bialgebra bicharacter. This notion is dual to the maybe more familiar term of quasitriangularity due to Drinfel'd [4].

One of the most important properties of dual quasitriangular bialgebras is that the comodules of such a bialgebra A form a quasi-tensor or *braided* category denoted $\mathcal{M}^A$. This means that the category $\mathcal{M}^A$ can be equipped with a bifunctor $\underline{\otimes} : \mathcal{M}^A \times \mathcal{M}^A \to \mathcal{M}^A$, which was called 'braided tensor product' in the introduction. This $\underline{\otimes}$ is required to satisfy some associativity conditions. Furthermore, for any two objects X, Y of $\mathcal{M}^A$ (i.e. for any two comodules) there is a natural isomorphism $\Psi_{X,Y} : X \underline{\otimes} Y \cong Y \underline{\otimes} X$, called the *braiding*. For $\mathcal{M}^A$, this braiding is given in terms of the dual quasitriangular structure $\Re$ and the coactions of the respective comodules. If we are now given two A comodule algebras C_1 and C_2 we can use Ψ to define their braided tensor product as $C_1 \underline{\otimes} C_2$ equipped with the new multiplication

$$(a \otimes b)(c \otimes d) := a\Psi(b \otimes c)d.$$

Due to the properties of $\underline{\otimes}$ and Ψ, the braided tensor product of two comodule algebras turns out to be a comodule algebra again. The braided tensor product provides a covariant way of combining two covariant systems.A braided bialgebra is an algebra B living in a braided category equipped with a braided coproduct $\underline{\Delta} : B \to B\underline{\otimes}B$ obeying axioms similar to the bialgebra axioms, but with $\underline{\Delta}$ a homomorphism to the braided tensor product $B\underline{\otimes}B$. All maps are morphisms, i.e. covariant under the coaction of the background quantum group A. The braided matrices mentioned in the introduction and which form our q-Minkowski space are of this type.

2 The Hopf algebras of polynomial functions on the Lorentz group and on Minkowski space

In this section, we reformulate some classical considerations about the Lorentz group and Minkowski space in an algebraic language suitable for later generalization. As familiar from other applications, we present Minkowski space as Hermitean 2×2 matrices. One usually chooses this description to give a simple exposition of the covering map $SL(2,\mathbf{C}) \to L_+^\uparrow$ of the subgroup of proper orthochronous Lorentz transformations. This map enables one to construct the well-known spinor decomposition of Lorentz tensors. If Minkowski space is given as Hermitean 2×2 matrices, the Lorentz metric can be expressed in terms of the $SL(2,\mathbf{C})$ spinor metric $\epsilon^{ab} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ as $C^{AB} := \epsilon^{a_0 b_0}\epsilon_{a_1 b_1}$, and the (full) Lorentz Group is given by

$$L = L(4,\mathbf{R})/(\Lambda^A_{\ C}\Lambda^B_{\ D}C^{CD} = C^{AB}),$$

where $L(4,\mathbf{R})$ denotes real $n \times n$ matrices. We use the convention $\epsilon^{ac}\epsilon_{ab} = \delta^c_{\ b}$ for the definition of the inverse spinor metric.

In order to q-deform these structures, we are in principle interested in $\mathcal{C}(X)$, the algebras of continuous $\mathbf{C}$-valued functions on subsets X of $\mathbf{R}^n$, such as the Lorentz group L or Minkowski Space M. However, to avoid the discussion of convergence problems and other complications due to the non-compact nature of these spaces, we consider only $\mathcal{P}(X)$, the algebra of polynomial functions, which is almost the same as $\mathcal{C}(X)$, since on arbitrarily large compact subsets $X' \subset X$, $\mathcal{P}(X')$ is dense in $\mathcal{C}(X')$.

For the application to the Lorentz group, we are particularly interested in the case where X is a subset of $L(n,k)$, $k = \mathbf{R}$ or $\mathbf{C}$. For clarity, we recall some simple results of this special case: It is easy to see that $\mathcal{P}(L(n,\mathbf{C}))$ is a commutative and associative $\mathbf{C}$-algebra generated by 1 and $t^a_{\ b}$, the linear coordinate functionals on $L(2,\mathbf{C})$, and their complex conjugates $\bar{t}^a_{\ b}$. It has the structure of a bialgebra with with pointwise multiplication, and comultiplication and counit

$$\Delta t^a_{\ b} = t^a_{\ c} \otimes t^c_{\ b}, \quad \varepsilon(t^a_{\ b}) = \delta^a_{\ b}.$$

A coalgebra structure of this form is called *matrix coalgebra structure*. If we are given a matrix group $G \subset L(n,\mathbf{C})$, then $\mathcal{P}(G)$ is a Hopf algebra and has the form of $\mathcal{P}(L(n,\mathbf{C}))$ divided by some further relation.

For the special case of $SL(2,\mathbf{C})$ one finds that $\mathcal{P}(SL(2,\mathbf{C})) = \mathcal{P}(L(2,\mathbf{C}))/(t^a{}_c t^b{}_d \epsilon^{cd} = \epsilon^{ab},\ \bar{t}^b{}_d \bar{t}^a{}_c \epsilon^{cd} = \epsilon^{ab})$ is a $*$-Hopf algebra with pointwise multiplication, matrix coalgebra structure, antipode defined by $St^a{}_b = \epsilon_{ia} t^i{}_j \epsilon^{jb}$ and $*$-structure $(t^a{}_b)^* = \bar{t}^b{}_a$. One obtains $\mathcal{P}(SU(2))$ as a real form of $\mathcal{P}(SL(2,\mathbf{C}))$:

$$\begin{aligned} \mathcal{P}(SU(2)) &= \mathcal{P}(L(2,\mathbf{C}))/(\tau^a{}_c \tau^b{}_d \epsilon^{cd} = \epsilon^{ab},\ \bar{\tau}^b{}_d \bar{\tau}^a{}_c \epsilon^{cd} = \epsilon^{ab},\ \bar{\tau}^a{}_b = S\tau^b{}_a) \\ &= \mathcal{P}(SL(2,\mathbf{C}))/(\bar{\tau}^a{}_b = S\tau^b{}_a) \end{aligned}$$

It is also a $*$-Hopf algebra with matrix coalgebra structure and with: $S\tau^a{}_b = \epsilon_{ia} \tau^i{}_j \epsilon^{jb}$, $(\tau^a{}_b)^* = S\tau^b{}_a$. Moreover, there is a $*$-Hopf algebra isomorphism (complexification)

$$\begin{aligned} \mathcal{P}(SL(2,\mathbf{C})) &\cong \mathcal{P}(\widetilde{SU}(2)) \otimes \mathcal{P}(SU(2)) \\ t^a{}_b &\mapsto 1 \otimes \tau^a{}_b \\ t^{*a}{}_b &\mapsto S\tilde{\tau}^b{}_a \otimes 1, \end{aligned}$$

where we equip the tensor product $\mathcal{P}(\widetilde{SU}(2)) \otimes \mathcal{P}(SU(2))$ with the non-standard $*$-structure $(1 \otimes \tau^a{}_b)^* := S\tilde{\tau}^b{}_a \otimes 1$ instead of the tensor product one.

Using these results we find for the algebra of polynomial functions on the Lorentz group:

Proposition 2.1 *Let $\lambda^A{}_B$ denote the linear coordinate functionals on $L(4,\mathbf{R})$. The polynomial functions on the Lorentz group $\mathcal{P}(L) = \mathcal{P}(L(4,\mathbf{R}))/(\lambda^A{}_C \lambda^B{}_D C^{CD} = C^{AB})$ form a commutative $*$-Hopf algebra with pointwise multiplication, matrix coalgebra structure and*

$$S\lambda^A{}_B = C_{DB} \lambda^D{}_C C^{CA}, \qquad (\lambda^A{}_B)^* = \lambda^{\overline{A}}{}_{\overline{B}}.$$

$\overline{A}$ is shorthand for the twisted multi-index $\overline{A} = (a_1 a_0)$.

We retrieve the covering map of the subgroup of proper orthochronous Lorentz transformations on the level of polynomial function algebras as an injective $*$-Hopf algebra homomorphism

$$\begin{aligned} \mathcal{P}(L) &\overset{\varphi}{\hookrightarrow} \mathcal{P}(\widetilde{SU}(2)) \otimes \mathcal{P}(SU(2)) \cong \mathcal{P}(SL(2,\mathbf{C})) \\ \lambda^A{}_B &\mapsto S(\tilde{\tau})^{b_0}{}_{a_0} \otimes \tau^{a_1}{}_{b_1}. \end{aligned}$$

This map is not surjective. Its image is the fixed-point set of the $\mathbf{Z}_2$-action σ given by $\sigma(1 \otimes \tau) := -(1 \otimes \tau)$ and $\sigma(\tilde{\tau} \otimes 1) := -(\tilde{\tau} \otimes 1)$:

$$\mathcal{P}(L) \cong (\mathcal{P}(\widetilde{SU}(2)) \otimes \mathcal{P}(SU(2)))^{\mathbf{Z}_2}.$$

Composition with the map φ defines a *push forward* of comodules, i.e. a covariant monoidal functor $\Phi : \mathcal{M}^{\mathcal{P}(L)} \longrightarrow \mathcal{M}^{\mathcal{P}(SL(2,C))}$, where $\mathcal{M}^{\mathcal{P}(SL(2,C))}$ and $\mathcal{M}^{\mathcal{P}(L)}$ denote the monoidal categories of right $\mathcal{P}(SL(2,\mathbf{C}))$- and $\mathcal{P}(L)$-comodules, respectively. Φ is the spinor decomposition of Lorentz tensors on the level of polynomial function algebras.

Next, we come to Minkowski space M in this algebraic form. Minkowski space has an additive group structure and not a multiplicative one as the matrix groups discussed so far. But as for the polynomial function algebras on matrix groups, this group structure is recovered in the coalgebra structure of $\mathcal{P}(M)$:

Proposition 2.2 *(i) $\mathcal{P}(M)$ is a commutative associative* C*-algebra generated by 1 and the linear coordinate functionals x_A, $A = (11)\ldots(22)$. $\mathcal{P}(M)$ is a $*$-Hopf algebra with*

$$\Delta x_A = x_A \otimes 1 + 1 \otimes x_A, \quad Sx_A = -x_A, \quad \varepsilon x_A = 0, \quad x_A^* := x_{\overline{A}}.$$

The comultiplication encodes the additive group structure of M on the level of $\mathcal{P}(M)$:

$$\xi + \zeta = \Delta x(\xi \otimes \zeta) \quad \forall \xi, \zeta \in M.$$

We call a coalgebra structure of this form coaddition .

(ii) There is an algebra homomorphism (spinor decomposition)

$$\mathcal{P}(M) \to \mathcal{P}(\mathbf{C}^2) \otimes \mathcal{P}(\mathbf{C}^{2\prime}),$$

where $\prime$ denotes the dual space.

(iii) $\mathcal{P}(M)$ is a right $\mathcal{P}(L)$ and $\mathcal{P}(SL(2,\mathbf{C}))$ comodule $*$-algebra with coactions

$$\begin{array}{rccc} \beta_L : & \mathcal{P}(M) & \to & \mathcal{P}(L) \otimes \mathcal{P}(M) \\ & x_I & \mapsto & x_J \otimes \lambda^J{}_I \end{array}$$

and $\beta_{SL(2,C)} := (id \otimes \varphi) \circ \beta_L$, using the functor Φ defined above.

(iv) $x_A x_B C^{AB} = 2 \det x$.

3 q-Lorentz group of function algebra type

In this section, we shall give a non-commutative version of $\mathcal{P}(M)$ and $\mathcal{P}(L)$, making use of the standard technique of deforming the commutative bialgebra of polynomial functions on a matrix group as a non-commutative dual quasitriangular bialgebra [8]. The resulting algebraic objects are called quantum matrix groups. The basic idea is to make the linear coordinate functionals $t^i{}_j$ commutative only up to conjugation by an invertible matrix solution $R = \Sigma\, R^{(1)} \otimes R^{(2)} \in GL(\mathbf{C}^n \otimes \mathbf{C}^n)$ of the quantum Yang-Baxter equation (QYBE): $R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}$. Explicitly, we define $A(R)$ to be the free associative C-algebra with n^2 generators $t^i{}_j$, $i, j = 1 \ldots n$ divided by the ideal generated by the relations $R_{12}t_1t_2 = t_2t_1R_{12}$ (i.e. $R^{i\,k}_{m\,n}t^m_j t^n_l = t^k{}_n t^i{}_m R^{m\,n}_{j\,l}$). It is known that $A(R)$ is a dual quasitriangular bialgebra with matrix coproduct and dual-quasitriangular structure $\Re : A(R) \otimes A(R) \to \mathbf{C}$ given by $\Re(t \otimes 1) = id = \Re(1 \otimes t)$ and $\Re(t_1 \otimes t_2) = R_{12}$ extended as bialgebra bicharacter [8]. $A(R)$ is a non-commutative version of the bialgebra of polynomial functions on the algebra of $n \times n$ matrices. One can obtain a non-commutative version of the Hopf algebra of polynomial functions on a matrix group by dividing $A(R)$ by some further 'quantum determinant'-like relations. The dual quasitriangularity of $A(R)$ follows from the fact that the so-called fundamental matrix representations

$$\rho_\pm : A(R) \to L(4, \mathbf{C})$$

defined by $\rho_+(t^i{}_j)^k{}_l = R^{ik}_{jl}$ and $\rho_-(t^i{}_j)^k{}_l = R^{-1ki}_{lj}$ (and two similar anti-representations) respect the relations in $A(R)$ and indeed extend to algebra homomorphisms [8]. This means that if we divide $A(R)$ by some further relations, it is sufficient to show that they are respected by the fundamental matrix relations in order to establish dual quasitriangularity of the quotient. Note that, by rescaling of R, we actually obtain a $\mathbf{C}\backslash\{0\}$-indexed family of dual quasitriangular structures on $A(R)$, whereas a quotient might have less freedom if the additional relation fixes the normalisation of R. Corresponding to the family of dual quasitriangular structures on $A(R)$, we obtain a family of braidings in $\mathcal{M}^{A(R)}$.

The aim of this section is to give a non-commutative version of $\mathcal{P}(L) = \mathcal{P}(L(4,\mathbf{R}))/(\lambda^A{}_C\lambda^B{}_D C^{CD} = C^{AB})$ as a dual quasitriangular $*$-Hopf algebra of the form

$$L_q = A(R_L)/(\text{ q-deformed metric relation}),$$

where R_L is an invertible solution of the four-dimensional QYBE which we introduce. L_q should generalize all features of $\mathcal{P}(L)$ presented in proposition 2.1. In order to obtain such an R_L-matrix and q-deformed metric, we make use of the important rôle $\mathcal{P}(SL(2,\mathbf{C}))$ plays as a building block of $\mathcal{P}(L)$. One can construct a non-commutative version of $\mathcal{P}(SL(2,\mathbf{C}))$ from the q-deformation of $\mathcal{P}(SU(2))$ [18]: Let

$$R = \sqrt{q}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & q^{-1} & 1-q^{-2} & 0 \\ 0 & 0 & q^{-1} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad q \in \mathbf{R}.$$

be the well-know invertible solution of the two-dimensional QYBE with the property $R^{ab}_{cd} = R^{dc}_{ba}$. $SU_q(2)$ is defined [18] to be $A(R)$ divided by the ideal generated by the relation $\tau^a{}_c\tau^b{}_d\epsilon^{cd} = \epsilon^{ab}$ equipped with the $*$-structure $(\tau^a{}_b)^* := S\tau^b{}_a$, where

$$\epsilon^{ab} = \begin{pmatrix} 0 & 1/\sqrt{q} \\ -\sqrt{q} & 0 \end{pmatrix}$$

is the q-deformed spinor metric. $SU_q(2)$ is a dual quasitriangular $*$-Hopf algebra with matrix coalgebra structure, antipode $St^a{}_b = \epsilon_{ib}\tau^i{}_j\epsilon^{ja}$, and standard dual quasitriangular structure defined by R.

We q-deform $\mathcal{P}(SL(2,\mathbf{C})) \cong \mathcal{P}(\widetilde{SU}(2))\otimes\mathcal{P}(SU(2))$ as $\widetilde{SU_q}(2) \bowtie SU_q(2)$, the double cross product Hopf algebra, as defined in [6], of two copies of $SU_q(2)$ acting on each other in a compatible way. $\widetilde{SU_q}(2) \bowtie SU_q(2)$ coincides with $\widetilde{SU_q}(2) \otimes SU_q(2)$ as a coalgebra, but has a different algebra structure, given by the compatible actions. Explicitly, one obtains $R^{ab}_{cd}(1\otimes\tau^c{}_e)(\tilde{\tau}^d{}_f\otimes 1) = (\tilde{\tau}^b{}_d\otimes 1)(1\otimes\tau^a{}_c)R^{cd}_{ef}$. This double cross product is a Hopf algebra with the tensor product coalgebra structure

$$\Delta_{\bowtie} = (id\otimes\tau\otimes id)(\Delta\otimes\Delta) \quad \varepsilon_{\bowtie} = \cdot\circ(\varepsilon\otimes\varepsilon), \quad S_{\bowtie}(a\otimes b) = (1\otimes Sb)(Sa\otimes 1)$$

Corresponding to the non-standard $*$-structure defined on $\mathcal{P}(SL(2,\mathbf{C})) \cong \mathcal{P}(\widetilde{SU}(2)) \otimes \mathcal{P}(SU(2))$, we define a $*$-structure on $SL_q(2,\mathbf{C}) = \widetilde{SU_q}(2) \bowtie SU_q(2)$ by $(\tilde{\tau}^a{}_b\otimes 1)^* = 1\otimes S\tau^b{}_a$, and obtain $SU_q(2)$ as a real form of $SL_q(2,\mathbf{C})$. The general algebraic construction underlying this is given in [14, Sec. 4].

After these preliminaries, we now present our new q-Lorentz group:

Definition 3.1 *The q-Lorentz group L_q is defined as*

$$L_q := A(R_L)/(\lambda^A{}_C \lambda^B{}_D C^{CD} = C^{AB}).$$

The preserved metric and the R-matrix are given by

$$C^{AB} := \epsilon_{a_0\alpha} \tilde{R}^{-1a_1\alpha}{}_{\beta b_0} \epsilon^{\beta b_1}, \qquad R_L{}^{BA}{}_{DC} := R^{d_0\alpha}{}_{\beta a_0} R^{a_1\beta}{}_{\gamma b_0} R^{b_1\gamma}{}_{\delta c_1} \tilde{R}^{\delta c_0}{}_{d_1\alpha},$$

where $\tilde{R} := ((R^{t_2})^{-1})^{t_2}$, and t_2 denotes transposition in the second tensor component. Explicitly, we find

$$C^{AB} = \begin{pmatrix} \frac{1-q^2}{q^3} & 0 & 0 & 1/q \\ 0 & 0 & -q & 0 \\ 0 & -1/q & 0 & 0 \\ 1/q & 0 & 0 & 0 \end{pmatrix}.$$

L_q is in fact a deformation of $\mathcal{P}(L)$. It obeys a non-commutative version of proposition 2.1:

Proposition 3.2 *(i) R_L is an invertible solution of the QYBE.*

(ii) L_q is a dual quasitriangular $*$-Hopf algebra with matrix coalgebra structure,

$$S(\lambda^A{}_B) = C_{IB} \lambda^I{}_J C^{JA}, \quad (\lambda^A{}_B)^* = \lambda^{\overline{A}}{}_{\overline{B}},$$

and dual quasitriangular structure induced by R_L.

(iii) There exist a $*$-Hopf algebra injection φ

$$\begin{array}{ccc} L_q & \overset{\varphi}{\hookrightarrow} & \widetilde{SU_q(2)} \bowtie SU_q(2) = SL_q(2, \mathbf{C}) \\ \lambda^A{}_B & \mapsto & S\tilde{\tau}^{b_0}{}_{a_0} \otimes \tau^{a_1}{}_{b_1}, \end{array}$$

where $\tilde{\tau}$ and τ denote two independent copies of generators of $SU_q(2)$. As in the commuting case, $Im(\varphi)$ is the fixed-point set of a $\mathbf{Z}_2$-action:

$$L_q \cong \widetilde{SU_q(2)} \bowtie SU_q(2))^{\mathbf{Z}_2}.$$

The Hopf algebra map φ induces a push forward of comodules, the q-spinor decomposition of q-Lorentz tensors.

Proof. (i) By explicit calculation. (ii) There are two non-trivial statements in this part of the proposition: firstly that the operation '$*$' does indeed determine a $*$-structure on a Hopf algebra, and, secondly, that $\Re$ defines a dual quasitriangular structure, i.e. that it is compatible with the metric relation. The rest amounts to a simple check of the Hopf algebra axioms. In order to show that '$*$' respects the algebra relations in L_q, note that, by virtue of the fact that R obeys $R^{ab}{}_{cd} = R^{dc}{}_{ba}$, and using $\epsilon^{ab} = -\epsilon_{ba}$, one can show the following two relations:

$$\begin{aligned} R_L{}^{BA}{}_{DC} &= R^{d_0\alpha}{}_{\beta a_0} R^{a_1\beta}{}_{\gamma b_0} R^{b_1\gamma}{}_{\delta c_1} \tilde{R}^{\delta c_0}{}_{d_1\alpha} \\ &= R^{a_0\beta}{}_{\alpha d_0} R^{b_0\gamma}{}_{\beta a_1} R^{c_1\delta}{}_{\gamma b_1} \tilde{R}^{\alpha d_1}{}_{c_0\delta} \\ &= R_L{}^{\overline{A}\,\overline{B}}{}_{\overline{C}\,\overline{D}}, \end{aligned}$$

$$\begin{aligned} C^{AB} &= \epsilon_{a_0\alpha} \tilde{R}^{-1a_1\alpha}{}_{\beta b_0} \epsilon^{\beta b_1} \\ &= \epsilon^{\alpha a_0} \tilde{R}^{-1b_0\beta}{}_{\alpha a_1} \epsilon_{b_1\beta} \\ &= C^{\overline{B}\,\overline{A}}. \end{aligned}$$

Therefore,

$$
\begin{aligned}
(R_L{}^{AB}_{CD}\lambda^C_E\lambda^D_F)^* &= R_L{}^{AB}_{CD}\lambda^{\overline{D}}_{\overline{F}}\lambda^{\overline{C}}_{\overline{E}} \\
&= R_L{}^{\overline{B}\,\overline{A}}_{\overline{D}\,\overline{C}}\lambda^{\overline{D}}_{\overline{F}}\lambda^{\overline{C}}_{\overline{E}} \\
&= \lambda^{\overline{A}}_{\overline{C}}\lambda^{\overline{B}}_{\overline{D}}R_L{}^{\overline{D}\,\overline{C}}_{\overline{F}\,\overline{E}} \\
&= \lambda^{\overline{A}}_{\overline{C}}\lambda^{\overline{B}}_{\overline{D}}R_L{}^{CD}_{EF} \\
&= (\lambda^A_C\lambda^B_D R_L{}^{CD}_{EF})^*,
\end{aligned}
$$

$$
\begin{aligned}
(\lambda^A_C\lambda^B_D C^{CD})^* &= \lambda^{\overline{B}}_{\overline{D}}\lambda^{\overline{A}}_{\overline{C}}C^{\overline{D}\,\overline{C}} \\
&= C^{\overline{B}\,\overline{A}} \\
&= (C^{AB})^*.
\end{aligned}
$$

Thus '$*$' can be extended as an anti-algebra map. The $*$-Hopf algebra axioms are easy to check.

In order to show that $\Re$ is dual quasitriangular, we only have to prove that the fundamental representations defined above respect the metric relations, i.e. that the can be extended to matrix representations of L_q. Thus it only remains to show that

$$
\begin{aligned}
\rho_+(C_{AB}\lambda^A_C\lambda^B_D)^E_F &= C_{AB}\rho_+(\lambda^A_C)^E_M\rho_+(\lambda^A_C)^M_F \\
&= C_{CD}R_L{}^{CE}_{AM}R_L{}^{DM}_{BF} \\
&= C_{AB}\delta^F_E \\
&= \rho_+(C_{CD})^E_F,
\end{aligned}
$$

$$
\begin{aligned}
\rho_-(C_{CD}\lambda^A_C\lambda^B_D)^E_F &= C_{CD}\rho_-(\lambda^A_C)^E_M\rho_-(\lambda^A_C)^M_F \\
&= C_{CD}R^{-1}{}_L{}^{EC}_{MA}R^{-1}{}_L{}^{MD}_{FB} \\
&= C_{AB}\delta^F_E \\
&= \rho_-(C_{AB})^E_F.
\end{aligned}
$$

These relations can be verified by explicit calculation. Similar for the anti-representations. Note, however, that these conditions fix the normalization of R_L. We call this normalisation the ***quantum group normalisation.*** The normalisation of R_L shall be of importance in the next section, when we discuss q-Minkowski space.

(iii) We have to prove that φ as defined above can be extended as a $*$-Hopf algebra map: Using the the algebra structure on $\widetilde{SU_q(2)} \bowtie SU_q(2)$, which can also be written as:

$$(S\tilde{\tau}^b_d \otimes 1)R^{ad}_{cf}(1 \otimes \tau^c_e) = (1 \otimes \tau^a_c)R^{cb}_{ed}(S\tilde{\tau}^d_f \otimes 1)$$

or

$$(S\tilde{\tau}^d_f \otimes 1)\tilde{R}^{cb}_{ed}(1 \otimes \tau^a_c) = (1 \otimes \tau^c_e)\tilde{R}^{ad}_{cf}(S\tilde{\tau}^b_d \otimes 1),$$

one can show:

$$
\begin{aligned}
\varphi(\lambda^E_A\lambda^F_B R_L{}^{BA}_{DC}) &= \varphi(\lambda^E_A)\varphi(\lambda^F_B)R^{d_0\alpha}_{\beta a_0}R^{a_1\beta}_{\gamma b_0}R^{b_1\gamma}_{\delta c_1}\tilde{R}^{\delta c_0}_{d_1\alpha} \\
&= (S\tilde{\tau}^{a_0}_{e_0} \otimes 1)(1 \otimes \tau^{e_1}_{a_1})R^{a_1\beta}_{\gamma b_0}(S\tilde{\tau}^{b_0}_{f_0} \otimes 1)(1 \otimes \tau^{f_1}_{b_1})R^{d_0\alpha}_{\beta a_0}R^{b_1\gamma}_{\delta c_1}\tilde{R}^{\delta c_0}_{d_1\alpha} \\
&= R^{d_0\alpha}_{\beta a_0}(S\tilde{\tau}^{a_0}_{e_0} \otimes 1)(S\tilde{\tau}^{\beta}_{b_0} \otimes 1)R^{e_1 b_0}_{a_1 f_0}(1 \otimes \tau^{a_1}_{\gamma})(1 \otimes \tau^{f_1}_{b_1})R^{b_1\gamma}_{\delta c_1}\tilde{R}^{\delta c_0}_{d_1\alpha} \\
&= R^{\beta a_0}_{b_0 e_0}R^{e_1 b_0}_{a_1 f_0}R^{f_1 a_1}_{b_1\gamma}(S\tilde{\tau}^{d_0}_{\beta} \otimes 1)(S\tilde{\tau}^{\alpha}_{a_0} \otimes 1)\tilde{R}^{\delta c_0}_{d_1\alpha}(1 \otimes \tau^{b_1}_{\delta})(1 \otimes \tau^{\gamma}_{c_1}) \\
&= R^{\beta a_0}_{b_0 e_0}R^{e_1 b_0}_{a_1 f_0}R^{f_1 a_1}_{b_1\gamma}\tilde{R}^{b_1\alpha}_{\delta a_0}(S\tilde{\tau}^{d_0}_{\beta} \otimes \tau^{\delta}_{d_1})(S\tilde{\tau}^{c_0}_{\alpha} \otimes \tau^{\gamma}_{c_1}) \\
&= \varphi(R_L{}^{FE}_{(\beta\delta)(\alpha\gamma)}\lambda^{(\beta\delta)}_D\lambda^{(\alpha\gamma)}_C)
\end{aligned}
$$

$$
\begin{aligned}
\varphi(C_{AB}\lambda^A_C\lambda^B_D) &= \epsilon^{a_0\alpha}(S\tilde{\tau}^{c_0}_{a_0} \otimes 1)(1 \otimes \tau^{a_1}_{c_1})\tilde{R}^{\beta b_0}_{a_1\alpha}(S\tilde{\tau}^{d_0}_{b_0} \otimes 1)(1 \otimes \tau^{b_1}_{d_1})\epsilon_{\beta b_1} \\
&= \epsilon^{a_0\alpha}(S\tilde{\tau}^{c_0}_{a_0} \otimes 1)(S\tilde{\tau}^{b_0}_{\alpha} \otimes 1)\tilde{R}^{a_1 d_0}_{c_1 b_0}(1 \otimes \tau^{\beta}_{a_1})(1 \otimes \tau^{b_1}_{d_1})\epsilon_{\beta b_1} \\
&= \epsilon^{c_0 b_0}\tilde{R}^{a_1 d_0}_{c_1 b_0}\epsilon_{a_1 d_1} \\
&= \varphi(C_{CD}).
\end{aligned}
$$

Since φ respects the 'q-metric relation', we also have $\varphi(1) = 1$, completing the proof that φ is an algebra map, and $\varphi(S\lambda) = S_{\bowtie}(\varphi(\lambda))$.

b) φ is a coalgebra map:

$$\begin{aligned}
\Delta_{\bowtie}(\varphi(\lambda^A{}_B)) &= \Delta(S\tilde{\tau}^{b_0}{}_{a_0} \otimes \tau^{a_1}{}_{b_1}) \\
&= S\tilde{\tau}^{c_0}{}_{a_0} \otimes \tau^{a_1}{}_{c_1} \otimes S\tilde{\tau}^{b_0}{}_{c_0} \otimes \tau^{c_1}{}_{b_1} \\
&= \varphi(\lambda^A{}_C) \otimes \varphi(\lambda^C{}_B) \\
&= (\varphi \otimes \varphi)(\Delta\lambda^A{}_B)
\end{aligned}$$

$$\begin{aligned}
\varepsilon_{\bowtie}(\varphi(\lambda^A{}_B)) &= \delta^A_B \\
&= \varphi(\varepsilon(\lambda^A{}_B))
\end{aligned}$$

c) φ is a $*$-homomorphism:

$$\begin{aligned}
\varphi((\lambda^A_B)^*) &= \varphi(\lambda^{\overline{A}}_{\overline{B}}) \\
&= S\tilde{\tau}^{b_1}{}_{a_1} \otimes \tau^{a_0}{}_{b_0} \\
&= (S\tilde{\tau}^{b_0}{}_{a_0} \otimes \tau^{a_1}{}_{b_1})^* \\
&= \varphi(\lambda^A_B)^*
\end{aligned}$$

This completes the proof that φ can be extended as a $*$-Hopf algebra map. **Q.E.D.**

Because φ is not surjective, i.e. not a Hopf algebra isomorphism, it is possible to construct many different R-matrices which all give rise to different q-Lorentz groups with properties as described in the last proposition. We list a few examples of possible R-matrices:

$R_L{}^{AB}{}_{CD} = R^{b_0\alpha}{}_{\beta c_0} R^{c_1\beta}{}_{\gamma d_0} R^{d_1\gamma}{}_{\delta a_1} \tilde{R}^{\delta a_0}{}_{b_1\alpha}$, our choice of R-matrix,

$R_1{}^{AB}{}_{CD} = R^{b_0\alpha}{}_{\beta c_0} R^{-1\beta c_1}{}_{d_0\gamma} R^{d_1\gamma}{}_{\delta a_1} \tilde{R}^{\delta a_0}{}_{b_1\alpha}$, the braided matrices R-matrix as described in [10],

$R_2{}^{AB}{}_{CD} = R^{-1b_1a_0}{}_{\alpha\beta} R^{-1\alpha a_1}{}_{d_1\gamma} R^{\beta b_0}{}_{d_0\delta} R^{\gamma\delta}{}_{c_1d_0}$, the quantum double R-matrix from [14],

$R_3{}^{AB}{}_{CD} = R^{a_0b_1}{}_{\alpha\beta} R^{\alpha b_0}{}_{c_0\gamma} R^{a_1\beta}{}_{\delta d_1} R^{-1\gamma\delta}{}_{d_0c_1}$, the R-matrix from [2].

Only the discussion of q-Minkowski space in the next section shall enable us to identify R_L as the natural choice of R-matrix, since it is the only R-matrix which gives rise to a satisfactory non-commutative version of $\mathcal{P}(M)$.

4 q-Minkowski space

To construct a q- Minkowski space with coaddition, we follow the general approach of [14]: Let R be an invertible solution of the n-dimensional QYBE and let R' be another matrix such that they satisfy the mixed QYBEs $R'_{12}R_{13}R_{23} = R_{23}R_{13}R'_{12}$ and $R_{12}R_{13}R'_{23} = R'_{23}R_{13}R_{12}$. Then the algebra of quantum covectors $V'(R')$ is defined as the free associative $\mathbf{C}$-algebra generated by 1 and n generators x_i with relations $x_ix_j = x_kx_lR'^{lk}{}_{ij}$. Similarly, one defines quantum vectors $V(R')$ as generated by 1 and elements v^i with relations $v^iv^j = R'^{ij}{}_{kl}v^lv^k$. The mixed QYBEs ensure that both algebras are right $A(R)$-comodule algebras. In the case of $V'(R')$, the coaction and braiding between two copies are given by $x_i \mapsto x_j \otimes t^j{}_i$ and $\Psi(x_1 \otimes x_2) = x_2 \otimes x_1R_{L12}$. For $V'(R')$ to be a braided Hopf algebra in $\mathcal{M}^{A(R)}$ with braided

coaddition

$$\underline{\Delta} x_A = x_A \underline{\otimes} 1 + 1 \underline{\otimes} x_A, \qquad \underline{\varepsilon} x_A = 0, \qquad \underline{S} x_A = -x_A,$$

the two matrices have to satisfy

$$0 = (PR + 1)(PR' - 1)$$

and some other relations stated in [14]. P denotes the permutation matrix. Given any R, we can construct an R' satisfying these relations out of the minimal polynomial

$$0 = \prod_{\lambda_i \in Spec\, PR} (PR - \lambda_i)^{n_i}$$

of PR as

$$R' = P(1 + \prod_{\lambda_i \in Spec\, PR;\, i \neq 1} (PR - \lambda_i))^{n_i} (PR + 1)^{n_1 - 1},$$

provided we have the freedom of rescaling PR such that one of its eigenvalues, say λ_1, is -1.

Applying this construction to the q-Lorentz group R-matrix R_L shows why we consider the q-Lorentz group presented in this paper as more promising than its rivals and maybe even th definitive one: by direct computation, we find

Proposition 4.1 *qR_L has the eigenvalue -1. From the minimal polynomial of qPR_L, we obtain $R' = R_M$, where*

$$R_M{}^{AB}_{CD} = R^{-1}{}^{\delta c_0}_{b_0 \alpha} R^{b_1 \alpha}_{\beta a_0} R^{a_1 \beta}_{\gamma d_1} \tilde{R}^{\gamma d_0}_{c_1 \delta},$$

is the braided matrices R-matrix from [10]. Thus using the general technique to construct an algebra of quantum covectors with braided coaddition law yields the algebra relations of the braided Hermitean matrices: $V'(R') = BM(2)$ as algebras.

The q-Lorentz group proposed in this paper equips the algebra $BM(2)$ with a braided coaddition law. One can certainly apply this general construction to the other proposed q-Lorentz group R-matrices, but the resulting algebras of quantum vectors with braided coaddition law do not coincide with $BM(2)$ as algebras.

We thus propose to take the braided matrices $BM(2)$ as q-Minkowski space, but we give it a new name, M_q, to avoid confusion: $BM(2)$ was constructed as the transmutation of $A(R)$; i.e. the braided group associated to a quantum group. As such, it has the structure of a braided bialgebra in $\mathcal{M}^{A(R_1)}$, the category of right $A(R_1)$-comodules with a braided matrix coalgebra structure. M_q on the other hand is a braided Hopf algebra in $\mathcal{M}^{A(R_L)}$ with a braided coaddition. It shares with $BM(2)$ only the algebra structure, which is sufficient [15] to ensures the existence of a q-spinor decomposition, i.e. an algebra map $M_q \rightarrow V(R) \underline{\otimes} V'(R)$.

The explicit algebra relation in M_q are given by [10]:

$$\begin{array}{lllllllll} ba & = & q^2 ab & ac & = & q^2 ca & da & = & ad \\ db & = & bd + (1 - q^{-2}) ab & cd & = & dc + (1 - q^{-2}) ca & bc & = & cb + (1 - q^{-2}) a(d - a) \end{array},$$

with$x = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. The elements $qd+q^{-1}a$ ('braided trace') and $\underline{det} = ad-q^2cb$ ('braided determinant') are central in this algebra. Reflecting the fact that ordinary determinants are multiplicative, but not additive, the braided determinant is grouplike only in $BM(2)$, but not in M_q.

The appropriate $*$-structure on $BM(2)$ was discussed in [11]. Its axioms are slightly different from the ones recalled in the preliminaries. The main difference is that one requires $\underline{\Delta} \circ \underline{*} = \tau \circ (\underline{*} \otimes \underline{*}) \circ \underline{\Delta}$. In [11] it was shown that $BM(2)$ is a braided $*$-Hopf algebra (with the slightly modified axioms) in $\mathcal{M}^{A(R_1)}$, where the $*$-structure $\underline{*}$ is defined by

$$\begin{pmatrix} a^{\underline{*}} & b^{\underline{*}} \\ c^{\underline{*}} & d^{\underline{*}} \end{pmatrix} = \begin{pmatrix} a & c \\ b & d \end{pmatrix}.$$

It is easy to see that this also defines a $*$-structure on M_q as a braided group with braided coaddition in $\mathcal{M}^{A(R_L)}$. Note that the braided determinant is real with respect to this $*$-structure: $(\underline{det})^{\underline{*}} = \underline{det}$.

Similar to the commuting case, where the norm on Minkowski space is given by the determinant of the corresponding Hermitean matrix, we find:

Proposition 4.2 *The q-norm on M_q given by the q-deformed metric C^{AB} is given in terms of the braided determinant as: $x_A x_B C^{AB} = q^{-3}(1+q^2)\underline{det}$. It is therefore real and central.*

M_q is a right L_q-comodule algebra, due to the fact that R_L and R_M obey the mixed QYBEs. But the normalisation of R_L we have to choose in order to provide for a braided coaddition law on M_q is not the quantum group normalisation from section 3. This means that M_q is a braided group in $\mathcal{M}^{A(R_L)}$ but not in $\mathcal{M}^{L_q}$. This problem already occurred in [14] and is discussed in detail there: We have to consider the slightly extended Hopf algebra $\tilde{L}_q := L_q \otimes \mathbf{C}Z$ with dual quasitriangular structure $\Re(g^a \otimes g^b) := q^{ab}$, where g denotes the generator of the group algebra $\mathbf{C}Z$. M_q is a braided bialgebra in $\mathcal{M}^{\tilde{L}_q}$ and both the algebra structure and the coaddition of q-Minkowski space are covariant under the coaction of $\tilde{L}_q$:

Proposition 4.3 *q- Minkowski space M_q is an L_q comodule $*$-algebra with coaction $\beta_{\tilde{L}_q} : x_A \mapsto x_B \otimes \Lambda^B_A$. Using the push forward defined by the $*$-Hopf algebra injection φ, M_q is also an $\widetilde{SU_q(2)} \bowtie SU_q(2)$ and an $SL_q(2,\mathbf{C})$ comodule $*$-algebra with coaction*

$$x_A \mapsto x_B \otimes (S\tilde{\tau}^{a_0}_{b_0} \otimes \tau^{b_1}_{a_1}) = x_B \otimes t^{*\,a_0}_{\ b_0} t^{b_1}_{\ a_1}.$$

Moreover, M_q is a braided Hopf algebra in the braided category $\mathcal{M}^{\tilde{L}_q}$.

In the beginning of this section, we introduced algebras of quantum *vectors* and quantum *covectors*, a notation which was motivated in [15] by their respective transformation properties. For our q-Minkowski space, we can show even more: not only do q-Minkowski vectors and covectors transform in the appropriate way, but these algebras are also related by raising and lowering of indices with the q-deformed Minkowski metric C^{AB}:

Proposition 4.4

$$\begin{array}{rcl} V'(R_M) & \cong & V(R_M) \\ x_A & \mapsto & v^A := x_B C^{BA} \end{array}$$

Proof. One can show explicitly: $R_{M\ EF}^{\ KL} = C_{PF} C_{QE} R_{M\ AB}^{\ QP} C^{AK} C^{BL}$, implying: $x_A x_B = x_C x_D R_{M\ AB}^{\ DC} \Leftrightarrow x_A C^{AK} x_B C^{BL} = x_C C^{CE} x_D C^{DF} C_{PE} C_{QF} R_{M\ AB}^{\ QP} C^{AK} C^{BL} \Leftrightarrow v^K v^L = R_{M\ EF}^{\ KL} v^F v^E$.
Q.E.D.

For completeness, we list the explicit form of the braiding on the generators of M_q, which is quite different from the braiding on $BM(2)$ with its multiplicative coalgebra structure [10]. Recall from the beginning of this section, that the braiding is given in terms of the q-Lorentz R-matrix as $\Psi(x_1 \otimes x_2) = x_2 \otimes x_1 R_{L\ 12}$. With $\alpha := 1 - q^{-2}$, we obtain:

$$\begin{array}{ll}
\Psi(a \otimes a) = a \otimes a & \Psi(a \otimes b) = q^{-2} b \otimes a \\
\Psi(a \otimes c) = c \otimes a + \alpha a \otimes c & \Psi(a \otimes d) = q^{-2} d \otimes a + \alpha(b \otimes c + \alpha a \otimes a) \\
\Psi(b \otimes a) = a \otimes b + \alpha b \otimes a & \Psi(b \otimes b) = b \otimes b \\
\Psi(b \otimes c) = q^{-2} c \otimes b + \alpha q^{-2}(d \otimes a + a \otimes d + \alpha b \otimes c + & (2 - q^{-2}) a \otimes a) \\
\Psi(b \otimes d) = q^{-2} d \otimes b + \alpha(b \otimes (d - q^{-2} a) + \alpha a \otimes b) & \\
\Psi(c \otimes a) = q^{-2} a \otimes c & \Psi(c \otimes b) = q^{-2} b \otimes c + \alpha q^{-2} a \otimes a \\
\Psi(c \otimes c) = c \otimes c & \Psi(c \otimes d) = d \otimes c + \alpha c \otimes a \\
\Psi(d \otimes a) = q^{-2} a \otimes d + \alpha(b \otimes c + \alpha a \otimes a) & \Psi(d \otimes b) = b \otimes d + \alpha a \otimes b \\
\Psi(d \otimes c) = q^{-2} c \otimes d + \alpha((d - q^{-2} a) \otimes c - \alpha c \otimes a) & \Psi(d \otimes d) = d \otimes d + \alpha(c \otimes b - q^{-2} b \otimes c - q^{-2} \alpha a \otimes a)
\end{array}$$

From the q-Minkowski space M_q and the q-Lorentz group $\tilde{L}_q$, one can construct a q-Poincaré group as a semidirect product $M_q \rtimes \tilde{L}_q$ [14]. Details of the q-Poincaré group resulting from our new q-Lorentz group will be given in a forthcoming paper.

5 q-Lorentz group of enveloping algebra type

Apart from the deformation as a function algebra, there exists another standard way of deforming a simple Lie group, namely the deformation of its universal enveloping algebra. The Lorentz group is not simple, but its enveloping algebra is given by $U(so(3,1)) \cong U(\widetilde{su(2)}) \otimes U(su(2))$. We shall use this relation and the standard q-deformation of the enveloping algebra of the simple Lie group $SU(2)$ to construct a q-deformation of $U(so(3,1))$. We also investigate how this algebra is related to the q-Lorentz group of function algebra type.

For clarity, we first have to recall some standard constructions: For $A(R)$ introduced above there is a well-known dual bialgebra $\tilde{U}(R)$ [18, 8] generated by 1 and $2n^2$ symbols $l_b^{a\ \pm}$ with relations $l_1^{\pm} l_2^{\pm} R_{12} = R_{12} l_2^{\pm} l_1^{\pm}$ and $l_1^{-} l_2^{+} R_{12} = R_{12} l_2^{+} l_1^{-}$. The dual pairing with $A(R)$ is given by $< t_1, l_2^{\pm} >= R_{12}^{\pm}$, where $R^{+} = R$ and $R_{12}^{-} = R_{21}^{-1}$. For the $SU(2)$ R-matrix from above, $\tilde{U}(R)$ is known to be related to a deformation of the universal enveloping algebra of $su(2)$: $U_q(su(2))$ is defined as $\tilde{U}(R)$ with the further relations implied by the ansatz

$$l^{+} = \begin{pmatrix} q^{\frac{H}{2}} & 0 \\ q^{-1/2}(q - q^{-1}) X_{+} & q^{-\frac{H}{2}} \end{pmatrix}, \quad l^{-} = \begin{pmatrix} q^{-\frac{H}{2}} & q^{1/2}(q - q^{-1}) X_{-} \\ 0 & q^{\frac{H}{2}} \end{pmatrix}.$$

We use M. Jimbo's convention [5] and the usual $*$-structure for $U_q(su(2))$:

$$q^{\frac{H}{2}} X_\pm q^{-\frac{H}{2}} = q^{\pm 1} X_\pm, \quad [X_+, X_-] = \frac{q^H - q^{-H}}{q - q^{-1}}, \quad H^* = H, \quad X_\pm^* = X_\mp.$$

This is a quasitriangular $*$-Hopf algebra with coalgebra structure of matrix form and a well-known quasitriangular structure $\mathcal{R}$ and it is dual to $SU_q(2)$ as $*$-Hopf algebras (i.e. not only as bialgebras). The FRT generators are given by $l^+ = \mathcal{R}^{(1)} < \tau, \mathcal{R}^{(2)} >$ and $l^- =< \tau, \mathcal{R}^{-1\,(1)} > \mathcal{R}^{-1\,(2)}$, and we also have $< \tau_1 \otimes \tau_2, \mathcal{R} >= R_{12}$. τ denotes the generators of $SU_q(2)$.

A natural generalization of $U(so(3,1)) \cong U(\widetilde{su(2)}) \otimes U(su(2))$ is $U_q(\widetilde{su(2)}) \otimes_{\mathcal{R}} U_q(su(2))$, the twisted product of two copies of $U_q(su(2))$ as introduced in [19]. It is defined as the tensor product as algebras of two copies of $U_q(su(2))$ equipped with a doubly twisted coproduct and antipode, and also has a $*$-structure

$$\Delta_{\mathcal{R}}(x \otimes y) := \mathcal{R}_{23}^{-1} \tilde{\Delta}_{13}(x) \Delta_{24}(y) \mathcal{R}_{23}, \qquad S_{\mathcal{R}} := \mathcal{R}_{21}(\tilde{S} \otimes S)\mathcal{R}_{21}^{-1}, \quad (\tilde{l}_1^{\pm} \otimes l_2^{\pm})^* = \mathcal{R}_{21}^{-1}(\tilde{l}_2^{\pm *} \otimes l_1^{\pm *})\mathcal{R}_{21}.$$

There is a known $*$-Hopf algebra pairing

$$< \, , \, >: \widetilde{SU_q(2)} \bowtie SU_q(2) \otimes U_q(so(3,1)) \to \mathbf{C}.$$

In [19], the twisted product is presented with a quasitriangular structure related to the quantum double. But since our q-Lorentz group R-matrix does not coincide with the quantum double R-matrix either this it is not the quasitriangular structure we are interested in. We give a new quasitriangular structure on $U_q(\widetilde{su(2)}) \otimes_{\mathcal{R}} U_q(su(2))$ which is more suitable for our purposes:

Proposition 5.1 *$U_q(so(3,1)) := U_q(\widetilde{su(2)}) \otimes_{\mathcal{R}} U_q(su(2))$ is a quasitriangular $*$-Hopf algebra with*

$$\mathcal{R}_L = \mathcal{R}_{41}^{-1} \mathcal{R}_{24} \mathcal{R}_{13} \mathcal{R}_{23}.$$

Proof.

We have to show that $\mathcal{R}_L = \mathcal{R}_{41}^{-1}\mathcal{R}_{24}\mathcal{R}_{13}\mathcal{R}_{23}$ obeys $\tau o \Delta = \mathcal{R}_L \Delta \mathcal{R}_L^{-1}$, $(id \otimes \Delta)(\mathcal{R}_L) = \mathcal{R}_{L\,13}\mathcal{R}_{L\,12}$ and $(\Delta \otimes id)(\mathcal{R}_L) = \mathcal{R}_{L\,13}\mathcal{R}_{L\,23}$. Using the fact that $\mathcal{R}$ obeys these axioms and that such an $\mathcal{R}$ automatically satisfies the QYBE $\mathcal{R}_{12}\mathcal{R}_{13}\mathcal{R}_{23} = \mathcal{R}_{23}\mathcal{R}_{13}\mathcal{R}_{12}$, we obtain:

$$\begin{aligned}
\mathcal{R}_L \Delta_{\mathcal{R}}(.) \mathcal{R}_L^{-1} &= \mathcal{R}_{41}^{-1}\mathcal{R}_{24}\mathcal{R}_{13}\mathcal{R}_{23}(\mathcal{R}_{23}^{-1}\Delta_{13}\Delta_{24}\mathcal{R}_{23})\mathcal{R}_{23}^{-1}\mathcal{R}_{13}^{-1}\mathcal{R}_{24}^{-1}\mathcal{R}_{41} \\
&= \mathcal{R}_{41}^{-1}(\mathcal{R}_{24}\Delta_{24}\mathcal{R}_{24}^{-1})(\mathcal{R}_{13}\Delta_{13}\mathcal{R}_{13}^{-1})\mathcal{R}_{41} \\
&= \mathcal{R}_{41}^{-1}\Delta_{42}\Delta_{31}\mathcal{R}_{41} \\
&= \tau \circ \Delta_{\mathcal{R}}
\end{aligned}$$

$$\begin{aligned}
(id \otimes \Delta_{\mathcal{R}})(\mathcal{R}_L) &= \mathcal{R}_{45}^{-1}(id_{12} \otimes \Delta_{35}\Delta_{46})\mathcal{R}_{41}^{-1}\mathcal{R}_{24}\mathcal{R}_{13}\mathcal{R}_{23}\mathcal{R}_{45} \\
&= \mathcal{R}_{45}^{-1}(id \otimes \Delta_{46})(\mathcal{R}_{41}^{-1}\mathcal{R}_{24})(id \otimes \Delta_{35})(\mathcal{R}_{13}\mathcal{R}_{23})\mathcal{R}_{45} \\
&= \mathcal{R}_{45}^{-1}\mathcal{R}_{61}^{-1}\mathcal{R}_{41}^{-1}\mathcal{R}_{26}\mathcal{R}_{24}\mathcal{R}_{15}\mathcal{R}_{13}\mathcal{R}_{25}\mathcal{R}_{23}\mathcal{R}_{45} \\
&= (\mathcal{R}_{61}^{-1}\mathcal{R}_{26}\mathcal{R}_{15}\mathcal{R}_{25})(\mathcal{R}_{41}^{-1}\mathcal{R}_{24}\mathcal{R}_{13}\mathcal{R}_{23}) \\
&= \mathcal{R}_{L\,13}\mathcal{R}_{L\,12}
\end{aligned}$$

The proof of $(\Delta \otimes id)(\mathcal{R}_L) = \mathcal{R}_{L\,13}\mathcal{R}_{L\,23}$ is similar. Thus $\mathcal{R}_L$ is a quasitriangular structure for $U_q(sl(2,\mathbf{C}))$. **Q.E.D.**

Using the general construction described above, we could construct a bialgebra $\widetilde{U}(R_L)$ dual to $A(R_L)$. After dividing both algebras by suitable further relations, we would expect a dual pairing between L_q and its 'canonical' dual Hopf algebra. After the experience with $SU(2)$ and $SL(2,\mathbf{C})$, one would expect this dual Hopf algebra to have some kind of relation with $U_q(so(3,1))$, the q-deformation of the universal enveloping algebra. This is indeed the case, but $U_q(so(3,1))$ does not coincide with L_q's canonical dual.

Proposition 5.2 *It is possible to divide $\widetilde{U}(R_L)$ by some further relations obtaining a $*$-Hopf algebra $\check{U}(R_L)$. We denote its generators by $L^{\pm}$. Using the general construction [8], one can define a $*$-Hopf algebra injection*

$$\psi : \check{U}(R_L) \hookrightarrow U_q(\widetilde{su(2)}) \otimes_{\mathcal{R}} U_q(su(2)).$$

This map is not surjective.

The restriction of the pairing $< \, , \, >$ to the subalgebras $Im(\phi)$ and $Im(\psi)$ recovers the standard pairing between L_q and its canonical dual $\widetilde{U}(R_L)$:

$$R_L^{\pm} =< \lambda, L^{\pm} >=< \varphi(\lambda), \psi(L^{\pm}) > .$$

Furthermore, due to the fact that φ and ψ are $*$-Hopf algebra injections, L_q and $\check{U}(R_L)$ are also dually paired with $U_q(so(3,1))$ and $SL_q(2,\mathbf{C})$, respectively.

Proof. We show first:

$$\begin{aligned}
< \varphi(\lambda^B_D \otimes \lambda^A_C), \mathcal{R}_L > &= < S\tilde{\tau}^{d_0}_{b_0} \otimes \tau^{b_1}_{d_1} \otimes S\tilde{\tau}^{c_0}_{a_0} \otimes \tau^{a_1}_{c_1}, \mathcal{R}^{1\ -1}_{41} \mathcal{R}^2_{24} \mathcal{R}^3_{13} \mathcal{R}^4_{23} > \\
&= < S\tilde{\tau}^{\beta}_{b_0}, \mathcal{R}^{1\ -1}_{(2)} >< S\tilde{\tau}^{d_0}_{\beta}, \mathcal{R}^3_{(1)} >< \tau^{b_1}_{\delta}, \mathcal{R}^2_{(1)} >< \tau^{\delta}_{d_1}, \mathcal{R}^4_{(1)} > \\
&\quad < S\tilde{\tau}^{\alpha}_{a_0}, \mathcal{R}^3_{(2)} >< S\tilde{\tau}^{c_0}_{\alpha}, \mathcal{R}^4_{(2)} >< \tau^{a_1}_{\gamma}, \mathcal{R}^{1\ -1}_{(1)} >< \tau^{\gamma}_{c_1}, \mathcal{R}^2_{(2)} > \\
&= < \tau^{a_1}_{\gamma} \otimes S\tilde{\tau}^{\beta}_{b_0}, \mathcal{R}^{1\ -1} >< \tau^{b_1}_{\delta} \otimes \tau^{\gamma}_{c_1}, \mathcal{R}^2 > \\
&\quad < S\tilde{\tau}^{d_0}_{\beta} \otimes S\tau^{\alpha}_{a_0}, \mathcal{R}^3 >< \tau^{\delta}_{d_1} \otimes S\tilde{\tau}^{c_0}_{\alpha}, \mathcal{R}^4 > \\
&= R^{a_1\beta}_{\gamma b_0} R^{b_1\gamma}_{\delta c_1} R^{d_0\alpha}_{\beta a_0} \tilde{R}^{\delta c_0}_{d_1\alpha} \\
&= R_L{}^{BA}_{DC},
\end{aligned}$$

using the property $\tilde{R}^{\delta c_0}_{d_1\alpha} = \epsilon_{i\alpha} R^{\delta i}_{d_1 j} \epsilon^{jc_0}$ of the $SU(2)$ R-matrix. Now it is easy to check that $\psi : \widetilde{U}(R_L) \to U_q(so(3,1))$ defined according to the general construction [8] by

$$\psi(L^{+\ A}{}_B) := \mathcal{R}_L^{(1)} < \lambda^A_B, \mathcal{R}_L^{(2)} >, \quad \psi(L^{-\ A}{}_B) :=< \lambda^A_B, \mathcal{R}_L^{-1\,(1)} > \mathcal{R}_L^{-1\,(2)}$$

is a bialgebra map. If we define $\check{U}(R_L) := \widetilde{U}(R_L)/Ker(\psi)$, then the induced map, which we also call ψ is a $*$-Hopf algebra injection: define $S_{\check{U}(R_L)} := \psi^{-1} \circ S_{\mathcal{R}} \circ \psi$ and $* := \psi^{-1} \circ * \circ \psi$. The other assertions are easy to verify. **Q.E.D.**

Acknowledgements I would like to thank S. Majid for many helpful discussions.

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