Fermilab Library
0 1160 0026065 7

DAMTP 93-37A

DAMTP 93-37A
July 1993

Generalised Oscillator Systems and Their Parabosonic Interpretation

A.J.Macfarlane,
Department of Applied Mathematics & Theoretical Physics
University of Cambridge, Cambridge CB3 9EW, U.K.

DEC 10 1993
LIBRARY

Manuscript based on lecture delivered at
International Workshop on Symmetry Methods in Physics,
in Memory of Professor Ya A Smorodinsky,
Dubna 5-10 July 1993.

1 Introduction

In this work, we consider the Fock space descriptions of and some interesting properties of various bosonic oscillator systems.

All are based on a single creation-annihilation pair. All are represented in a suitable Fock space of states $|n\rangle$ of form $(a^\dagger)^n|0\rangle$, $n = 0, 1, 2\cdots$, with a ground state $|0\rangle$ such that $a|0\rangle = 0$ and a number operator N such that $N|n\rangle = n|n\rangle$.

From the quantum harmonic oscillator with its well-known commutation relations

$$[a, a^\dagger] = 1.$$

we pass to the study of a generalisation of it that we have elsewhere [1] called modification. This gives rise to the Calogero-Vasiliev oscillator [2],[3],[4] governed by

$$[a, a^\dagger] = 1 + 2\nu K, \tag{1}$$

where $\nu \in \mathbf{R}$, and $K = (-)^N$ obeys

$$K = K^\dagger, \quad K^2 = 1, \quad aK + Ka = 0, \quad a^\dagger K + Ka^\dagger = 0.$$

Below we review the construction of its Fock space $\mathcal{F}_\nu$, for which $2\nu > -1$ emerges as a condition sufficient to ensure the hermiticity properties implied by use of the dagger in $a^\dagger$, and also the $su(1,1)$ and $osp(1|2)$ properties of $\mathcal{F}_\nu$. A notable result, believed new here, is then explained. It states that for $\nu = (p-1)/2$, $\mathcal{F}_\nu$ coincides with the Fock space of a single paraboson of order $p = 1, 2\cdots$. An important consequence of this observation is that for Fock space work, one may use the bilinear commutation rule

(1), with $2\nu = p + 1$, instead of the more awkward trilinear commutation rules [5] that characterises paraboson systems of order p.

Then we go on to consider the q-deformation of the Calogero-Vasiliev oscillator. Setting out from the Fock space description of this, it is easy to follow the familiar method of replacing round brackets suitably by square brackets of the type

$$[x; r] = (q^{xr} - q^{-xr})/(q^r - q^{-r}), \quad [x; 1] = [x]. \tag{2}$$

This leads to a system involving a pair b and $b^\dagger$ governed by

$$bb^\dagger - q^{\pm(1+2\nu K)}b^\dagger b = [1 + 2\nu K]q^{\mp(N+\nu-\nu K)}, \tag{3}$$

in which q and $\nu \in \mathbf{R}$, and K and N have the meanings indicated above. While (3) might seem unduly complicated, it does describe a very natural q-deformation of the Calogero-Vasiliev oscillator and calculations in the Fock space $\mathcal{F}_{q\nu}$ in which it is represented are easy enough to do.

Our Fock space methods allow us to derive a variety of interesting simple formulas that hold true in $\mathcal{F}_{q\nu}$: these can be used to show clearly its relationship to the various simpler systems that can be reached from it as special or limiting cases. We show how $\mathcal{F}_{q\nu}$ interpolates between the spaces, defined for $\nu = (p-1)/2$, that describe q-deformed parabosons of order $p = 1, 2\cdots$. We discuss also the relationship of $\mathcal{F}_{q\nu}$ to representations of $su(1,1)_q$ and $osp(1|2)_q$.

We discuss next some relationship between the content of the present paper and previous work. Firstly we mention our recent paper [1] in which a different approach was followed to the combination of q-deformation and ν-modification. There, rather than effecting a q-deformation of (1) as here, we considered the simplest modification of the now familiar commutation relation [6] [7],[8]

$$aa^\dagger - qa^\dagger a = q^{-N}, \tag{4}$$

namely

$$aa^\dagger - qa^\dagger a = q^{-N}(1 + 2\nu K). \tag{5}$$

The consequences of (5) – its Fock space, $su(1,1)_q$ and $osp(1|2)_q$ properties, and hidden supersymmetry – show similarities with features of the present work, but have no parabose aspects.

We uesd the term Calogero-Vasiliev oscillator in relation to (1), because the latter author introduced it first explicitly [2] and gave its Fock space representation, and because it plays a crucial role in the modern progress in understanding [3],[4],[9] of the integrable many body model discovered by the former author [10].

Previous treatment of q-analogues of parabose (and also of parafermi) oscillators appears in [11]. This paper also discsses their $su(1,1)_q$ or $sp(2\mathbf{R})_q$ and $osp(1|2)_q$ properties, but not the relationship to ν-modification. We remark here that our approach gives rise to various interesting formulas not given explicitly in [11].

We mention here two works on general deformation schemes, including parabose and parafermi oscillators, namely the paper [12], which surveys earlier work, and the paper [13].

There is also the article [14] useful especially for $su(1,1)_q$ questions.The paper [15] contains information of q-deformation of parabose and parafermi operators, and mentions the relationship of $osp(1|2)_q$ representations to the Fock space of parabose oscillators.

Finally we note a matter which attracted debate at the International Workshop at which the work described in this paper was presented. The matter concerns the understanding of systems of the type in focus here within the context of Lie-admissible algebras. An assessment may be sought on the basis of [16] and [17] and from references given in the former source.

2 The Calogero-Vasiliev Oscillator.

We review here work on this system, partly to describe its relation to parabosons of order p, and partly to provide background for our discussion of q-deformation of the system later.

In seeking Fock space description of a and $a^\dagger$ subject to (1), we use the obvious facts that a and $a^\dagger$ reduce and increase the quantum number n of the kets $|n\rangle$ by 1, i.e. that

$$[N,a] = -a, \quad [N,a^\dagger] = a^\dagger.$$

Because $K = (-)^N$ features in (1), it is convenient to treat even and odd kets $|2n\rangle$ and $|2n+1\rangle$ separately. Thus we seek to represent (1) by means of

$$\langle 2n+1|a^\dagger|2n\rangle = p_n^{1/2}, \quad \langle 2n|a^\dagger|2n-1\rangle = q_n^{1/2}, \tag{6}$$

with $\langle n|n\rangle = 1$ for $n = 0,1,2\cdots$. By taking diagonal matrix elements of (1) for states $|2n\rangle$ and $|2n+1\rangle$, one is led to difference equations which can be seen to possess the following solutions

$$p_n = 2n + 2\nu + 1, \quad q_n = 2n. \tag{7}$$

In view of the former we impose the restriction $(2\nu+1) > 0$ or $\nu > -1/2$ on the modification parameter ν of (1). It follows from (6) and (7) that

$$aa^\dagger|2n\rangle = (2n+2\nu+1)|2n\rangle, \quad aa^\dagger|2n-1\rangle = 2n|2n-1\rangle. \tag{8}$$

Hence we deduce the operator identity

$$aa^\dagger = N + 1 + \nu(1+K), \tag{9}$$

valid on the Fock space $\mathcal{F}_\nu$ of the kets $|n\rangle$. Similarly

$$a^\dagger a = N + \nu(1-K). \tag{10}$$

Addition and subtraction of (9) and (10) now gives the expression

$$N = \{a,a^\dagger\}/2 - (2\nu+1)/2$$

for the number operator of $\mathcal{F}_\nu$, and the expected verification that (1) holds as an operator identity on $\mathcal{F}_\nu$. Use of results of the type (9) and (10) is crucial to our methods here and elsewhere. The idea of adjoining K to the families of operators with whose algebras we work offers valuable simplifications. The same applies to the more general use of exchange operators [3], [4], [9] in the study of Calogero systems.

Turning to $su(1,1)$ or $sp(2\mathbf{R})$ properties, we note that a^2, $a^{\dagger 2}$ and $\{a,a^\dagger\}$ have commutation rules

$$[a,a^{\dagger 2}] = 2a^\dagger, \quad [a^2,a^\dagger] = 2a = [a,\{a,a^\dagger\}] \tag{11}$$

that do not involve ν or$K$. Then it is easy to show that

$$K_+ = a^{\dagger 2}/2, \quad K_- = -K_+^\dagger = -a^2/2, \quad K_z = \{a,a^\dagger\}/4$$

obey an $su(1,1)$ algebra, and that its Casimir operator

$$\mathbf{K}^2 = K_z^2 + \{K_+,K_-\}/2,$$

commutes with $K_z, K_\pm$. One may then use (1) to show that $\mathbf{K}^2$ takes the value

$$-3/16 + \nu(\nu-K)/4$$

on $\mathcal{F}_\nu$. The fact that this gives the values $-3/16+\nu(\nu\pm1)$ on its even and odd subspaces can also be verified directly. Thus $\mathcal{F}_{\nu\pm}$ carry unitary irreducible representations of $su(1,1)$ with distinct eigenvalues of the Casimir operator of $su(1,1)$. This generalises the situation well-known [18] for the simple harmonic oscillator (with $\nu = 0$) itself, and one goes beyond as follows, recalling the work of [19] which drew attention to the relevance of osp-algebras to ordinary harmonic oscillator systems. To reach an algebra whose Casimir operator commutes with the entire Calogero-Vasiliev algebra of (1), i.e with a and $a^\dagger$ as well as $\mathbf{K}$, we evaluate $[\mathbf{K}^2,a]$. When we do this calculation using only (11), i.e. without assuming any knowledge of $[a,a^\dagger]$, we find that

$$[\mathbf{K}^2,a] = -[[a,a^\dagger],a]/8,$$

so that

$$C = \mathbf{K}^2 + [a,a^\dagger]/8 \tag{12}$$

commutes with a and hence $a^\dagger$ as well as $\mathbf{K}$. In fact it is well- known [20],[21] that $a, a^\dagger, \mathbf{K}$ can serve as a set of generators of $osp(1|2)$ and that C, given by (12), is its Casimir operator. It takes the value

$$c = -3/16 + \nu(\nu-K)/4 + (1+2\nu K)/8$$

on $\mathcal{F}_\nu$, with the pieces that depend on K cancelling, so that

$$c = -1/16 + \nu^2/4 \tag{13}$$

throughout $\mathcal{F}_\nu$. Thus the Fock space of the Calogero-Vasiliev oscillator carries a single irreducible unitary representation of $osp(1|2)$.

To conclude the section, we describe briefly the connection of the work of this section to the Calogero many body system. Let a_0 and $a_0^\dagger$ be a pair of ordinary harmonic oscillator variables and write

$$a_0 = (x+ip)/\sqrt{2}, \quad a_0^\dagger = (x-ip)/\sqrt{2}.$$

Then

$$a = a_0 - \frac{\nu}{\sqrt{2}x}K, \quad a^\dagger = a_0^\dagger + \frac{\nu}{\sqrt{2}x}K,$$

obey(1), and define the Hamiltonian

$$H = \{a,a^\dagger\}/2 = (p^2+x^2)/2 + \nu(\nu-K)/(2x^2).$$

This Hamiltonian describes the relative motion of a two-particle Calogero system. It follows from (9) and (10) that on $\mathcal{F}_\nu$ we have $H = 2N+1+2\nu$. Thus $\mathcal{F}_\nu$ provides a complete set of eigenstates of H

$$H|n\rangle = (2n+1+2\nu)|n\rangle. \quad n = 0,1,2\cdots.$$

3 The Parabose Oscillator of Order p.

Parabose oscillators emerge as a serious quantum mechanical possibility from asking the question: what are the weakest commutation rules that can be imposed on an a, $a^\dagger$ pair that make

$$[H,a] = -a, \quad [H,a^\dagger] = a^\dagger,$$

follow from use of the Hamiltonian

$$H = \{a,a^\dagger\}/2.$$

The answer is evident: we should impose only

$$[\{a,a^\dagger\},a] = \quad -a \quad = [a^\dagger,a^2] \tag{14}$$

and its adjoint. We observe immediately that these results have already been seen to hold in the Calogero-Vasiliev system, cf. (11). To obtain the Fock space representation of (14), see e.g. [5], it is necessary to supplement the specification $a|0\rangle = 0$ of the ground state by the demand that it satisfy

$$aa^\dagger|0\rangle = p|0\rangle, \quad p = 1,2,3\cdots. \tag{15}$$

The definition $N = \{a,a^\dagger\}/2$+const. makes $[N,a] = -a$ follow from (14). Then the explicit form

$$N = \{a,a^\dagger\}/2 - p/2,$$

is in agreement with $N|0\rangle = 0$, because of (15). It follows from this point that the completion of the description of the Fock space is contained in

$$\langle 2n+1|a^\dagger|2n\rangle = \sqrt{2n+p}, \langle 2n|a^\dagger|2n-1\rangle = \sqrt{2n}.$$

This agrees with the results (6) and (7) for $p = 2\nu+1$. The results given for the $su(1,1)$ and $osp(1|2)$ properties of the Calogero- Vasiliev oscillator also reproduce, for $\nu = (p-1)/2$, results already known for the parabose oscillator of order p. We note in particular that (13) now reads

$$c = (p^2 - 2p)/16. \tag{16}$$

The fact that

$$[a,a^\dagger] = 1 + (p-1)K,$$

holds in the Fock space of a parabose oscillator of order seems to be a new result here. Of course it does not follow from the algebra (14) alone; it requires also the use of (15) and is true only as an operator identity on the Fock space. It follows therefore that we should have a natural and direct derivation of it in the context of the Green ansatz [22] for the parabose oscillator of order p. We do; it is to be found in a forthcoming paper [23] on the algebraic description of parabose and parafermi systems based on a new explicit and technically convenient version of the Green ansatz.

4 The q-Deformed Calogero-Vasiliev Oscillator.

This is a system, described in a Fock space $\mathcal{F}_{q\nu}$, by a pair b and $b^\dagger$, which have the non-vanishing matrix-elements

$$\langle 2n+1|b^\dagger|2n\rangle = \sqrt{[2n+2\nu+1]}, \quad n = 0,1,2\cdots,$$
$$\langle 2n|b^\dagger|2n-1\rangle = \sqrt{[2n]}, \quad n = 1,2,3\cdots. \tag{17}$$

This specification arises from the equations of section two by the insertion of square brackets in the sense of (2). We note that

$$[N,b] = -b, \quad [N,b^\dagger] = b^\dagger,$$

are built into our approach, and that (17) yields also

$$bb^\dagger = [N+1+\nu(1+K)], \quad b^\dagger b = [N+\nu(1-K)]. \tag{18}$$

It can be shown that (18) implies

$$bb^\dagger - q^{\pm(1+2\nu K)}b^\dagger b = [1+2\nu K]q^{\mp(N+\nu-\nu K)}. \tag{19}$$

Thus, although this result specifies the algebra with whose representation we work, we did not proceed by postulation of it and by construction thence of the representation: rather it simply emerged in the form displayed in (19) from our obvious starting point.

We now deduce some algebraic consequences. Firstly we have

$$[b^2,b^\dagger] = b(q^{N+\nu+\nu K} + q^{-(N+\nu+\nu K)}). \tag{20}$$

This notable result follows most easily from (18) via

$$b^2b^\dagger = b(bb^\dagger) = b[N+1+\nu(1+K)],$$
$$b^\dagger b^2 = (b^\dagger b)b = b[N-1+\nu(1+K)].$$

The fact that the trilinear commutation relations have come out so simply may be considered to point towards an expected parabose interpretation of our analysis. Also, as $\nu \to 0$, (19) reduces to (4), and (20) reduces to its useful consequence

$$[a^2,a^\dagger] = a(q^N + q^{-N}),$$

here written in terms of a rather than b for the q-deformed oscillator.

If we put $\nu = (p-1)/2$ in our results, it reproduces findings of [11] for the q-deformation of the parabose oscillator of order p. It can be said, as in the undeformed situation, that $\mathcal{F}_{q\nu}$ interpolates between the Fock spaces of these oscillators. Our work however offers easy access to more explicit formulas than were given in [11]. One such is the version of (20) that arises for $\nu = (p-1)/2$ and is valid for any integer p; it agrees with a result for $p = 2$ quoted in [11]. Another arises by evaluating (19) separately in the even and odd subspaces of $\mathcal{F}_{q\nu}$. For $K = 1$ and $\nu = (p-1)/2$, this yields

$$bb^\dagger - q^{\mp p}b^\dagger b = [p]q^{\pm N}.$$

When $p = 1$ this is the usual q-deformed oscillator formula, but, for $p > 1$, contains a generalisation of it that looks very suitable for a parabose oscillator of higher order. For $K = -1$, on the other hand, and $\nu = (p-1)/2$, we get

$$bb^\dagger - q^{\mp(2-p)}b^\dagger b = [2-p]q^{\pm(N+p-1)}.$$

For $p = 1$, this is the familiar q-deformed oscillator result, while for higher integers p it reflects the fact that results for the even and odd subspaces often turn out to be of different appearance, a feature which can be avoided only by allowing K to appear explicitly.

We can directly discuss the $su(1,1)_q$ or $sp(2\mathbf{R})_q$ and $osp(1|2)_q$ properties of $\mathcal{F}_{q\nu}$. The definitions

$$B_+ = (q+q^{-1})^{-1}b^{\dagger 2}, \quad B_- = -B_+^\dagger, \quad 4B_z = 2N + 2\nu + 1,$$

are the same as those used in the limit $\nu \to 0$ and lead to the same algebra. One proves

$$[B_+, B_-] = [2B_z; 2],$$

in which the notation of (2) is used, most easily by considering actions of the two sides on the even and odd subspaces $\mathcal{F}_{q\nu\pm}$ of $\mathcal{F}_{q\nu}$ separately. It requires very little more calculation to show that

$$\mathbf{B}^2 = B_+B_- + [B_z - 1/2; 2]^2,$$

takes distinct values

$$[(2\nu \pm 1)/4; 2]^2,$$

independent of n, on the subspaces. Thus $\mathbf{B}^2$ commutes with $B_\pm, B_z$, but not with b or $b^\dagger$. To find the Casimir operator C_q of $osp(1|2)_q$ which commutes with b, one uses information from calculations already done to compute $[\mathbf{B}^2, b]$ on $\mathcal{F}_{q\nu\pm}$ separately and castes the result into the form

$$[\mathbf{B}^2, b] = -f(\nu)[K, b].$$

with

$$2f(\nu) = [(2\nu+1)/4; 2]^2 - [(2\nu-1)/4; 2]^2.$$

It follows that we may use the expression

$$C_q = \mathbf{B}^2 + f(\nu)K - [1/2; 2]^2/2$$

for the required Casimir operator. Here the last ν-independent term has been included so that our results take on the expected form in the limit $q \to 0$. Also one might reasonably ask for a more explicit alternative expression for the Casimir operator, one in which K did not appear explicitly. One can show however that, acting upon any state of $\mathcal{F}_{q\nu}$ whatever, that it takes the value

$$c_{q\nu} = ([(2\nu+1)/4; 2]^2 + [(2\nu-1)/4; 2]^2 - [1/2; 2]^2)/2.$$

When $\nu = (p-1)/2$, i.e. for the q-deformed paraboson of order p, this reduces to

$$c_{qp} = ([p/4; 2]^2 + [(2-p)/4; 2]^2 - [1/2; 2]^2)/2.$$

Finally as $q \to 1$, i.e. for the undeformed paraboson of order p, we get

$$c_p = (p^2 - 2p)/16,$$

which coincides with the result (16) given above.

NAME | I D No.
RETURN TO FERMILAB LIBRARY

References

[1] T.Brzeziński, I.L.Egusquiza and A.J.Macfarlane. Generalised Harmonic Oscillator Systems and Their Fock Space Description. *Preprint*, DAMTP 93-10,PAR-LPTHE 93-13.

[2] M.A.Vasiliev *Int. J. Mod. Phys.* A6:1115, 1991.

[3] L. Brink, T.H.Hanson and M.A.Vasiliev. *Phys. Lett.* B286:109, 1992.

[4] L. Brink, T.H.Hanson, S.Konstein and M.A.Vasiliev. The Calogero Model-Anyonic Representation, Fermionic Extension and Supersymmetry. *Preprint*, USITP-92-14, ITP-92-53, 1992.

[5] O.W.Greenberg and A.M.L.Messiah. *Phys. Rev.* 138B:1155,1965.

[6] M. Arik and D.D.Coon. *J. Math. Phys.* 17:524,1976.

[7] A.J.Macfarlane. *J. Phys. A: Math. Gen.* 22:4581,1989.

[8] L.C.Biedenharn. *J. Phys. A: Math. Gen.* 22:L873,1989.

[9] A.P.Polychronakos. *Phys. Rev. Lett.* 69:71,1991.

[10] F. Calogero. *J. Math. Phys.* 10:2191 and 2197,1969 and 12:419,1971.

[11] R.Floreanini and L.Vinet. *J. Phys. A: Math. Gen.* 23:L1019,1990.

[12] D.Bonatsos and C.Daskaloyannis. *Phys. Lett.* B307:100,1993.

[13] K.Odaka, T.Kishi and S.Kamefuchi. *J. Phys. A: Math. Gen.* 24:L591,1991.

[14] V. Spirodonov. Deformation of Supersymmetric and Conformal Quantum Mechanics. *Preprint* UdeM-LPN-TH 94-92, 1992.

[15] E.Celeghini, T.D.Palev and M.Tarlini. *Mod. Phys. Lett.* B5:187,1991.

[16] A.Jannussis. Genotopic and Isotopic Structure of Quantum Groups. Paper presented to International Workshop on Symmetry Methods in Physics, Dubna, 5-10 July, 1993.

[17] A.Jannussis. *J. Phys. A: Math. Gen.* L233,1993.

[18] C. Itzykson. *Commun. Math. Phys.* 4:162,1967.

[19] M.de Crombrugghe and V.Rittenberg. *Ann. Phys.* 151:99,1983.

[20] M. Scheunert,W.Nahm and V.Rittenberg. *J. Math. Phys.* 18:146 and 155,1977.

[21] P.P.Kulish and N.Yu.Reshitikhin. *Lett. Math. Phys.* 18:143,1989.

[22] H.S.Green. *Phys. Rev.* 90:270,1953.

[23] A.J.Macfarlane. Algebraic Structure of Parabose and Parafermi Fock Space, I: The Green Ansatz Revisited. *Preprint* DAMTP 93-40, in preparation.