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Kahler-Einstein metrics with SU(2) action.

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1. Introduction.

The aim of this paper is to analyse Kähler-Einstein metrics of real dimension four admitting an isometric action of $SU(2)$ with generically three-dimensional orbits. In the case when the Einstein constant is zero the metrics are hyperkähler and have been classified [BGPP], [GP],[AH]. We shall take the Einstein constant to be nonzero.

We derive a system of ordinary differential equations whose solutions correspond precisely to such metrics, and we determine which trajectories of the equations give complete metrics. There are two families of complete metrics with negative Einstein constant; one of these families is that found in [GP], [Pd].

2. Kahler forms.

Suppose that we have an Einstein metric in four real dimensions with an isometric action of $SU(2)$. We also suppose that the generic orbit is three-dimensional. The Einstein condition implies that the metric can be expressed in the form

$$g = (abc)^2 dt^2 + a^2\sigma_1^2 + b^2\sigma_2^2 + c^2\sigma_3^2 \tag{1}$$

where t is a coordinate transverse to the $SU(2)$ orbits, a, b and c are functions of t, and the σ_i are the invariant one-forms satisfying $d\sigma_1 = \sigma_2 \wedge \sigma_3$ etc. Explicitly

$$\begin{aligned}
\sigma_1 &= -\cos\psi\, d\theta - \sin\psi \sin\theta\, d\phi \\
\sigma_2 &= \sin\psi\, d\theta - \cos\psi \sin\theta\, d\phi \\
\sigma_3 &= -d\psi - \cos\theta\, d\phi
\end{aligned}$$

where θ, ϕ, ψ are Euler angles on the $SU(2)$ orbits.

The vector fields dual to $dt, \sigma_1, \sigma_2, \sigma_3$ are $\frac{\partial}{\partial t}, X_1, X_2, X_3$ where

$$\begin{aligned}
X_1 &= -\cos\psi \frac{\partial}{\partial\theta} - \frac{\sin\psi}{\sin\theta}\frac{\partial}{\partial\phi} + \cot\theta \sin\psi \frac{\partial}{\partial\psi} \\
X_2 &= \sin\psi \frac{\partial}{\partial\theta} - \frac{\cos\psi}{\sin\theta}\frac{\partial}{\partial\phi} + \cot\theta \cos\psi \frac{\partial}{\partial\psi} \\
X_3 &= -\frac{\partial}{\partial\psi}.
\end{aligned}$$

We have the relations

$$\left[\frac{\partial}{\partial t}, X_i\right] = 0$$

and

$$[X_1, X_2] = -X_3 \quad \text{and cyclically.}$$

Let us assume that the metric is Kählerian. We further assume that the Einstein constant is nonzero so the metric is not hyperkähler. Therefore the space of covariant constant two-forms will have dimension less than three. However this space is a real representation of the isometry group $SU(2)$ and hence will be acted on trivially by this group.

Now the Kähler form on a surface is always self-dual, so we deduce from the above remarks that our Kähler form Ω is given by

$$\Omega = A(t)\Omega_1^+ + B(t)\Omega_2^+ + C(t)\Omega_3^+ \tag{2}$$

where A, B, C are functions of t only, and

$$\begin{aligned}\Omega_1^+ &= e_0 \wedge e_1 + e_2 \wedge e_3 \\ \Omega_2^+ &= e_0 \wedge e_2 + e_3 \wedge e_1 \\ \Omega_3^+ &= e_0 \wedge e_3 + e_1 \wedge e_2.\end{aligned}$$

Here $e_0 = abc\, dt, e_1 = a\sigma_1, e_2 = b\sigma_2, e_3 = c\sigma_3$ defines an orthonormal coframe for g.

Now the condition $d\Omega = 0$ is equivalent to the equations

$$(Abc)' = Aa^2bc \tag{3}$$

$$(Bac)' = Bab^2c \tag{4}$$

$$(Cab)' = Cabc^2. \tag{5}$$

If we now introduce the standard variables $w_1 = bc, w_2 = ac, w_3 = ab$ and define functions α, β, γ by

$$w_1' = w_2 w_3 + \alpha w_1 \tag{6}$$

$$w_2' = w_1 w_3 + \beta w_2 \tag{7}$$

$$w_3' = w_1 w_2 + \gamma w_3 \tag{8}$$

then the equations (3-5) become

$$A' = -\alpha A \tag{9}$$

$$B' = -\beta B \tag{10}$$

$$C' = -\gamma C. \tag{11}$$

We see that for each metric g there is a 3-dimensional space of closed, selfdual, $SU(2)$-invariant two-forms, which are candidates for Kähler forms.

3. Integrability.

Given such a form Ω we can use the metric to define an endomorphism I of the tangent bundle by

$$g(IX, Y) = \Omega(X, Y).$$

With Ω given by (2) the endomorphism is defined by

$$I\frac{\partial}{\partial t} = Aw_1X_1 + Bw_2X_2 + Cw_3X_3 \tag{12}$$

$$IX_1 = -\frac{A}{w_1}\frac{\partial}{\partial t} + \frac{Cw_2}{w_1}X_2 - \frac{Bw_3}{w_1}X_3 \tag{13}$$

$$IX_2 = -\frac{B}{w_2}\frac{\partial}{\partial t} - \frac{Cw_1}{w_2}X_1 + \frac{Aw_3}{w_2}X_3 \tag{14}$$

$$IX_3 = -\frac{C}{w_3}\frac{\partial}{\partial t} + \frac{Bw_1}{w_3}X_1 - \frac{Aw_2}{w_3}X_2. \tag{15}$$

Moreover I is an almost complex structure if and only if $A^2 + B^2 + C^2 = 1$.

If the above constraints on A, B, C are satisfied then Ω is a Kähler form precisely when I is integrable. Let us now check when integrability holds.

Assume first that A is not identically ± 1.

Then we can take

$$\chi_1 = \frac{\partial}{\partial t} - iAw_1X_1 - iBw_2X_2 - iCw_3X_3$$

and

$$\chi_2 = \frac{iA}{w_1}\frac{\partial}{\partial t} + X_1 - \frac{iCw_2}{w_1}X_2 + \frac{iBw_3}{w_1}X_3$$

as independent (1,0) vector fields.

Their Lie bracket is given by

$$[\chi_1, \chi_2] = P\frac{\partial}{\partial t} + QX_1 + RX_2 + SX_3$$

where

$$
\begin{aligned}
P &= i(\frac{A}{w_1})' \\
Q &= -\frac{A}{w_1}(Aw_1)' - (B^2+C^2)\frac{w_2w_3}{w_1} \\
R &= -i(\frac{Cw_2}{w_1})' + (AB+iC)w_3 - \frac{A}{w_1}(Bw_2)' \\
S &= i(\frac{Bw_3}{w_1})' + (AC-iB)w_2 - \frac{A}{w_1}(Cw_3)'.
\end{aligned}
$$

Using the equations (6-11) we can simplify these expressions to

$$
\begin{aligned}
P &= -\frac{iA}{w_1}(2\alpha + \frac{w_2w_3}{w_1}) \\
Q &= -\frac{w_2w_3}{w_1} \\
R &= \frac{iCw_2}{w_1}(\gamma - \beta + \alpha + \frac{w_2w_3}{w_1}) \\
S &= \frac{iBw_3}{w_1}(\gamma - \beta - \alpha - \frac{w_2w_3}{w_1}).
\end{aligned}
$$

Now $[\chi_1, \chi_2]$ is a (1,0) vector field if and only if the following equations hold.

$$
\begin{aligned}
-iP - \frac{A}{w_1}Q - \frac{B}{w_2}R - \frac{C}{w_3}S &= 0 \\
Aw_1P - iQ - \frac{Cw_1}{w_2}R + \frac{Bw_1}{w_3}S &= 0 \\
Bw_2P + \frac{Cw_2}{w_1}Q - iR - \frac{Aw_2}{w_3}S &= 0 \\
Cw_3P - \frac{Bw_3}{w_1}Q + \frac{Aw_3}{w_2}R - iS &= 0.
\end{aligned}
$$

On substituting in our expressions for P, Q, R, S these equations reduce to

$$
\begin{aligned}
\alpha A &= -iBC(\gamma - \beta) && (16) \\
\alpha(1+A^2) &= (B^2 - C^2)(\gamma - \beta) && (17) \\
(C - iAB)(\alpha - \beta + \gamma) &= 0 && (18) \\
(B + iAC)(\gamma - \beta - \alpha) &= 0 && (19)
\end{aligned}
$$

From (18) we see that $\beta = \alpha + \gamma$ or $C = iAB$.

In the former case equation (19) implies that $\alpha = 0$ or $B = -iAC$. If $\alpha = 0$ then $\beta = \gamma$. If, on the other hand, $\alpha \neq 0$ and $B = -iAC$, then (16),(17) imply that $\alpha = C^2(\beta - \gamma)$ and hence $C^2 = 1$. It follows from (11) that $\gamma = 0$ and $\alpha = \beta$.

Similar manipulations in the case $C = iAB$ lead to the conclusion that either $\alpha = 0, \beta = \gamma$ or $\beta = 0, \alpha = \gamma$.

In the case $A \equiv 1$ we can take

$$\frac{\partial}{\partial t} - iw_1X_1 \ , \ X_2 - i\frac{w_3}{w_2}X_3$$

as independent (1,0) vector fields. Imposing the integrability condition leads to the equation

$$w_1 - (\frac{w_3}{w_2})' = \frac{w_1w_3^2}{w_2^2}$$

which implies that $\beta = \gamma$. We also have $\alpha = 0$ because $A \equiv 1$. The case $A \equiv -1$ is similar.

Theorem 3.1

If the metric (1) is Kählerian and not hyperkählerian then one of the following three conditions hold.

(i) $\alpha = 0, \beta = \gamma$

(ii) $\beta = 0, \alpha = \gamma$

(iii) $\gamma = 0, \alpha = \beta$.

Conversely if (i),(ii) or (iii) is true then the metric is Kählerian. □.

In fact, it is straightforward to show that in case (i) we must have either $B \equiv C \equiv 0$ or else $\alpha = \beta = \gamma = 0$ (in which case our metric is hyperkähler). Similar statements hold, with appropriate permutations, in cases (ii) and (iii).

4. The Einstein equations.

We now consider what constraints on a, b, c the Einstein condition imposes.

The Einstein equations in terms of a, b, c are

$$\begin{aligned}
2(\frac{a'}{a})' + 2\Lambda(abc)^2 &= a^4 - (b^2 - c^2)^2 \\
2(\frac{b'}{b})' + 2\Lambda(abc)^2 &= b^4 - (c^2 - a^2)^2 \\
2(\frac{c'}{c})' + 2\Lambda(abc)^2 &= c^4 - (b^2 - a^2)^2 \\
4(\frac{a'b'}{ab} + \frac{b'c'}{bc} + \frac{c'a'}{ca}) + 4\Lambda(abc)^2 &= -a^4 - b^4 - c^4 + 2a^2b^2 + 2b^2c^2 + 2c^2a^2.
\end{aligned}$$

As we are assuming that the metric is Kählerian, from Theorem 3.1 we can without real loss of generality take $\alpha = \beta$ and $\gamma = 0$. Using (6-8) the Einstein equations reduce to the constancy of Λ and the relation

$$\alpha = -\Lambda w_3^2$$

so we obtain the following result.

Theorem 4.1

The Kähler-Einstein metrics of the form (1) which are not hyperkähler are given, up to permutations, by solutions of the equations

$$\begin{aligned}
w_1' &= w_2 w_3 - \Lambda w_3^2 w_1 \\
w_2' &= w_3 w_1 - \Lambda w_3^2 w_2 \\
w_3' &= w_1 w_2
\end{aligned}$$

where Λ is a nonzero constant.

In terms of the original variables a, b, c, the equations are

$$\begin{aligned}
a' &= \frac{1}{2}a(b^2 + c^2 - a^2) \\
b' &= \frac{1}{2}b(c^2 + a^2 - b^2) \\
c' &= \frac{1}{2}c(a^2 + b^2 - c^2 - 2\Lambda a^2 b^2).
\end{aligned}$$

□

In some cases we can write down the metric explicitly, and recover known metrics.

(i) If $w_1 = w_2$ (that is, $a = b$) then the solution to the equations is

$$w_1 = w_2 = \sqrt{(w_3')}$$

where

$$w_3' = w_3^2 - \frac{2}{3}\Lambda w_3^3 + \delta$$

and δ is a constant.

After making the substitution $w_3 = \rho^2/4$ we arrive at the Kähler-Einstein metric

$$g = (1 - \frac{\Lambda}{6}\rho^2 + \frac{16\delta}{\rho^4})^{-1} d\rho^2 + \frac{1}{4}\rho^2(\sigma_1^2 + \sigma_2^2) + \frac{1}{4}\rho^2(1 - \frac{\Lambda}{6}\rho^2 + \frac{16\delta}{\rho^4})\sigma_3^2 \qquad (20)$$

which was obtained in **[GP]** and **[Pd]**. This metric has isometry group $U(2)$ rather than $SU(2)$ because $a = b$.

(ii) We can also obtain the Bianchi IX form of the product metric on $\mathbb{CP}^1 \times \mathbb{CP}^1$ by making the ansatz

$$a = \sqrt{\frac{2}{\Lambda}},\ b = \sqrt{\frac{2}{\Lambda}}\sin F,\ c = \sqrt{\frac{2}{\Lambda}}\cos F$$

where F is a function of t. After taking F as the new radial coordinate we can write the metric as

$$\frac{2}{\Lambda}(dF^2 + \sigma_1^2 + \sin^2 F\, \sigma_2^2 + \cos^2 F\, \sigma_3^2).$$

(iii) Finally, the ansatz

$$a^2 = \frac{3}{\Lambda}(1+\cos G),\ b^2 = \frac{3}{\Lambda}(1-\cos G),\ c^2 = \frac{6}{\Lambda}\cos^2 G$$

leads to the metric

$$\frac{6}{\Lambda}\,(d(\frac{G}{2})^2 + \frac{1}{2}(1+\cos G)\sigma_1^2 + \frac{1}{2}(1-\cos G)\sigma_2^2 + \cos^2 G\sigma_3^2)$$

which is the form of the Fubini-Study metric on $\mathbb{C}\mathbb{P}^2$ obtained by writing it in an $SU(2)$-invariant form with respect to the maximal embedding of $SO(3)$ in the isometry group $SU(3)$ [BG].

Note

A calculation of the Weyl tensor for the metrics of theorem 4.1 shows that the only self-dual examples are the Fubini-Study metric on $\mathbb{C}\mathbb{P}^2$ and the pseudo-Fubini- Study metric on the open ball in $\mathbb{C}^4$

Note

If $\Lambda = 0$ the equations of Theorem 4.1 give us the hyperkähler metrics with triholomorphic $SU(2)$ action. The equations are now solvable in terms of Jacobi elliptic functions, and the resulting metrics were analysed in [BGPP].

Remark

We can also derive systems of equations for Kähler-Einstein metrics with other Bianchi type A isometry groups preserving the Kähler structure.

5. The differential equations.

Our next aim is to see which solutions of our differential equations give rise to complete metrics. As remarked above, the equations are completely understood when $\Lambda = 0$.

If we take the Einstein constant Λ to be positive then we can argue as follows, without needing to study the equations directly.

Any complete Kähler-Einstein manifold M of real dimension four with $\Lambda > 0$ is compact, by Myers's Theorem. Moreover, considered as a complex manifold M has $c_1 > 0$, so is isomorphic to either $\mathbb{P}^1 \times \mathbb{P}^1$ or the blowup of $\mathbb{P}^2$ at n points, where $0 \leq n \leq 8$. If the metric is $SU(2)$-invariant then, as explained earlier, the Kähler form is also $SU(2)$-invariant so the $SU(2)$ action is holomorphic. Hence the complex automorphism group of M has a subgroup isomorphic to $SL(2,\mathbb{C})$. Using the descriptions of the automorphism groups of blowups of $\mathbb{P}^2$ in Chapter VII of [F] we see that n must be less than 3. In fact the blowup of $\mathbb{P}^2$ at one or two points cannot admit any Kähler-Einstein metric [F] as the Lie algebra of the automorphism group is not reductive. So M must be isomorphic to $\mathbb{P}^2$ or $\mathbb{P}^1 \times \mathbb{P}^1$.

Now it is a theorem of Matsushima [B],[M] that the Lie algebra of the isometry group of a compact Kähler-Einstein space with positive Einstein constant is a compact real form of the Lie algebra of the automorphism group. It follows that any such metric on $\mathbb{P}^2$ has isometry group $SU(3)$ (or a finite quotient) so is the symmetric space metric (using the classification of compact Einstein four-manifolds with isometry group of dimension at least four given in [BB]). Similarly any such metric on $\mathbb{P}^1 \times \mathbb{P}^1$ is the standard product metric.

¿From now on, therefore, we shall just consider the case when the Einstein constant is negative. It is clear that by rescaling t, a, b, c we can set Λ equal to -1, so the equations we shall be looking at are

$$a' = \frac{1}{2}a(b^2+c^2-a^2) \tag{21}$$

$$b' = \frac{1}{2}b(c^2+a^2-b^2) \tag{22}$$

$$c' = \frac{1}{2}c(a^2+b^2-c^2+2a^2b^2). \tag{23}$$

Consider a solution of (21-23), analytic on a maximal interval (ξ,η). It is immediate from the equations that if a is zero at any point of (ξ,η) then a is identically zero throughout this interval. The corresponding statements also hold for b, c. As we are assuming that $SU(2)$ acts with three-dimensional orbits we can exclude this case, so

we can assume that a, b, c are nowhere zero in the interval. The metric will therefore be defined on (ξ, η); to check whether it is complete we need to study its behaviour as t tends to ξ and as t tends to η.

Now our expressions for the metric, as well as the equations, are invariant under changes of sign of a, b or c, so we shall take a, b, c to be positive.

We summarise certain useful properties of the equations below.

Lemma 5.1

The critical points of the equations (21-23) are given by

$a = b, c = 0$

$b = c, a = 0$

$a = c, b = 0.$

The linearisation about a critical point which is not the origin has one positive, one negative and one zero eigenvalue. Hence there is at least one unstable curve for this critical point.

□

Lemma 5.2

$$\begin{aligned}
(ab)' &= abc^2 \\
(bc)' &= bca^2(1+b^2) \\
(ac)' &= acb^2(1+a^2) \\
(\frac{a}{b})' &= \frac{a}{b}(b^2-a^2) \\
(a^2-b^2)' &= (a^2-b^2)(c^2-a^2-b^2) \\
(a^2-c^2)' &= (a^2-c^2)(b^2-a^2-c^2)-2(abc)^2.
\end{aligned}$$

□

It follows that ab, bc, ac are increasing on (ξ, η). Moreover either a is identically equal to b in (ξ, η) or else a is never equal to b. In the latter case the symmetry

between a and b means that we can take $a > b$. Therefore we can assume from now on that $a \geq b$ and hence that b is increasing and $\frac{a}{b}$ is decreasing.

We shall now analyse the trajectories of the equations (21-23). Let us first deal with the case when the maximal interval is (ξ, η) where ξ is finite.

Recall that b, ab, bc, ac are monotonic increasing, so they tend to finite nonnegative limits as t tends to ξ. Now, as ξ is a singularity of the solution we see that a, b, c cannot all tend to finite limits at ξ, so it must be the case that b tends to zero at ξ.

(i) Assume first that $b^2 + c^2 - a^2$ is negative at some u. It easily follows from the equations that $(b^2 + c^2 - a^2)'$ is then positive at u. Hence $b^2 + c^2 - a^2$ is negative on (ξ, u), so a is decreasing and c is increasing on (ξ, u). We deduce that

$$a \to \nu, b \to 0, c \to \mu \text{ as } t \to \xi.$$

Here μ is finite but ν may be infinite.

But ξ is a singularity of the solution, so in fact $\nu = \infty$. Recalling that ac tends to a finite limit as t tends to ξ, we see that

$$a \to \infty, b \to 0, c \to 0 \text{ as } t \to \xi.$$

So near ξ the equations are approximately

$$\begin{aligned}
a' &= -\frac{1}{2}a^3 \\
b' &= \frac{1}{2}ba^2 \\
c' &= \frac{1}{2}c(a^2+2\lambda^2)
\end{aligned}$$

where $\lambda = \lim_{t\to\xi} ab$

We deduce that

$$\begin{aligned}
a &\simeq (t-\xi)^{-\frac{1}{2}} \\
b &\simeq \tau_1(t-\xi)^{\frac{1}{2}} \\
c &\simeq \tau_2(t-\xi)^{\frac{1}{2}}
\end{aligned}$$

where τ_1, τ_2 are nonzero constants. Substituting into our expression for the metric shows that the metric has a singularity at $t = \xi$.

(ii) Now assume instead that $c^2 - a^2 - b^2 - 2a^2b^2$ is positive at some u. It easily follows that the derivative of this expression is negative at u, so $c^2 - a^2 - b^2 - 2a^2b^2$ is positive on (ξ, u). Hence a, b are increasing and c is decreasing on (ξ, u). Arguing as above we find that

$$a, b \to 0, c \to \infty \text{ as } t \to \xi.$$

The equations tell us that near $t = \xi$ we have

$$\begin{aligned} a &\simeq \tau_1(t-\xi)^{\frac{1}{2}} \\ b &\simeq \tau_2(t-\xi)^{\frac{1}{2}} \\ c &\simeq (t-\xi)^{-\frac{1}{2}} \end{aligned}$$

for nonzero constants τ_1, τ_2. Again this leads to a singular metric.

(iii) The remaining possibility is that

$$a^2 \leq b^2 + c^2, c^2 \leq a^2 + b^2 + 2a^2b^2$$

on (ξ, η). This implies that a, c are increasing , so tend to finite limits as $t \to \xi$. Also recall that $b \to 0$ as $t \to \xi$. So this contradicts the assumption that ξ is a singularity.

We summarise the results of this section as follows.

Proposition 5.3.

If the maximal interval is (ξ, η) where ξ is finite, then the metric is incomplete. □

6. Complete metrics.

The results above show that we need now only consider solutions to (21-23) defined on $(-\infty, \eta)$ We remark that such solutions certainly exist, because the unstable curves of critical points of the equations provide examples.

Consider an arbitrary solution to (21-23) defined on $(-\infty, \eta)$. Now, as b, ab, bc, ac are increasing on $(-\infty, \eta)$ we see that

$$b \to \lambda_1, ab \to \lambda_2, bc \to \lambda_3, ac \to \lambda_4 \text{ as } t \to -\infty.$$

If $\lambda_1 > 0$ then we deduce that $a \to \frac{\lambda_2}{\lambda_1}, c \to \frac{\lambda_3}{\lambda_1}$ as $t \to -\infty$. It follows that (a, b, c) tends to a critical point as t tends to $-\infty$. As we are assuming that the limiting value of b is positive, we see from Lemma 5.1 that this point is $(q, q, 0)$ for some $q > 0$. However Lemma 5.2 shows that $\frac{a}{b}$ is decreasing and ≥ 1, so this situation only occurs if a is identically equal to b.

Suppose, on the other hand, that $\lambda_1 = 0$, that is, b tends to 0 as t tends to $-\infty$.

Now we can argue as in section 5. We again distinguish three cases.

(i) Assume $b^2 + c^2 - a^2$ is negative at u. Then, as before, we find that a is decreasing while b, c are increasing on $(-\infty, u)$ so

$$a \to \nu, b \to 0, c \to \mu \text{ as } t \to -\infty.$$

Here ν may be ∞ but μ is finite. In fact a'' is positive so $\nu = \infty$. But ac tends to a finite limit, so $\mu = 0$, and

$$a \to \infty, \ b, c \to 0 \text{ as } t \to -\infty.$$

The equations now show that

$$a' \simeq -\frac{1}{2}a^3$$

so in fact a blows up in finite time , giving a contradiction.

(ii) Similar arguments show that if $c^2 > a^2 + b^2 + 2a^2b^2$ at some u, then

$$a, b \to 0, c \to \infty \text{ as } t \to -\infty$$

and moreover c blows up in finite time. So again we have a contradiction.

(iii) The remaining possibility is that

$$a^2 \leq b^2 + c^2, c^2 \leq a^2 + b^2 + 2a^2b^2. \tag{24}$$

These inequalities mean that a, c are monotonic increasing so converge to finite limits as t tends to $-\infty$. As we know that b tends to 0 it follows that (a, b, c) converges to a critical point of the equations (21)-(23), which must be $(q, 0, q)$ for some q. If $q = 0$, that is, if a, b, c all converge to 0 as t tends to $-\infty$, then we have

$$a, b, c \simeq (-t)^{-\frac{1}{2}}$$

for t large and negative. Hence $\frac{a}{b}$ tends to 1 as t tends to $-\infty$, and we see that $a \equiv b$. A direct analysis of the equations with $a \equiv b$ shows that we do indeed have a unique trajectory with a, c positive and $a \equiv b$ converging to the origin as t tends to $-\infty$.

We have proved the following result.

Proposition 6.1

Trajectories of solutions to (21-23) (with $a, b, c \geq 0$ and $a \geq b$) defined on $(-\infty, \eta)$ are precisely the unstable curves of critical points $(q, 0, q)$ or $(q, q, 0)$. If the critical point is $(q, 0, q)$ where q is positive then we have $a > b$. Otherwise we have $a \equiv b$. □

Let us now analyse the behaviour as $t \to -\infty$ of the metrics corresponding to such trajectories.

First consider the unstable curve of $(q, q, 0)$ where $q > 0$. The metric defined by this trajectory will have $a \equiv b$ and hence will be one of the metrics of [GP],[Pd].

For t large and negative we have

$$\begin{aligned} a &\simeq q \\ b &\simeq q \\ c &\simeq ke^{(q^2+q^4)t}. \end{aligned}$$

The metric is now asymptotically

$$q^4k^2e^{2(q^2+q^4)t}dt^2 + q^2(\sigma_1^2 + \sigma_2^2) + k^2e^{2(q^2+q^4)t}\sigma_3^2.$$

Letting

$$y = \frac{k}{1+q^2}e^{(q^2+q^4)t}$$

the metric is asymptotically

$$dy^2 + q^2(\sigma_1^2 + \sigma_2^2) + (1+q^2)^2y^2\sigma_3^2$$

as y tends to zero, and this has a bolt if $1 + q^2$ is a half-integer (see [GP] for a discussion of nuts and bolts).

It is easy to check that the metric corresponding to a trajectory converging to the origin has a nut at $t = -\infty$. This is the pseudo-Fubini-Study metric on the open ball in $\mathbb{C}^4$.

It can be checked that in the above examples of complete metrics the Kähler structure extends to the manifold obtained by adding in the nut or bolt.

Moreover, using the form (20) of the metric, it follows that there are no completeness problems for any of the above metrics as t approaches η. We have established the following proposition.

Proposition 6.2

The complete metrics with $a \equiv b$ corespond precisely to the unstable curves of critical points $(q, q, 0)$ where $1 + q^2$ is a half-integer. □

Remark

This analysis agrees with the discussion of completeness in [Pd].

Now let us examine the metrics given by unstable curves of $(q, 0, q)$ where q is positive.

As t tends to $-\infty$ we have

$$\begin{aligned} a &\simeq q \\ b &\simeq ke^{q^2t} \\ c &\simeq q \end{aligned}$$

for some constant k, so the metric is asymptotically

$$q^4k^2e^{2q^2t}dt^2 + q^2(\sigma_1^2+\sigma_3^2) + k^2e^{2q^2t}\sigma_2^2.$$

Putting $v = ke^{q^2t}$ brings the metric asymptotically into the form

$$dv^2 + q^2(\sigma_1^2+\sigma_3^2) + v^2\sigma_2^2$$

as $v \to 0$, so the metric has a bolt and is nonsingular.

In fact as we approach the bolt the metric is asymptotic to the Eguchi-Hanson metric. Moreover our Kähler form Ω is asymptotic to one of the two-sphere of Kähler forms on Eguchi-Hanson; this is *not* the form corresponding to the complex structure on Eguchi-Hanson with respect to which the bolt is a complex submanifold. We deduce that our Kähler structure extends over the bolt, but that the bolt is not a complex manifold.

We have seen that as t tends to $-\infty$ the metric and Kähler structure are well-behaved. What happens as t tends to η ?

Recall from Lemma 5.2 that $(a^2-c^2)' = (a^2-c^2)(b^2-a^2-c^2) - 2(abc)^2$, so if a^2-c^2 is positive at u, then $(a^2-c^2)'$ is negative, so a^2-c^2 is bounded away from zero on $(-\infty, u)$. This however, contradicts our earlier result that a^2-c^2 tends to zero as t tends to $-\infty$. So we have shown that $a^2 \le c^2$. It is clear that we cannot have $a \equiv c$ so the metric is *triaxial*, that is, no two of a, b, c are identically equal.

Recalling that $a > b$ we see from the equations that

$$b' \ge \frac{1}{2}b^3$$

so for each $s \in (-\infty, \eta)$ there exists K_s such that

$$b \ge (K_s - t)^{-\frac{1}{2}} \text{ for } t \ge s.$$

Hence our solution blows up in finite time and the upper bound η of our interval is finite.

Next, recall that $\frac{a}{b}$ is decreasing , so tends to a finite limit $L \ge 1$ as t tends to η.

We can adapt an argument of [PP] to find the asymptotics of the metric as $t \to \eta$.

If we let $r = 2\sqrt{ab}$ the metric becomes

$$g = W^{-1}dr^2 + \frac{1}{4}r^2(V\sigma_1^2 + V^{-1}\sigma_2^2 + W\sigma_3^2)$$

where $W = \frac{c^2}{ab}$ and $V = \frac{a}{b}$. Remember that ab is strictly increasing so this is an allowable change of variables. Moreover r tends to ∞ as t tends to η.

Now

$$\begin{aligned}\frac{dW}{dr} &= \frac{dW}{dt}\frac{dt}{dr} \\ &= (\frac{c^2}{ab})'(\frac{dr}{dt})^{-1} \\ &= (\frac{c^2}{ab})'(ab)^{-\frac{1}{2}}c^{-2}\end{aligned}$$

and after simplifying we arrive at

$$\frac{dW}{dr} = \frac{2}{r}(\frac{a}{b}+\frac{b}{a}) + r - \frac{4}{r}W.$$

Now asymptotically $\frac{a}{b} = L$ and so

$$\frac{dW}{dr} = \frac{2}{r}(L+L^{-1}) + r - \frac{4}{r}W.$$

Therefore the metric is asymptotically

$$g = W^{-1}dr^2 + \frac{1}{4}r^2(V\sigma_1^2 + V^{-1}\sigma_2^2 + W\sigma_3^2)$$

where

$$W = \frac{1}{2}(L+L^{-1}) + \frac{r^2}{6} + \frac{\kappa}{r^4}$$

for some constant of integration κ, and

$$V = L.$$

Hence the metric is complete. We finally have the theorem.

Theorem 6.3

The trajectories of solutions to (21-23) which give complete metrics with $a > b$ are precisely the unstable curves of the critical points $(q, 0, q)$ where q is positive. □

We have shown that there are precisely two families of $SU(2)$-invariant complete Kähler-Einstein four-metrics with negative Einstein constant, where the generic $SU(2)$ orbit is three-dimensional. One of these families, given by Proposition 6.2, consists of the $U(2)$-invariant metrics of [GP],[Pd]. The other family, given by Theorem 6.3, consists of triaxial metrics.

Jensen [J] has classified the homogeneous four-dimensional Einstein spaces, and the only Kähler examples with negative Einstein constant on his list are the pseudo-Fubini-Study metric on the homogeneous space $SU(1,2)/U(2)$ and the product metric on $H^2 \times H^2$. Here H^2 denotes hyperbolic 2-space. In the case of pseudo-Fubini-Study the only way of embedding $SU(2)$ in the isometry group is via a $U(2)$ embedding, so this cannot lead to triaxial metrics. Moreover, the isometry group of $H^2 \times H^2$ admits no $SU(2)$ embedding. We deduce that the metrics given in Theorem 6.3 are not homogeneous.

Remark

We do not know if the equations of Theorem 4.1 describing Kähler-Einstein metrics can be solved explicitly. The fact that the metrics will not in general be self-dual or anti-self-dual suggests that this may not be possible. A further indication of this is that the equations do not satisfy the strong Painlevé property. However neither of these points is conclusive, and the solvability or otherwise of the equations remains an open question.

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