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Inaccuracy and Spontaneous Symmetry Breaking in Quantum Measurements

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Abstract

Conditions are formulated under which a representation of an intrinsic C*-algebra of (often quasilocal) observables of an infinite system is appropriate to describe measurement-type processes: such a representation should allow for the description of inaccurate experiments, it should be separable, and the pointer observable should be in its weak closure. If the pointer values are discrete the existence of such a measurement representation can be proved. If the pointer can take continuously many values, then the existence can only be proved under the additional assumptions of having an asymptotically abelian system or dealing with type I representations. In the constructed measurement representations the pointer observable turns out to be classical. The structure of the representation suggests that spontaneous symmetry breaking might be a physical explanation of the emergence of the classical pointer.

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1 Introduction

The application of the C*-algebraic formalism to the discussion of quantum mechanical measurement processes was originated by Hepp [15]. In this formalism superselection rules and classical observables can be described in a natural way. The essential feature of Hepp's approach to the measurement problem is the use of disjoint final states.

The virtues of the approach are obvious: due to Hepp's disjointness theorem a coherent superposition of different pointer states is always equivalent to an incoherent mixture of such states. Therefore, if one is satisfied with a statistical description, the projection postulate is not necessary. Furthermore, the central decomposition of this mixture into the different disjoint pointer states is unique. This justifies the application of the ignorance interpretation of the mixed statistical state in exactly the same way as in classical statistical mechanics.

The vices of the approach are obvious as well: firstly, the emergence of a classical pointer observable in a quantum system has to be explained. If this is achieved one has to recognise that secondly an automorphic time evolution can only map two states of the same folium into disjoint states in infinite time (or, using highly singular time-evolutions, instantaneously [15]). Real measurements, however, take finite time.

Leaving aside these two problems for the moment, the first part of this paper examines the relation between the factual inaccuracy of our experiments and the classical character of the pointer. In section 2 a notion of finite accuracy for experiments is introduced which will be used in the sequel. In section 3 it is shown for W*-algebras that, assuming the existence of a pointer observable, an experiment can be inaccurate if and only if there exists a classical observable which takes the same value as the pointer on all typical final states. Unfortunately this result is not satisfactory for the description of continuous pointer observables.

Section 4 addresses the first major problem, namely the origin of the classical character of the pointer. We start from a phenomenological characterisation of mea-

surement situations: one is given a set of typical final states, an interpretative rule associating to each of the typical final states a real number (the measurement result), and one knows that experiments are inaccurate. The existence of a representation which reflects these phenomenological features of measurement situations is proved. It turns out that the pointer observable is in the centre of the weak closure of this representation. Therefore even if the intrinsic C*-algebra of quasilocal observables has trivial centre, there is a classical pointer observable in appropriate reducible representations. The result of section 4 is, however, not entirely satisfactory because the measurement representation constructed there is non-separable if the pointer observable has a continuous range of possible values. But it at least implies that the typical final states are disjoint.

In section 5 we suggest a more physical explanation of the origin of the classical pointer, namely spontaneous symmetry breaking. In the case of a discrete pointer, the pointer observable can be shown to arise from the breaking of an abstract symmetry relabelling the different folia of the typical final states. The crucial point here is that the abstract symmetry of permutation of certain sets of folia is equivalent to a physical symmetry, the breaking of which gives rise to the classical observable which distinguishes the sets of folia. In the case of a continuous range of pointer values the same can be shown only under some additional assumptions, for example having an asymptotic abelian system, or dealing only with type I representations. Under either of these assumptions, the resulting representation overcomes the difficulties with the description of continuous pointers clouding the previous results. In general, without either of these assumptions, it is not clear under which circumstances the spontaneous breaking of a continuous symmetry (in a representation whose weak closure has separable predual) generates a classical observable. The question which physical symmetry corresponds to the abstract symmetry can only be answered case by case.

In the outlook attention is drawn to the fact that long-range interactions might indicate an answer to both major problems related to the use of a classical pointer. Firstly, under a non-automorphic time evolution states of the same folium may evolve

in finite time into disjoint ones. This suggestion [17] is fundamentally different from attempts to deduce the non-automorphic character of the time evolution from the influence of the environment. Secondly, long-range interactions can be a physical reason for spontaneous symmetry breaking and thus provide a mechanism for the generation of a classical pointer in an apparatus described by quantum theory.

2 Inaccurate experiments

In this section we describe two ways in which experiments can be inaccurate, and then abstract from the examples a notion of inaccuracy which we think fits most realistic experiments.

First example: consider the case where a digital pointer is used to measure a quantity A with continuous spectrum. After the experiment we register one number k out of a discrete set K as result, but say that the actual value of A might be in the interval $[k-\epsilon, k+\epsilon]$. There is a state ψ_k in which A really has the dispersionfree value k. (ψ_k is not normal if A is represented on a separable Hilbert space, because it is a pure state in which a quantity with continuous spectrum has a dispersionfree value.) But all the states in which A has a value in $[k-\epsilon, k+\epsilon]$ would as well have led us to register the result k. So all states ψ in $\mathcal{U}_{A,\epsilon}(\psi_k) := \{\psi : |\psi(A) - \psi_k(A)| < \epsilon\}$ lead to the pointer reading k. $\mathcal{U}_{A,\epsilon}(\psi_k)$ contains the set of states ψ with $\|\psi - \psi_k\| := \sup_B |\psi_k(B) - \psi(B)|/\|B\| < \epsilon$, where the supremum is over all observables B. *A fortiori* all states ψ with $\|\psi - \psi_k\| < \epsilon$ lead to the pointer reading k.

It may seem that, if we are restricting ourselves to pure final states, this kind of inaccuracy is excluded when we are measuring a discrete quantity where the value k is isolated by more than ϵ from the other possible results. This can for example be the case if we measure the spin of a particle in a given direction z. Still—and this is the second example—a Stern-Gerlach experiment can be inaccurate in the following way: The counter registers particles leaving a magnetic field whose direction might deviate by a small angle from z. Therefore these particles give rise to the same pointer reading as particles with spin in z. Primas [25] described such a kind of inaccuracy by a *finite partition of the Hilbert space* on which the measured observable operates.

(Note also that such an inaccurate Stern-Gerlach experiment can only be described by a positive-operator valued measure, but not by a projection valued measure.) This example displays an aspect of a more general phenomenon: ignorance about the measured observable leads to ignorance about the final state of the observed system (the former ignorance may originate in the ambiguities inherent in quantization theory: which operator represents the quantization of a given classical observable?). It should be emphasised that this kind of inaccuracy can occur even if the pointer has a continuous range of possible values.

These examples lead to the following notion of inaccuracy. There are some "typical" final states ψ_k, in which the quantity we want to measure really has the value k. After an inaccurate experiment, however, all states in a sufficiently small ϵ-neighbourhood of one ψ_k give rise to the same pointer reading as ψ_k.

Let us try to formulate this notion of inaccuracy in a more general framework. In a statistical description one assumes that the observables of the measured system and of the apparatus generate W^*-algebras $\mathcal{A}_S$, $\mathcal{A}_M$ respectively. Let ρ_k be the state of the joint system after an experiment which leaves the system in the state $\rho_k|_s$. (The states $\rho_k|_s$ are what was called ψ_k above.) K is an index set for the typical final states; in the example of a Stern-Gerlach experiment on a spin 1/2 particle it would be a set with two elements. The typical final states ρ_k are assumed to be normal and factorial. Furthermore, assume that there is one observable $P \in \mathcal{A}_S\overline{\otimes}\mathcal{A}_M$ specified which associates the measurement results $\rho_k(P)$ to the typical final states ρ_k; this is the pointer observable.

Finite measurement accuracy (FMA): *There exists a $\delta > 0$ and a non-empty family $\{\rho_k\}_{k\in K}$ of normal factor states of the joint system, such that the pointer observable P has the same expectation value in all factor states ρ of the joint system satisfying $\|\rho - \rho_k\| < \delta$ for some k.*

Note that this notion of inaccurcy is a modification of the one used in [9]. The above formulation of (FMA) is appropriate for a statistical descrription. It is also possible to give a formulation suitable for an individual description. Then $\mathcal{A}_S$, $\mathcal{A}_M$ are taken to be C^*-algebras and the states ρ_k should be pure. Such a formulation

of (FMA) will be used in section 4.

3 The main result for W*-algebras

The following proposition starts from a given observable $P \in \mathcal{A}_S\overline{\otimes}\mathcal{A}_M$, which is interpreted as the pointer observable. It shows that (the W^*-algebraic version of) FMA can be fulfilled if and only if $\mathcal{A}_S\overline{\otimes}\mathcal{A}_M$ has non-trivial centre and a classical observable $\overline{P} \in \mathcal{Z}(\mathcal{A}_S\overline{\otimes}\mathcal{A}_M)$ yields the same pointer reading as P for the typical final states. The typical final states cannot distinguish between P and $\overline{P}$. Furthermore it shows that then the typical final states are disjoint.

The drawback of the proposition is that as long as the typical final states ρ_k are assumed to be normal there cannot be a continuous observable on which they take different dispersionfree values. Therefore, this pure W*-framework is not suitable for the description of a continuous pointer observable, except perhaps in Piron's approach [24] which always uses a discrete topology but can represent "continuous"observables only on non-separable Hilbert spaces.

Proposition 1 *Let $\mathcal{A}_S, \mathcal{A}_M$ be W^*-algebras and $\{\rho_k\}_{k\in K}$ be a set of normal factor states on $\mathcal{A}_S\overline{\otimes}\mathcal{A}_M$. Let $P \in \mathcal{A}_S\overline{\otimes}\mathcal{A}_M$.*

(A) *Assuming that for all $Z \in \mathcal{Z}(\mathcal{A}_S\overline{\otimes}\mathcal{A}_M)$ there is a ρ_k with $\rho_k(Z) \neq 0$, (FMA) can be fulfilled if and only if the pointer observable $P \in \mathcal{Z}(\mathcal{A}_S\overline{\otimes}\mathcal{A}_M)$.*

(B) *(FMA) can be fulfilled if and only if there is an observable $\overline{P}$ lies in $\mathcal{Z}(\mathcal{A}_S\overline{\otimes}\mathcal{A}_M)$ with $\rho_k(P) = \rho_k(\overline{P})$, $\forall k \in K$. In this case the states $\{\rho_k\}_{k\in K}$ are mutually disjoint because for $i \neq k$ and a $\overline{P} \in \mathcal{Z}(\mathcal{A}_S\overline{\otimes}\mathcal{A}_M)$, $\rho_k(\overline{P}) \neq \rho_i(\overline{P})$.*

Proof of (A): Denote by $\pi_{\rho_k}(\mathcal{A}_S\overline{\otimes}\mathcal{A}_M)$ the GNS-representation with respect to the state ρ_k of $\mathcal{A}_S\overline{\otimes}\mathcal{A}_M$ on the Hilbert space $\mathcal{H}_{\rho_k}$, where there is a cyclic vector Ω_{ρ_k} such that $\rho_k(A) =< \Omega_{\rho_k}|\pi_{\rho_k}(A)|\Omega_{\rho_k} >$ for all $A \in \mathcal{A}_S\overline{\otimes}\mathcal{A}_M$.

Step 1: We show that $\bigcap_{k\in K} \ker \pi_{\rho_k} = \{0\}$.

Take $A \in \bigcap_{k\in K} \ker \pi_{\rho_k}$. First of all we observe that A cannot be in $\mathcal{Z}(\mathcal{A}_S\overline{\otimes}\mathcal{A}_M)$: If A were in $\mathcal{Z}(\mathcal{A}_S\overline{\otimes}\mathcal{A}_M)$, then there would be a ρ_k with $\rho_k(A) \neq 0$ and therefore $A \notin \ker \pi_{\rho_k}$.

Since the states ρ_k are normal, $\bigcap_{k\in K} \ker \pi_{\rho_k}$ is a weakly closed two-sided ideal of $\mathcal{A}_S\overline{\otimes}\mathcal{A}_M$. It follows that there is a projection $Q \in \mathcal{Z}(\mathcal{A}_S\overline{\otimes}\mathcal{A}_M)$ such that $\bigcap_{k\in K} \ker \pi_{\rho_k} = Q(\mathcal{A}_S\overline{\otimes}\mathcal{A}_M)Q$ (see [8], proposition 2.4.22). So $\bigcap_{k\in K} \ker \pi_{\rho_k} \neq \{0\}$ would imply that $\bigcap_{k\in K} \ker \pi_{\rho_k} \cap \mathcal{Z}(\mathcal{A}_S\overline{\otimes}\mathcal{A}_M) \neq \{0\}$. But since there is no central element in $\bigcap_{k\in K} \ker \pi_{\rho_k}$, $\bigcap_{k\in K} \ker \pi_{\rho_k} = \{0\}$.

Step 2: $P \notin \mathcal{Z}(\mathcal{A}_S\overline{\otimes}\mathcal{A}_M)$ implies $(\forall\delta)(\exists k \in K)(\exists\rho$ factorial$) : \|\rho - \rho_k\| < \delta$, $\rho(P) \neq \rho_k(P)$.

$P \notin \mathcal{Z}(\mathcal{A}_S\overline{\otimes}\mathcal{A}_M)$ implies that there exists a $B \in \mathcal{A}_S\overline{\otimes}\mathcal{A}_M$ such that $[P,B] \neq 0$. From the first step it follows that there is a k with $\pi_{\rho_k}([P,B]) \neq 0$. So $\pi_{\rho_k}(P) \notin \mathcal{Z}(\pi_{\rho_k}(\mathcal{A}_S\overline{\otimes}\mathcal{A}_M))$. Since $\pi_{\rho_k}(\mathcal{A}_S\overline{\otimes}\mathcal{A}_M)$ is a factor, $\pi_{\rho_k}(P)$ cannot be constant. Take any vector state $\psi \in \mathcal{H}_{\rho_k}$ with $< \psi|\pi_{\rho_k}(P)|\psi > \neq < \Omega_{\rho_k}|\pi_{\rho_k}(P)|\Omega_{\rho_k} > = \rho_k(P)$. Define $\psi_\alpha := \alpha\Omega_{\rho_k} + (1-\alpha)\psi$ for $0 < \alpha < 1$. Take $\rho_\alpha(A) := < \psi_\alpha|\pi_{\rho_k}(A)|\psi_\alpha >$ for all $A \in \mathcal{A}_S\overline{\otimes}\mathcal{A}_M$. Now it can be checked that all ρ_α are pure (and therefore factorial) states satisfying $\rho_\alpha(P) \neq \rho_k(P)$. If we choose α close enough to one, then $\|\rho_\alpha - \rho_k\| < \delta$.

Step 3: $P \in \mathcal{Z}(\mathcal{A}_S\overline{\otimes}\mathcal{A}_M)$ implies $(\exists\delta)(\forall k \in K)(\forall\rho$ factorial$) : \|\rho - \rho_k\| < \delta \Rightarrow \rho(P) = \rho_k(P)$.

Take $\delta < \inf_{i\neq k}\|\rho_i - \rho_k\|/2 < 2$ and let ρ be a pure state with $\|\rho - \rho_k\| < \delta < 2$ for some k. From [13] it follows that the representations π_ρ and π_{ρ_k} are quasi-equivalent. All classical observables have the same value in the states ρ and ρ_k, so $\rho(P) = \rho_k(P)$.

Proof of (B):

The family $\{\rho_k\}_{k\in K}$ defines a two-sided ideal $\mathcal{I} \subset \mathcal{A}_S\overline{\otimes}\mathcal{A}_M$ by $\mathcal{I} := \{A|\rho_k(B^*AC) = 0, \forall B,C \in \mathcal{A}_S\overline{\otimes}\mathcal{A}_M, \forall k \in K\} = \bigcap_{k\in K} \ker \pi_{\rho_k}$. There exists *central* projection $Q \in \mathcal{A}_S\overline{\otimes}\mathcal{A}_M$ such that $\mathcal{I} = (\mathcal{A}_S\overline{\otimes}\mathcal{A}_M)Q = Q(\mathcal{A}_S\overline{\otimes}\mathcal{A}_M)Q$. $\mathcal{A}_S\overline{\otimes}\mathcal{A}_M$ can be written as $\mathcal{A}_S\overline{\otimes}\mathcal{A}_M = (\mathcal{A}_S\overline{\otimes}\mathcal{A}_M)(1-Q)\oplus(\mathcal{A}_S\overline{\otimes}\mathcal{A}_M)Q =: \mathcal{A}_1\oplus\mathcal{A}_2$. Since for every $Z' \in \mathcal{Z}(\mathcal{A}_1)$ there is a ρ_k with $\rho_k(Z') \neq 0$, we have $\bigcap_{k\in K} \ker \pi_{\rho_k}(\mathcal{A}_1 \oplus 0) = \{0\}$. Every observable $A \in \mathcal{A}_S\overline{\otimes}\mathcal{A}_M$ can be written as $A = A' \oplus A''$ with $A' \in \mathcal{A}_1, A'' \in \mathcal{A}_2$; every state ρ on $\mathcal{A}_S\overline{\otimes}\mathcal{A}_M$ as $\rho = \rho' + \rho''$.

As $\bigcap_{k\in K} \ker \pi_{\rho_k}(\mathcal{A}_1 \oplus 0) = \{0\}$, everything now follows from the proof of (A): If

there is a $\overline{P} \in \mathcal{Z}(\mathcal{A}_S\overline{\otimes}\mathcal{A}_M)$ with $(\forall k \in K) : \rho_k(P) = \rho_k(\overline{P})$, then all factor states ρ with $\|\rho - \rho_k\| < \delta$ satisfy $\rho(\overline{P}' \oplus 0) = \rho'(\overline{P}') = \rho'_k(\overline{P}') = \rho_k(\overline{P}' \oplus 0)$. So FMA is satisfied for the pointer observable $\overline{P}' \oplus 0 =: \overline{P}$. If there is no $\overline{P} \in \mathcal{Z}(\mathcal{A}_S\overline{\otimes}\mathcal{A}_M)$ with $(\forall k \in K) : \rho_k(P) = \rho_k(\overline{P})$, then $P' \notin \mathcal{Z}(\mathcal{A}_1)$. From the second step of the proof of (A) it follows that $(\forall\delta)(\exists k \in K)(\exists\rho'$ factor state on $\mathcal{A}_1) : \|\rho' - \rho'_k\| < \delta$ but $\rho'_k(P') \neq \rho'(P')$. Taking $\rho := \rho' + \rho''_k$ it follows that $\|\rho - \rho_k\| < \delta$ but $\rho(P) \neq \rho_k(P)$.

4 The measurement representation

Proposition 1 uses a pure W*-framework and is tailored for a statistical description. The assumption that the typical final states are normal is sensible in a statistical description because otherwise these states would not be σ-additive and therefore would not induce probabiblity measures.

A C*-algebraic version of Propostition 1 meets two difficulties: Firstly, the assumption that the typical final states ρ_k are normal cannot be justified. In fact normality is not a very natural concept for states on general C*-algebras. Therefore, the C*-framework is not suitable for a statistical description but rather for an individual one [25]. In an individual description it is justified to assume the typical final states to be pure. Secondly, the basic C*-algebra of quasilocal observables usually has trivial centre so that we do not have any classical observables. In this section we try to weld together these two problems and give a C^*-algebraic version of Proposition 1.

As a C*-analogue of Proposition 1 one could show the following.

Let $\{\rho_k\}_{k\in K}$ be a family of pure states on a C^*-algebra $\mathcal{A}_S \otimes \mathcal{A}_M$. This family defines a two-sided ideal $\mathcal{I}$ by

$$\mathcal{I} := \{A \in \mathcal{A}_S \otimes \mathcal{A}_M : \rho_k(B^*AC) = 0 \text{ for all } k \in K, \text{ for all } B,C \in \mathcal{A}_S \otimes \mathcal{A}_M\}.$$

Then (FMA) can be satisfied if and only if there exists an observable $\overline{P} \in \mathcal{Z}(\mathcal{A}_S \otimes \mathcal{A}_M/\mathcal{I})$ such that $\rho_k(P) = \rho_k(\overline{P})$ for all $k \in K$.

We will not prove this proposition because it is not very useful. The algebra of

quasilocal observables usually has trivial centre and is simple. Since it has trivial centre we cannot find a classical pointer observable in it. Since it is simple the ideal $\mathcal{I}$ defined by the $\{\rho_k\}_{k\in K}$ is the null-ideal, so $\mathcal{Z}(\mathcal{A}_S \otimes \mathcal{A}_M/\mathcal{I})$ is again trivial. Therefore in most quantum theories the above proposition is empty.

This brings our attention to a fundamental problem rooted in the use of a classical pointer observable: how does a classical observable arise in a quantum system? Usually the algebra of quasilocal observables is simple. Classical observables are often only in the weak closure of a particular representation appropriate to describe the particular situation under consideration. In the same way, the pointer observable will usually not be an element of the basic C^*-algebra $\mathcal{A}_S \otimes \mathcal{A}_M$. Rather it will occur in the weak closure of a representation appropriate to describe measurement situations. The main goal is to find such a representation.

Phenomenology of measurement situations. First of all let us describe the typical features of realistic measurement situations. The basic C^*-algebra $\mathcal{A}_S \otimes \mathcal{A}_M$ generated by the local observables of the joint system is given. Furthermore, we have a family $\{\rho_k\}_{k\in K}$ of typical final states. In an individual description we will require them to be pure. There is a pointer value functional p on the states (so p is an element of the bidual $(\mathcal{A}_S \otimes \mathcal{A}_M)^{**}$) which has different values on the typical final states. Furthermore, we know that realistic measurements are inaccurate in the way described in section 2: there is a $\delta > 0$ such that for all pure states ρ on $\mathcal{A}_S \otimes \mathcal{A}_M$ we have $\|\rho - \rho_k\| < \delta$ for some $k \Rightarrow < p, \rho_k > = < p, \rho >$.

The existence of a representation describing this situation is guaranteed by the following

Proposition 2 *Let $\mathcal{A}_S \otimes \mathcal{A}_M$ be the simple C^*-algebra of observables of the joint system, $\otimes$ denoting the minimal *-tensor product. Assume that we have a set $\{\rho_k\}_{k\in K}$ of pure typical final states on $\mathcal{A}_S \otimes \mathcal{A}_M$ and a pointer functional $p \in (\mathcal{A}_S \otimes \mathcal{A}_M)^{**}$ on the states such that FMA is fulfilled:*

$$(\exists \delta > 0)(\forall \rho \text{ pure states}) : \|\rho - \rho_k\| < \delta \text{ for some } k \Rightarrow < p, \rho_k > = < p, \rho > .$$

Then there is a representation π_0 of $\mathcal{A}_S \otimes \mathcal{A}_M$ such that the states ρ_k can be extended to normal states $\overline{\rho_k}$ on $\pi_0(\mathcal{A}_S \otimes \mathcal{A}_M)''$, and that there is an observable $P \in \pi(\mathcal{A}_S \otimes \mathcal{A}_M)''$ with $\overline{\rho_k}(P) = < p, \rho_k >$ for all k. Furthermore, P is in the centre of $\pi_0(\mathcal{A}_S \otimes \mathcal{A}_M)''$ and therefore classical. The states $\overline{\rho_k}$ (as well as the ρ_k) are mutually disjoint, because $\overline{\rho_k}(P) = < \rho_k, p > \neq \overline{\rho_i}(P) = < \rho_i, p >$ for $i \neq k$.

Proof: Embed $\mathcal{A}_S \otimes \mathcal{A}_M$ canonically into its enveloping von Neumann algebra $\mathcal{B}$, which is the weak closure of its universal representation. As a Banach space $\mathcal{B}$ is isomorphic to the bidual $(\mathcal{A}_S \otimes \mathcal{A}_M)^{**}$. To this imbedding there corresponds an isometric isomorphism of $(\mathcal{A}_S \otimes \mathcal{A}_M)^*$ onto the predual of $\mathcal{B}$, associating to each state ρ on $\mathcal{A}_S \otimes \mathcal{A}_M$ a normal state $\tilde{\rho}$ on $\mathcal{B}$. Since the states ρ_k on $\mathcal{A}_S \otimes \mathcal{A}_M$ are pure, the W*-algebraic version of FMA is satisfied for the pointer functional p and the typical final states $\widetilde{\rho_k}$. From Proposition 1, (B), it follows that there is a central projection $Q \in \mathcal{Z}(\mathcal{B})$ such that $\mathcal{Z}(Q\mathcal{B})$ contains an observable P on which the restrictions of the $\widetilde{\rho_k}$ take the same values as the pointer functional P does on the ρ_k. π_0 now is the subrepresentation of the universal representation specified by $\pi_0(\mathcal{A}_S \otimes \mathcal{A}_M)'' = Q\mathcal{B}$. Since $\mathcal{A}_S \otimes \mathcal{A}_M$ is simple it is isomorphic to $\pi_0(\mathcal{A}_S \otimes \mathcal{A}_M)$. π_0 satisfies the conditions we mentioned in the proposition: the states ρ_k extend to states $\overline{\rho_k}$ which are defined by $\overline{\rho_k}(QA) = \widetilde{\rho_k}(A)$, $A \in \mathcal{B}$, the pointer observable P is in $\pi_0(\mathcal{A}_S \otimes \mathcal{A}_M)''$ and fulfils $\widetilde{\rho_k}(P) = < p, \rho_k >$, and FMA is satisfied.

Remarks. (1) Even if $\mathcal{A}_S \otimes \mathcal{A}_M$ has a trivial centre, $(\mathcal{A}_S \otimes \mathcal{A}_M)^{**}$ does not if $\mathcal{A}_S \otimes \mathcal{A}_M$ has disjoint representations. This is for example the case if $\mathcal{A}_S \otimes \mathcal{A}_M$ is the C*-algebra of quasilocal observables in a field theory.

(2) Knowing from experimental experience that measurements are inaccurate, the proposition guarantees the existence of a representation which can describe this situation. It turns out that in such a representation the pointer observable is classical.

Proposition 2 is not satisfactory because from a physical point of view it would have been justified to require the representation π to be separable. Proposition 2 does not prove the existence of a separable representation fulfilling all the other requirements. If the pointer functional has a continuous range of possible values,

then the representation π_0 is surely not separable. Another drawback of Proposition 2 is that the states $\overline{\rho_k}$ are normal. This is not what is needed in the case of a continuous pointer observable, because normal states cannot take a dispersionfree value on an observable with continuous spectrum. The consequence of Proposition 2 primarily used in the sequel is that the typical final states ρ_k are disjoint.

So neither Proposition 1 nor Proposition 2 can offer a satisfactory description of a continuous pointer. Proposition 1 had to assume the normality of the typical final states, which is a condition that cannot be satisfied for a continuous pointer observable. The C*-algebraic version of FMA cannot be satisfied if the C*-algebra of quasilocal observables has trivial centre. Trying to satisfy FMA in a particular representation (in which the typical final states should be normal) led to non-separable representations. In the following section we will blend the purely W*-algebraic and the purely C*-algebraic formulation of FMA and will arrive—under some additional assumptions—at a measurement representation satisfying the following requirements: (1) FMA is satisfied, (2) the representation is separable, (3) there exists a pointer observable in the weak closure of the representation which takes the same values as the pointer functional on the typical final states.

Proposition 2 will imply that the pointer observable is classical and that the typical final states are disjoint. The typical final states will not be normal on the weak closure of the representation, but their singular character will become apparent only on the centre of the representation.

5 Symmetry Breaking

In many physical examples spontaneous symmetry breaking is linked to the emergence of a classical observable. For example, the two-dimensional infinitely extended Ising model below the critical temperature has degenerate and disjoint ground states, which break the original rotation symmetry of the Hamiltonian. This leads to the emergence of magnetisation as a classical observable of a ferromagnet. Chirality as a classical observable of molecules in optically active substances amounts to a breakdown of the space inversion symmetry possessed by the hamiltonian. (A rigorous discussion of chirality is still missing, but some features have been sketched, see [23, 2, 26].)

A group $\mathcal{G}$ is said to be a symmetry group of a C*-algebra (or a W*-algebra) $\mathcal{A}$ if there is a representation of $\mathcal{G}$ as a pointwise norm-continuous[3] (or a σ-weakly continuous) automorphism group of the algebra. A symmetry $\beta : \mathcal{G} \to \mathrm{Aut}(\mathcal{A})$ is said to be broken in a representation π if the representation $\pi \circ \beta$ is not equivalent to π. Equivalently, a symmetry β is broken in a representation π if there is no automorphism $\overline{\beta}$ of the generated von Neumann algebra $\pi(\mathcal{A})''$ such that $\pi \circ \beta = \overline{\beta} \circ \pi$.[4] Accordingly, if π_ρ is a GNS-representation with respect to a factor state ρ, then the symmetry β is spontaneously broken iff ρ is not invariant under β. (Note that there is an alternative definition of symmetry breaking which requires in addition to the above also that a classical observable is generated. But since the question when the above implies the generation of a classical observable is very interesting in the case of a continuous symmetry group, we will stick to our definition.)

In this section we want to suggest a way in which the pointer observable can be considered to be arising from spontaneous symmetry breaking. The starting point is the abstract C*-algebra $\mathcal{A}_S \otimes \mathcal{A}_M$ and the set $\{\rho_k\}_{k\in K}$ of pure typical final states. From Proposition 2 we know that the ρ_k are mutually disjoint if the described experiment is of finite accuracy. Assume also that the Hilbert spaces $\mathcal{H}_{\rho_k}$ of the GNS-representations π_{ρ_k} are all isomorphic to a Hilbert space $\mathcal{H}_\rho$.

Let us first consider the case in which the set $\{\rho_k\}_{k\in K}$ of typical final states is discrete. Consider the direct sum $\pi_d := \sum_{k\in K}^{\oplus} \pi_{\rho_k}$ of disjoint factor representations (d stands for discrete) which operates on the Hilbert space $\mathcal{H}_d = \sum_{k\in K}^{\oplus} \mathcal{H}_{\rho_k}$. The

[3]Even though the requirement of pointwise norm-continuity is weaker than norm-continuity, it is not fulfilled for many bosonic systems, see [3].

[4]In the traditional (non-algebraic) formulation of quantum field theories, spontaneous symmetry breaking is said to occur if the "formal" charge $Q(x_0) = \int_{\mathbf{R}^3} dx j_0(x)$ associated with a conserved current j_μ does not exist as an operator and therefore cannot not be the generator of a symmetry, i.e of a unitary group of transformation generated by a self-adjoint operator commuting with the momentum operator and the S-matrix. The algebraic and the traditional definition are not in a straightforward relation, but they are connected by the idea that in the representation specified by a vacuum of the traditional theory there is no automorphism group although the existence of a conserved current suggests a certain symmetry of the hamiltonian. For a review of spontaneous and intrinsic symmetry breaking in the traditional formalism see [22], and for a discussion in the algebraic framework cf. [29, 10].

centre $\mathcal{Z}(\pi_\omega(\mathcal{A})'')$ contains exactly the diagonal operators of the form $\sum_{k\in K}\lambda_k \mathbf{1}_k$.

This centre can be considered to be arising from the breaking of the following "abstract" symmetry. If K is infinite (but still discrete), assume K to be the set of integers and take the symmetry group $\mathcal{G} = \{G_k : k \in K\}$ to be the one generated by the shifts $k' : k \mapsto k - k'$. If K is finite, a similar constrcuction works by imposing a periodicity condition on K. The direct sum representation π_d is stable under the permutation or shifts of indices, but each representation π_{ρ_k} is not stable. The symmetry is broken in each of the states ρ_k. The centre $\mathcal{Z}(\pi_\omega(\mathcal{A})'')$ can be considered arising from the breaking of a symmetry associated to a relabelling of the folia. Note that π_d is a representation satisfying all the conditions for a measurement representation mentioned at the end of section 4. In particular π_d is separable as long as the factor representations π_{ρ_k} are separable. Therefore, in the case of a discrete pointer, π_d is really a satisfactory measurement representation. This is not remarkable since Propositions 1 and 2 already give satisfactory results for the case of a discrete pointer.

Let us hasten to emphasise that these considerations are empty as long as it is not clear which physical symmetry corresponds to the relabelling of the folia. The crucial point here is that the abstract symmetry of permutation of certain sets of folia is equivalent to a physical symmetry the breaking of which gives rise to the classical observable which distinguishes the sets of folia. In some special cases it is easy to find the corresponding physical symmetry: the two-valued classical observable chirality arises from a breaking of space inversion symmetry. This corresponds to a permutation of the two sectors containing left- and right-handed states respectively. In general, the symmetry group $\mathcal{G}$ acts transitively on the set of folia defined by the disjoint final states of a measurement; this set may differ from $\mathcal{G}$ if part of the symmetry is unbroken.

It is perhaps worth relating this situation to spontaneous symmetry breaking in quantum field theories. The essential point there is whether the vacuum (or ground) state breaks the symmetry. The typical final states correspond not to the different ground states, but rather to *scattering states*. Scattering states in Hilbert spaces

built from different ground states are disjoint. The similarity of scattering states and final states is one of the basic ideas in Hepp's paper and from his point of view makes the infinite time limit seem natural.

Note that in quantum field theories the analogue of the physical symmetry broken in a measurement-type situation is a spatial symmetry, not an internal one. The disjoint vacua occurring in the spontaneous breaking of an internal symmetry would not have a counterpart in measurement-type situations: because of the occurrence of different vacua in a system with broken internal symmetry we have a broken symmetry even if we remove all particles from the system. But if we were to remove all particles from the apparatus the symmetry would not be broken any more. Similarily, if in a system with a broken spatial symmetry, for example in an infinite spin chain, one removes all the particles, then in the ground state at which one arrives the symmetry is not broken any more.

Before proceeding to the case of a continuous pointer observable, let us come back to one of the drawbacks of Proposition 2. The extended states $\overline{\rho_k}$ constructed there were normal. This is fine in the case of a discrete pointer, but not for a continuous pointer: the points in the continuous phase space M of a classical system do not correspond to normal states on $L^\infty(M)$. So a representation can only be appropriate for the description of an experiment with continuous pointer if the states ρ_k can be extended to states which are not normal on the centre of the representation.

In the discrete case the direct sum representation has the functions on the discrete set K as the centre. The representation is isomorphic to the tensor-product of this centre with a factor π_{ρ_k}. In the case of a continuous pointer we will expect something similar, except that the commutative part of the representation will correspond to functions on a continuous configuration space. In this case the states ρ_k will not be extendable to normal states on the representation. The singular character of the extensions is, however, of a harmless kind. Only their restrictions to the the commutative part are singular, on the factor they are normal. (It would be much worse to have a singular state on a type II or III factor, since these states do not

fulfil the Jauch-Piron condition [4].)

We now try to construct a representation appropriate for a continuous pointer observable. As in the case of the discrete pointer, the configuration space of the pointer observable should determine the choice of the representation: in the end the centre of the representation should be isomorphic to a space of functions on the configuration space of the pointer.

For a continuous pointer the index set K of typical final states contains continuously many elements, so we may assume it to be the real line $\mathbf{R}$. Taking μ to be a probability measure on the real line, for example $d\mu(k) = \pi^{-1/2}\exp(-k^2)dk$, define $\omega := \int_{\mathbf{R}} d\mu(k)\rho_k$. Then

$$\pi_\omega = \int_{\mathbf{R}}^{\oplus} d\mu(k)\pi_{\rho_k}$$

is the direct integral of the mutually disjoint factor representation π_{ρ_k}.

Define U_t on the space $C_0(\mathbf{R}, \mathcal{H}_\rho)$ of continuous functions from $\mathbf{R}$ to $\mathcal{H}_\rho$ vanishing at infinity by $(U_t\Psi)(k) := \exp(kt - t^2/2)\Psi(k - t)$. Since $C_0(\mathbf{R}, \mathcal{H}_\rho)$ is a dense subset of $\mathcal{H}_\omega = \int_{\mathbf{R}}^{\oplus} d\mu(k)\mathcal{H}_\rho$, U_t can be extended continuously to all of $\mathcal{H}_\omega$. The automorphism group τ_t on the von Neumann algebra $\pi_\omega(\mathcal{A}_S \otimes \mathcal{A}_M)''$ is defined by $(\tau_t A)\Psi := U_t A U_t^* \Psi$. As in the discrete case, the states ρ_k are not invariant unter U_t and so the symmetry cannot be implemented unitarily in $\pi_{\rho_k}(\mathcal{A}_S \otimes \mathcal{A}_M)''$. The symmetry τ_t is broken in each of the factor states ρ_k.

Under which conditions does this symmetry breaking generate a continuous classical observable? Certainly not in general, because in contradistinction to the discrete case the diagonal operators (that is, those of the form $\int_{\mathbf{R}}^{\oplus} d\mu(k) f(k)\mathbf{1}_k$) are not necessarily in the weak closure of the direct integral representation π_ω (although in the example above they are, for reasons to become clear shortly). There are cases known in which the direct integral of disjoint factor representations has trivial centre (see [7]). This phenomenon is also exhibited in the relatively transparent context of group representation theory [18]. Dixmier [11] even showed that for every non-type I C*-algebra there is a factor representation which can be decomposed into non-factorial representations. So the direct integral of disjoint representations

with non-trivial centre may have trivial centre. In such cases the breaking of the symmetry does not give rise to a classical observable. But under some additional assumptions the centre of the direct integral representation π_ω can shown to be non-trivial.

Firstly, this is the case if $\pi_\omega(\mathcal{A}_S \otimes \mathcal{A}_M)''$ is of type I or II_1 [14]: *If the direct integral over irreducible (or type II_1) and mutually disjoint representations of a C*-algebra is again of type I (resp. II_1), then the centre of the direct integral representation is non-trivial.* (Conversely, the components in a direct integral decomposition of a von Neumann algebra of type A which contains the diagonal operators, is again of type A ($A = I$, II_1; in fact this converse holds for $A = II_\infty$, III) as well.) In our special case one can give a simple argument to show that $\mathcal{Z}(\pi_\omega(\mathcal{A}_S \otimes \mathcal{A}_M)'') \cong L^\infty(\mathbf{R})$ if $\pi_\omega(\mathcal{A}_S \otimes \mathcal{A}_M)''$ is of type I. If $\pi_\omega(\mathcal{A}_S \otimes \mathcal{A}_M)''$ is of type I then it is quasi-equivalent to a multiplicity-free representation ([12], 5.4.6). As the π_{ρ_k} in the integral π_ω are mutually disjoint and irreducible representations, π_ω is itself multipicity-free. Therefore ([12], 5.4.4) the commutant $\pi_\omega(\mathcal{A}_S \otimes \mathcal{A}_M)'$ is abelian and the composition of π_ω into irreducible representations is unique (see [8], Theorem 4.2.3). Since $\pi_\omega(\mathcal{A}_S \otimes \mathcal{A}_M)'$ is abelian every primary representation is irreducible, so we have a unique decomposition into primary representations as well. Making this decomposition in our special case of an integral representation over the real line, one sees that $\pi_\omega(\mathcal{A}_S \otimes \mathcal{A}_M)' \cong L^\infty(\mathbf{R})$. Since the commutant is abelian it equals the centre. Thus $\mathcal{Z}(\pi_\omega(\mathcal{A}_S \otimes \mathcal{A}_M)'') \cong L^\infty(\mathbf{R})$.

If the direct integral representation $\pi_\omega(\mathcal{A}_S \otimes \mathcal{A}_M)''$ is not of type I, then it may be misbehaved. In case $\pi_\omega(\mathcal{A}_S \otimes \mathcal{A}_M)''$ is of types II or III the above argument that the centre is not trivial breaks down. The commutant of such representations cannot be abelian since the commutant is always of the same type as the representation and abelian von Neumann algebras are of type I.

Under the assumption that $\pi_\omega(\mathcal{A}_S \otimes \mathcal{A}_M)''$ is of type I, this representation is indeed appropriate for the descriptions of measurements with continuous pointer. Besides fulfilling the conditions on a measurement representation in section 4, the

extensions of the states ρ_k are singular on the centre $L^\infty(\mathbf{R})$. Since $L^2(\mathbf{R},\mu)$ is separable, $\mathcal{H}_\omega = \int_{\mathbf{R}}^{\oplus} d\mu(k)\mathcal{H}_{\rho_k}$ is separable if the $\mathcal{H}_{\rho_k}$ are. Furthermore, if the representations π_{ρ_k} are separable, then the direct integral representation π_ω is separable. But the assumption that the direct integral representation is of type I is very strong. In many applications to thermodynamics or field theories one deals with factors of type III.

Secondly, one can obtain by integration over disjoint factors a non-trivial centre if the system is asymptotically abelian. More precisely, it may be shown [1] that if the system is *asymptotically abelian* in the norm topology ($\lim_{|q|\to\infty}\|[\alpha_{|q|}(A), B]\| = 0$, where $\alpha : \mathbf{R}^3 \to \mathrm{Aut}(\mathcal{A}_S \otimes \mathcal{A}_M)$ is the representation of the group $\mathbf{R}^3$ of space translations), and if there is a space-translation invariant state ϕ, and if the action of the symmetry group commutes with the space translations, then the centre of the direct integral representation obtained from averaging the factor π_ϕ over the symmetry group $\mathcal{G}/\mathcal{H}$ is isomorphic to the abelian von Neumann algebra $L^\infty(\mathcal{G}/\mathcal{H})$. (Here $\mathcal{H}$ is the subgroup of $\mathcal{G}$ leaving ϕ invariant.) We have $\pi_\omega(\mathcal{A}_S \otimes \mathcal{A}_M)'' \cong L^\infty(\mathcal{G}/\mathcal{H}) \otimes \pi_\phi(\mathcal{A}_S \otimes \mathcal{A}_M)''$.

The assumption of asymptotic abelianess is essential in the proof of this result. This assumption is widely used in quantum statistical mechanics but still lacks a good motivation. It is fulfilled for systems of free fermions, but for more realistic systems with interactions even weaker forms of asymptotic abelianess have not been proved.

To summarise, the main problem in generating continuous classical observables by spontaneous symmetry breaking is the question under which conditions the direct integral of disjoint factor representations is not a factor. Two cases in which this is guaranteed rely on the assumptions of having a type I representation or an asymptotic abelian system. Under either of these assumptions, and if the factor representations π_{ρ_k} are separable, the direct integral representation π_ω is indeed a representation satisfying all requirements for measurement representation with a continuous pointer. A complete answer to the above question would also suggest

conditions under which the breaking of a continuous symmetry fails to produce a superselection rule.

6 Conclusions and Outlook

There are two major problems concerning the use of a classical pointer observable in quantum measurements. The emergence of a classical pointer observable in a quantum system has to be explained, and a non-automorphic time evolution of the joint system has to be derived if the final pointer states should be reached in finite time. It seems possible that long-range interactions are one possible key to the solution of *both* problems: they could explain the emergence of classical observables and lead to the right time evolution.

Under a non-automorphic time evolution states of the same folium may evolve in finite time into disjoint ones [15, 17]. In the presence of long-range interactions observables at infinity are not constant in the course of time. As Morchio and Strocchi [19] show, this prevents the definition of an automorphic time evolution on the intrinsic C*-algebra of quasilocal observables. An algebraic, automorphic time evolution can only be defined in the weak closure of some representation with non-trivial centre. Since the algebra of quasilocal observables is not stable under this time evolution, states can evolve from one folium into another in finite time. This motivation of the non-automorphic character of the time evolution is fundamentally different from attempts to deduce it from the influence of the environment, because it even applies to isolated, closed system which include the "environment". It follows from a theorem of Shultz [28] that if the time evolution of the algebra is not automorphic, then either (i) the corresponding Schrödinger evolution is not a group of bijections of the set of pure states, or (ii) the corresponding Schrödinger-picture evolution is a group of bijections, but the bijections are not uniformly continuous. In an individual description it would be desirable to have the second alternative.

Long-range interactions as for example the Coulomb force are omnipresent in nature. Not only do they give rise to non-automorphic time evolutions, they also are one possible physical reasons for spontaneous symmetry breaking [27, 19]. (Note

that this situation is different from what is usually described by the Goldstone theorem, where spontaneous symmetry breaking is a reason for long-range correlations: symmetry breaking in the presence of *short*-range interactions generates massless Goldstone bosons which give rise to long-range correlations in a theory where originaly just short-range ones were present.)

The desired features are exhibited in the general class of lattice models with mean-field interaction described by Hepp and Lieb [16], Rieckers [27], Bóna [5, 6] and Unnerstall [30]. In [27] the connection of symmetry breaking with the emergence of classical observables in the model was pointed out. The terms subtracted from the microscopic hamiltonian in order to arrive at a reduced thermodynamic description (with temperature, chemical potential and other macroscopic contact variables fixed) generate an internal symmetry group acting automorphically on the algebra $\mathcal{A}$ of quasilocal observables. Classical structures now arise if this symmetry is broken in a grand canonical equilibrium state, and these classical structures are the same in all the representations of different temperatures where the symmetry is broken. The time evolution of the algebra of classical observables was described in [5]. There it is shown that if a symmetry group $\mathcal{G}$ of the quasilocal algebra is a Lie group, then the observables of the corresponding classical system $\mathcal{N}$ are global means of the generators of the unitary representation of $\mathcal{G}$. The corresponding phase space is the dual space of the Lie algebra of $\mathcal{G}$ and the time evolution of the classical system is described by a Poisson flow on this phase space. Conversely, a time evolution of $\mathcal{A} \otimes \mathcal{N}$ is determined by any norm continuous unitary representation of $\mathcal{G}$ and by a differentiable real function (the Hamiltonian function) on phase space. The quasilocal algebra $\mathcal{A}$ is not invariant under the dynamics; an automorphic time evolution can only be defined if the classical observables are added to $\mathcal{A}$, as pointed out for long-range interactions in general by [19].

Note also that the generalised Goldstone bosons generated by spontaneous symmtery breaking in the presence of long-range interactions are quasi-particles with non-vanishing mass [19]. For fixed momentum k the energy spectrum consists of a pure point spectrum (plus possibly a continuous background). If the system is asymptotically abelian, then the algebra at infinity is abelian. The quasi-particles are the classical system described by the variables at infinity. In a measurement situation, this classical system can serve as an apparatus. Examples for this kind of quasi-particles are the the Cooper pairs emerging in the BCS-model of superconductivity [20] or the plasmons occuring in a Coulomb Fermi gas with uniform background [21].

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