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THE HIGGS MECHANISM AS A PROBLEM OF INEQUIVALENT REPRESENTATIONS

I. The Landau gauge problem

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ABSTRACT

Several rigorous results obtained over the last two decades in lattice field theory indicate that the Higgs field vacuum expectation value is zero in theories which are generally believed to show spontaneous symmetry breaking. After reviewing the situation, we show that all apparent inconsistencies we are aware of can be removed by a classification of the possible Faddeev-Popov gauge fixings, used in the path integral formalism. We explain how these different kinds of Faddeev-Popov gauge fixing correspond to different direct integral representations of the field algebra. As a by-product, we obtain an argument that the cluster decomposition property does not hold in the Landau gauge. Furthermore, we show that in a large class of direct integral representations, the vacuum expectation value corresponds to an element of the center of the field algebra. The possible relevance of this observation for mass-generation through the Higgs mechanism is discussed briefly.

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1. INTRODUCTION

Any theory of the weak interaction has to incorporate two fundamental features: it must give masses to the intermediate vector bosons and it must yield finite radiative corrections. The Higgs mechanism[1-6] promises to fulfill both requirements: if a massless gauge field is coupled to some scalar matter field ϕ, then the assumption of a non-zero vacuum expectation value $\langle\phi\rangle$ implies that the vector field develops a mass. Furthermore, the mechanism leads to renormalizable theories, thereby implying finite radiative corrections.[7] However, the assumption of a non-zero vacuum expectation value is highly non-trivial. Indeed, it will be a major part of the present work to point out that this assumption only holds true in certain representations of the field algebra.

To clarify the context of our discussion, it is worth beginning with some textbook knowledge about the Higgs mechanism: for a non-zero vacuum expectation value, both local and global gauge invariance are broken. We say: they are "spontaneously broken", since the ground state does not exhibit the symmetry of the Lagrangian. However, there is in general no massless particle for this spontaneously broken symmetry, since a necessary condition for Goldstone's theorem to hold in relativistic theories is manifest covariance. Relativistic gauge theories, however, are not manifestly covariant and positive definite at the same time.[4] Either they are formulated in non-covariant gauges (e.g. Coulomb gauge) where commutators do not always vanish outside the light cone and hence Goldstone's theorem fails to apply, or they are described in the Gupta-Bleuler indefinite metric formalism (e.g. Lorentz gauge) where Goldstone's theorem applies but the zero mass particles appear in the unphysical modes of the Hilbert space.[4] We say: "the Goldstone bosons are eaten by the gauge fields".

As is well known, this Higgs mechanism plays a crucial role in the experimentally successful standard electroweak model,[8,9] allowing to describe massive vector bosons in a theory which is free from Goldstone bosons. Despite its striking success, however, attempts to reach a deeper understanding of the Higgs mechanism face serious difficulties. As pointed out by Wightman[10], the following apparently inconsistent results and assumptions exist in literature:

[i] In the conventional formulation of continuum gauge field theories with Faddeev-Popov gauge fixing, a non-zero vacuum expectation value is assumed. This assumption appears to be necessary for the experimental success of the electroweak theory.

[ii] In latticised versions of gauge theories which are believed to show spontaneous symmetry breaking, the vacuum expectation value of the scalar field is generically zero. [11,12] This is nothing but a simple consequence of the manifest gauge invariance of the measure in the usual lattice formulation without Faddeev-Popov gauge fixing (cf. section I.1).

[iii] In lattice gauge theories with Faddeev-Popov gauge fixing, it has been shown that the gauge-invariant two-point function of the scalar field decays exponentially in the radiation gauge [13,14] and in all Gupta-Bleuler gauges.* [16] This result holds formally in the continuum limit. [13]

[iv] In lattice gauge theories with Faddeev-Popov gauge fixing of the Landau type, the gauge-invariant two point function of the scalar field approaches a non-zero value in the large distance limit. [16]

In this work, we present an explanation of these results. Our discussion is organized as follows: In section I.1, we review lattice gauge theories. Introducing the notions of Faddeev-Popov gauge fixing of the first and second kind,+ we argue in I.2 that

[v] Lattice gauge theories with Faddeev-Popov gauge fixing of the first kind have a zero vacuum expectation value while

[vi] Lattice gauge theories with Faddeev-Popov gauge fixing of the second kind can have a non-zero vacuum expectation value.

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* Covariant indefinite metric gauges are specified by a gauge fixing term $\lambda(\partial.A)^2$ in the lagrangian. Formulations with finite λ are called Gupta-Bleuler gauges while the Landau gauge is defined as the limiting $\lambda \longrightarrow \infty$ case.

\+ We say: a Faddeev-Popov gauge fixing is of the first kind if the Faddeev-Popov term is invariant under global gauge transformations, whereas it is of the second kind if both local and global gauge invariance are completely broken by the Faddeev-Popov term (see Definitions 1 and 2 in I.2).

We proceed formally to obtain [v] and [vi], i.e., we assume the existence of the infinite volume limit as well as the continuum limit, and we assume that these limits are independent of the specification of boundary conditions. Though these assumptions are satisfied by many examples in the literature, we regard the results obtained merely as a useful guidance but not as final answer to our problem. All we conclude at this stage is that if our formal results hold true, then [i]-[iv] are consistent with each other but the cluster decomposition property fails in the Landau gauge.

This guides us in section II to an investigation of the cluster decomposition property in definite and indefinite metric spaces. Following Strocchi [35], we argue that the cluster decomposition property holds true in all Gupta-Bleuler gauges. In the Landau gauge however, the cluster decomposition property cannot be established by the same methods.

In section III, we adopt an algebraic point of view. Our main result is that the vacuum expectation value corresponds to an element of the center of the field algebra in a large class of integral representations. There is an obvious connection between these integral representations and the usual path integral formulation. Exploiting this connection, we are able to confirm the main results of section I under a different set of technical assumptions. We conclude with some remarks on the possible relevance of our observations for mass-generation in the Higgs mechanism.

I. EXPECTATION VALUES IN LATTICE GAUGE THEORIES

I.1. Formulations without gauge fixing

In this section, we discuss euclidean gauge theories, given by the Lagrangian

$$\mathcal{L} = tr(F_{\mu\nu}F_{\mu\nu}) + \mid D_\mu\phi \mid^2 - V(\phi), \tag{1.1}$$

where $F_{\mu\nu}$ denotes the field tensor, D_μ the covariant derivative of the scalar matter field ϕ and the potential V a polynomial in the scalar field. We work on a lattice with lattice sites x. Following a method of Osterwalder and Seiler, [18] we introduce the matter field $\phi(x)$ by associating with each lattice site x a pair of generalized polar

coordinates $[R(x), g_x]$. Here $R(x)$ takes values in $R_+ = [0, \infty)$ and g_x is an element of the gauge group G. We denote an n-dimensional vector $R(x) = (R(x), 0, 0, ..., 0)$ and an nxn-matrix g_x in an arbitrary representation of the global gauge group G by the same symbols. For the matter field, we write in this representation

$$\phi(x) = g_x R(x). \tag{1.2}$$

The gauge field is introduced in terms of the variables $g_{x,x+\mu}$, taking values on the links between neighbouring sites x and $x+\mu$. Formally, we can write these link variables as

$$g_{xy} = \exp\left[ie \int_x^y A dl\right], \tag{1.3}$$

where e denotes a coupling constant, the vector potential A (geometrically: the connection) is an element of the Lie algebra $L(\mathbf{G})$, associated with the gauge group G and the integral is taken along the path from x to y. Obviously, the g_{xy} are elements of G.

Now, the latticised theory, corresponding to the Lagrangian (1.1) is given by the action S_Λ in a finite volume Λ: [18,19]

$$\begin{aligned} S_\Lambda =& \beta \sum\nolimits_P (\chi_p(g) - \chi(I)) \\ &+ \beta' \sum\nolimits_x \left(R(x) g_x^{-1} g_{x,x+\mu} g_{x+\mu} R(x+\mu)\right) - \sum\nolimits_x V'(R(x)) \end{aligned} \tag{1.4}$$

where

$$V'(R(x)) = V(R(x)) + a(x) R^2(x), \tag{1.5}$$

and $a(x)$ is essentially the number of bonds in the finite volume Λ with one endpoint at x. Here I is the identity in G, the β's are constants which we do not need to specify for our purpose, $\chi_P(g)$ is the plaquette character

$$\chi_P(g) = Tr\left(g_{x,x+\mu} g_{x+\mu,x+\mu+\nu} g_{x+\mu+\nu,x+\nu} g_{x+\nu,x}\right), \tag{1.6}$$

and $\chi(I)$ denotes the dimension of the choosen representation. The sums are taken over lattice sites x and elementary plaquettes P of the finite volume Λ.

On this lattice, we introduce the finite volume Gibbs measure [18]

$$d\mu_\Lambda = Z_\Lambda^{-1} e^{S_\Lambda} d\nu_\Lambda, \tag{1.7}$$

where

$$Z_\Lambda = \int e^{S_\Lambda} d\nu_\Lambda \tag{1.8}$$

and

$$d\nu_\Lambda = \prod\nolimits_{xy} dg_{xy} \prod\nolimits_x dg_x \prod\nolimits_x R^k(x) dR(x). \tag{1.9}$$

The positive integer k in the last expression is equal to the dimension of the Lie algebra

$$k = \dim L(\mathbf{G}) \tag{1.10}$$

(e.g. $k = 1$ for $\mathbf{G} = U(1)$, $k = 3$ for $\mathbf{G} = SU(2)$).

This measure is invariant under the joint gauge transformation γ of gauge field and matter field:

$$g_x \longrightarrow g_x^\gamma = \gamma_x g_x, \tag{1.11}$$

$$g_{xy} \longrightarrow g_{xy}^\gamma = \gamma_x g_{xy} \gamma_y^{-1}. \tag{1.12}$$

Introducing the gauge-invariant link variables

$$h_{xy} = g_x^{-1} g_{xy} g_y \tag{1.13}$$

allows us to factorize the measure

$$d\nu_\Lambda = \prod\nolimits_{xy} dh_{xy} \prod\nolimits_x R^k(x) dR(x) \prod\nolimits_x dg_x = d\hat{\nu}_\Lambda \prod\nolimits_x dg_x. \tag{1.14}$$

With this measure, the expectation values $\langle f(x) \rangle_\Lambda$ of arbitrary functions $f(x)$ in the fields

$$f(x) = f\left(R(x), g_x, g_{xy}\right) \tag{1.15}$$

are given by

$$\langle f(x) \rangle_\Lambda = \int f(x) d\mu_\Lambda. \tag{1.16}$$

It follows directly from this formulation and the gauge invariance of the measure (1.6) that the expectation values of all gauge-dependent components f_c of f must vanish, since

$$\langle f_c^\gamma(x) \rangle_\Lambda = \int f_c(R^\gamma(x), g_x^\gamma, g_{xy}^\gamma) d\mu_\Lambda$$

$$= \int f_c(R(x), g_x, g_{xy}) d\mu_\Lambda^{\gamma^{-1}}$$

$$= \int f_c(x) d\mu_\Lambda$$

$$= \langle f_c(x) \rangle_\Lambda. \tag{1.17}$$

Since the scalar matter field has no gauge-invariant component, we conclude that the vacuum expectation value vanishes in lattice gauge theories without gauge fixing:

$$\langle \phi(x) \rangle_\Lambda = \langle \phi^\gamma(x) \rangle_\Lambda = 0. \tag{1.18}$$

Of course, this approach is somehow formal. One might argue that the specification of boundary conditions in (1.17) can upset the argument. That this is not the case has been shown at least for some models (e.g. scalar electrodynamics, Abelian Higgs model) by de Angelis, de Falco and Guerra[12], following an earlier work of Elitzur[11].

I.2 Formulations with Faddeev-Popov gauge fixing

Any lattice gauge fixing F is a modification of the finite volume Gibbs measure (1.6)

$$e^{S_\Lambda} \longrightarrow e^{S_\Lambda} F, \tag{1.19}$$

which leaves the expectation value (1.16) for gauge-invariant functions unchanged. This condition is equivalent to the Faddeev-Popov gauge condition

$$\int \prod_x dg_x F(R(x), h_{xy}, g_x) = 1. \tag{1.20}$$

This is sufficient to ensure that the physical content of the theory is unaffected by the gauge fixing procedure. Of course, the expectation values of gauge-dependent functions change in general with the gauge fixing. To get some control over this change, we characterize F further. Introducing the set of global gauge transformations $\check{g}$ (e.g. $\check{g} = e^{ia}$, where $a \in [0, 2\pi)$ for $\mathrm{G} = U(1)$), we define

Definition 1 F_1 is called *Faddeev-Popov gauge fixing of the first kind* if

$$\int \prod_x dg_x F_1(R(x), h_{xy}, g_x) f(R(x), h_{xy}, g_x) = \int d\check{g} f(R(x), h_{xy}, \check{g}\bar{g}_x) \tag{1.21}$$

for any function f of the fields. Here $\bar{g}_x$ denotes the set of fixed field variables g_x.

Definition 2 F_2 is called *Faddeev-Popov gauge fixing of the second kind* if

$$\int \prod_x dg_x F_2(R(x), h_{xy}, g_x) f(R(x), h_{xy}, g_x) = f(R(x), h_{xy}, \bar{g}_x) \tag{1.22}$$

for any function f of the fields and some $\bar{g}_x$ as in definition 1.

All Gupta-Bleuler and Landau gauges are of the first kind. They leave the global gauge invariance unbroken. The following propositions clarify that such a gauge fixing leads to a zero vacuum expectation value for lattice gauge theories in a finite lattice volume Λ. With a gauge fixing of the second kind, one obtains a non-zero vacuum expectation value.

Proposition 1 For a lattice gauge theory with Faddeev-Popov gauge fixing of the first kind, the vacuum expectation value is zero:

$$\langle \phi(z') \rangle_\Lambda^{F_1} = 0. \tag{1.23}$$

Proposition 2 For a lattice gauge theory with Faddeev-Popov gauge fixing of the second kind, the vacuum expectation value is non-zero:

$$| \langle \phi(z') \rangle_\Lambda^{F_2} | = R_\Lambda > 0. \tag{1.24}$$

Proof of Proposition 1 To prove proposition 1, we use the factorization of the measure (1.14) and definition 1 to write

$$\langle \phi(z') \rangle_\Lambda^{F_1} = \int g_{z'} R(z') F_1 d\mu_\Lambda$$

$$= \int g_{z'} R(z') F_1 e^{S_\Lambda} d\hat{\nu}_\Lambda \prod_x dg_x$$

$$= \int \check{g} \bar{g}_{z'} R(z') e^{S_\Lambda} d\hat{\nu}_\Lambda d\check{g}. \tag{1.25}$$

The action S_Λ and the reduced measure $d\hat{\nu}_\Lambda$ are left invariant by the global gauge transformation $\check{g}$. However in all non-trivial representations of the gauge group G

$$\int \check{g} d\check{g} = 0. \tag{1.26}$$

Hence the vacuum expectation value $\langle\phi(z')\rangle_\Lambda^{F_1}$ in (1.25) is zero.* □

Proof of Proposition 2 Using the factorization of the measure (1.14), we write

$$\langle\phi(z')\rangle_\Lambda^{F_2} = \int g_{z'} R(z') F_2 d\mu_\Lambda$$

$$= \int g_{z'} R(z') F_2 e^{S_\Lambda} d\hat{\nu}_\Lambda \prod_x dg_x$$

$$= \bar{g}_{z'} Z_\Lambda^{-1} \int \prod_{xy} dh_{xy} \prod_z dR(z) R^k(z) R(z') e^{S_\Lambda}. \tag{1.27}$$

Let us recall that $dR(z)$ is a Lebesgue measure on R_+ while dh_{xy} is a Haar measure on the compact group G. This allows us to write

$$\int \prod_{xy} dh_{xy} \exp[S_\Lambda(h_{xy}, R(z))] = K_\Lambda(R) > 0 \tag{1.28}$$

for every arbitrary set of fixed values $R(z)$, since $e^{S_\Lambda} > 0$. Hence

$$|\langle\phi(z')\rangle_\Lambda^{F_2}| = Z_\Lambda^{-1} \int \prod_z dR(z) K_\Lambda(R) R^k(z) R(z') = R_\Lambda \tag{1.29}$$

is greater than zero since Z_Λ is finite for finite volume Λ and $dR(z)$ is a Lebesgue measure on R_+, i.e. all contributions to the integral are finite and positive. □

II.1.3. Consequences for the continuum limit

The relevance of finite volume lattice gauge theories for the continuum theory relies on the assumption that there exists both an infinite volume limit and a scaling limit for the infinite volume lattice theory. Within the program of constructive quantum field theory, the existence of the infinite volume limit and of the scaling limit has been shown so far for an two-dimensional euclidean $U(1)$-theory only.[20,21] However, the existence of a cluster expansion, uniformily converging in the lattice volume, has been shown for various models.[18] In this expansion, the expectations of local observables have an infinite volume limit.

* (1.26) really means: $\int d\check{g} U(\check{g}) = 0$ where U denotes a non-trivial representation of G.

Hence we take the following approach: We exhibit the volume dependence of $|\langle\phi\rangle_\Lambda|$ for propositions 1 and 2. Then we assume the following:

A) The finite volume expectation values converge uniformily in Λ. This assumption is motivated by the existence of a cluster expansion with the corresponding property.[18]

B) The scaling limit exists and does not change the results of the infinite volume limit essentially (i.e. non-zero infinite volume expectations do not converge to zero in the scaling limit). This is an assumption about the absence of phase transitions.

These assumptions are consistent with the rigorous results, obtained so far.[18,20,21] They are usual working assumptions and allow a statement about the size of the vacuum expectation value in the continuum limit.

For Proposition 1, this approach is straightforward. According to (1.23), $\langle\phi\rangle_\Lambda^{F_1}$ is zero for all finite lattice volumes, and hence

$$\lim_{\Lambda\to\infty} \langle\phi\rangle_\Lambda^{F_1} = 0. \tag{1.30}$$

This establishes in our approach the vacuum expectation value of the corresponding continuum theory.

For Proposition 2, the question whether the corresponding statement

$$\lim_{\Lambda\to\infty} R_\Lambda = R_\infty > 0 \tag{1.31}$$

holds true is nontrivial. It should be clear that (1.31) depends on the choice of the potential $V(R(x))$ in (1.5) and there might be choices of V such that the set of all R_Λ converges to zero in the infinite volume limit. Here we argue that there are choices of V such that $R_\infty \neq 0$. E.g. for a finite volume lattice theory of the type discussed in section I.1, with $V_\lambda = 0$ for $\bar{R} - \lambda \leq R \leq \bar{R} + \lambda$, $V_\lambda = \infty$ elsewhere, $\bar{R}$ large enough and λ small enough, it is possible to show that subject to the above assumptions

$$\lim_{\Lambda\to\infty} |\langle\phi(z')\rangle_\Lambda^{F_2}| = R_\infty > 0. \tag{1.32}$$

For more realistic models, technical difficulties might make such arguments hard to obtain. Here we conclude that the above arguments justify statement [vi] of the introduction: Lattice gauge theories with Faddeev-Popov gauge fixing of the second kind can have a non-zero vacuum expectation value.

As already pointed out, the results of this section have been obtained by a formal procedure. Neither did we specify boundary conditions for finite volume lattice theories, nor do we have rigorous results about the existence of the infinite volume limit and the continuum limit. Despite of the technical draw-backs of our derivation, it is interesting to check how far these results allow us to clarify the apparent inconsistencies, mentioned in the introduction. Subject to the technical caveats mentioned above, we conclude the following:

1) The conventional formulation of continuum gauge field theories in Gupta-Bleuler gauges leaves the global gauge symmetry unbroken. It is a gauge fixing of the first kind which leads to a zero vacuum expectation value according to (1.30).* This is inconsistent with the usual assumption [i] of a non-zero vacuum expectation value in these formulations. To match the experimentally successful assumption of a non-zero vacuum expectation value with the path integral formulation, a gauge fixing of the second kind is required. Therefore, we have to modify the path integral measure analogous to (1.19) with the corrected Faddeev-Popov gauge fixing term

$$F_2 = F_1 F_G, \tag{1.33}$$

where F_G is the global gauge fixing term, satisfying

$$\int d\check{g} F_G(R(x), h_{xy}, g_x) f(R(x), h_{xy}, \check{g}\bar{g}_x) = f(R(x), h_{xy}, \bar{g}_x) \tag{1.34}$$

[cf. (1.21) and (1.22)]. Replacing F_1 by F_2 affects only the calculation of gauge-dependent quantities. Since the vacuum expectation of the Higgs field is usually the only relevant gauge-dependent quantity, it is clear why one can get away with a Faddeev-Popov gauge fixing of the first kind plus the assumption of a non-zero vacuum expectation value: all calculations of gauge-invariant quantities do not depend on the kind of Faddeev-Popov gauge fixing in the path integral formalism.

* For formulations with gauge fixings which contain global unbroken subgroups of G (e.g. Coulomb gauge), the vacuum expectation value is zero too. This is easy to check with the method of proof 1, though these gauge fixings are not of the first kind.

2) As argued in (1.18), the vacuum expectation value is zero in formulations without gauge fixing. Hence [ii] is correct. This holds true in the infinite volume limit.

3) [iii] is correct but not at all astonishing in the light of our results. As we have argued in (1.30), gauge fixings of the first kind lead to zero vacuum expectation values. However, due to the possibility of gauge fixings of the second kind and their relation to gauge fixings of the first kind (1.33), this does not contradict formulations of the same theory with non-zero vacuum expectation values.

4) The non-zero large distance behaviour of the gauge-invariant two-point function in the Landau gauge is difficult to embed in this picture. The Landau gauge is Lorentz invariant and classifies as a Faddeev-Popov gauge fixing of the first kind. Hence we should obtain a zero vacuum expectation value in this gauge. As we shall argue in the next section, a contradiction with our results exists if and only if the cluster decomposition property holds. However, all existing proofs of the cluster decomposition property will be shown to break down in the Landau gauge.

II. THE CLUSTER DECOMPOSITION PROPERTY IN THE LANDAU GAUGE

So far, we have seen that lattice gauge theories in a Faddeev-Popov formulation of the first kind have zero vacuum expectation values and that this does not contradict the electroweak theory, based on a non-zero vacuum expectation value. On the other hand, the large distance behaviour of the (gauge-invariant) two-point functions is known for Faddeev-Popov formulations without global gauge fixing: they decay exponentially in the temporal gauge but they do not decay in the Landau gauge. Hence, either the vacuum expectation value of the scalar Bose field is non-zero in the Landau gauge,

$$\langle\phi\rangle \neq 0, \tag{2.1}$$

or the cluster decomposition property fails, i.e.

$$\lim_{x\to\infty} | \langle\phi(0)\overline{\phi}(x)\rangle - \langle\phi(0)\rangle\langle\overline{\phi}(x)\rangle | \neq 0. \tag{2.2}$$

Obviously, (2.2) makes it impossible to determine $\langle\phi\rangle$ from the large distance behaviour of the two-point function. It is the possibility (2.2) which we investigate in this section. Two arguments motivate our approach:

i) Although our definition of Faddeev-Popov gauge fixings of the first and second kind oversimplifies the problem somehow by concentrating on the effect of the global symmetry fixing only, all features of the Landau gauge indicate that it can be classified as a Faddeev-Popov gauge fixing of the first kind. According to proposition 6 this means that (2.1) is wrong and hence (2.2) is correct.

ii) It is well known that the cluster decomposition property may fail in certain indefinite metric gauge field theories. [28]

This section is organized as follows: In II.1, we review the cluster decomposition property shortly, highlighting the assumptions on which its proof is based as well as the additional problems arising in an indefinite metric formalism. In II.2, the lattice two-point functions are introduced on which Fröhlich, Morchio and Strocchi [13,14] and Kennedy and King [16] based their proofs. It becomes apparent that the assumptions

of the cluster decomposition property are not necessarily fulfilled for these objects. II.3 proves that the cluster decomposition property holds in all Gupta-Bleuler gauges. Furthermore, some arguments are presented which indicate that the cluster theorem might fail to apply in the Landau gauge.

II.1. The Cluster Theorem

The cluster theorem gives an upper bound on the correlations between spacelike separated local observables in the vacuum sector. It can be stated in the following way, covering the positive metric case as well as the indefinite metric one: [28,22]

Cluster Theorem: Let B_1, B_2 be bounded local operators, localized in space-time regions $\mathcal{O}_1$, $\mathcal{O}_2$. Denote by τ the spacelike distance between $\mathcal{O}_1$ and $\mathcal{O}_2$ and denote the vacuum sector by ψ_0. Then the following statement holds true:

[CT-I] If the Wightman functions $W(p_1,...,p_n)$ of the theory have a mass gap μ, i.e. if their truncated form is always zero if $p_i^2 < \mu^2$ for some i, then

$$| \langle\psi_0, B_1B_2\psi_0\rangle - \langle\psi_0, B_1\psi_0\rangle\langle\psi_0, B_2\psi_0\rangle | \leq c\frac{1}{\tau^{\frac{3}{2}}}e^{-\mu\tau}\tau^{2N}. \tag{2.3}$$

[CT-II] If the Wightman functions exhibit no mass gap*, then

$$| \langle\psi_0, B_1B_2\psi_0\rangle - \langle\psi_0, B_1\psi_0\rangle\langle\psi_0, B_2\psi_0\rangle | \leq c\frac{1}{\tau^2}\tau^{2N}. \tag{2.4}$$

Here, c is a constant independent of τ but dependent on the norm of B_1 and B_2, and N is a non-negative integer. In the case of a positive metric quantum field theory, $N = 0$ and we are left with the classical result of Araki, Hepp and Ruelle. [22] In the indefinite metric case, $N \geq 1$ is possible. [28] This opens the possibility for a violation of the cluster decomposition property in indefinite metric quantum field theories. Since what follows strongly depends on this result, we feel it is necessary to trace back its origin.

* In an indefinite metric qft, even if the physical spectrum has a mass gap, there might be no mass gap of the Wightman functions due to the presence of unphysical fields.

The classical proof of the cluster theorem [22,28] starts by considering functions of the form

$$f_{1,2}(x) = \langle \psi_0, B_1 U(x) B_2 \psi_0 \rangle, \tag{2.5}$$

where $U(x)$ is the space-time translation operator, translating B_2 by a distance x (remark that $U(-x)\psi_0 = \psi_0$). The essential idea is then to write $f_{1,2}$ in a Jost-Lehmann-Dyson representation and to use the known large space-like distance behaviour of the spectral functions in this representation. [22] This leads directly to (2.3) and (2.4) for $N = 0$. It is possible if and only if the Fourier transform $\hat{f}(p)$ of

$$f(x) = f_{1,2}(x) - f_{2,1}(-x) \tag{2.6}$$

defines a measure. [22] Loosely speaking, this means that the singularities of $\hat{f}(p)$ are not worse than δ-functions. [28,23] In the positive metric case, it is well known that the matrix elements of the translation operator $U(p)$ define measures and hence the proof goes through. [24] In the indefinite metric case however, the analogous statement does not hold in general and $\hat{f}(p)$ denotes symbolically a tempered distribution which is not necessarily a measure. [28,23] However, one can use the Bros-Epstein-Glaser theorem that every tempered distribution with the support and covariance properties of $\hat{f}(p)$ can be written as

$$\hat{f}(p) = \Box_p^N \hat{g}(p), \tag{2.7}$$

where $\Box$ denotes the Laplace operator, N is a non-negative integer and $\hat{g}(p)$ is a continuous function of at most polynomial increase. [25] Since $\hat{g}(p)$ has the same support properties as $\hat{f}(p)$ and defines a measure, the Jost-Lehmann-Dyson representation exists for the objects $g_{1,2}$ and $g_{2,1}$ defined correspondingly. [28] The Fourier transform of (2.7) leads then to the extra factor τ^{2N} in (2.3) and (2.4). We conclude that

Criterion I the cluster decomposition property can only fail if the Fourier transform of (2.5) is not a measure and if the expectation value has no mass gap.

To close this subsection, we mention a criterion which characterizes those indefinite metric formulations for which $N = 0$.

Criterion II (Strocchi [28]) If $[U(x), \eta] = 0$, then $N = 0$.

Here, η is the metric operator. To clarify our notation, let us recall that two different bilinear forms are introduced in indefinite metric formulations of quantum field theory. There is the indefinite product $(*,*)$ on the Hilbert space $\mathcal{H}$, generated via a GNS-construction by the operation of all elements of the field algebra on ψ_0. Secondly, there is the positive sesquilinear form $\langle *,* \rangle = (*, \eta *)$. Constructing the maximal subspaces $\mathcal{H}' = \{\psi \in \mathcal{H}; \langle \psi, \psi \rangle \geq 0\}$ and $\mathcal{H}'' = \{\psi \in \mathcal{H}'; \langle \psi, \psi \rangle = 0\}$, the physical state space is obtained as the quotient space $\mathcal{H}_p = \mathcal{H}'/\mathcal{H}''$. If $[U(x), \eta] = 0$, then it turns out that the Araki-Hepp-Ruelle analysis can be applied, which implies criterion II.

II.2. Lattice two-point functions

In the light of the above statements about the cluster property, we can investigate the behaviour of the two-point functions $G(x,y)$ anew. In this section, we introduce these objects in a lattice formulation. There are two of them which we distinguish with subscripts: G_1, G_2.

Fröhlich, Morchio and Strocchi [13,14] took the latticised two-point Green's function

$$G_1(x_0, x_n) = \phi(x_0) g_{x_0 x_1} g_{x_1 x_2} \cdots g_{x_{n-1} x_n} \bar{\phi}(x_n) \tag{2.8}$$

as the starting point of their analysis. They proved the exponential decay for G_1 in the Coulomb gauge. $G_1(x_0, x_n)$ is gauge-invariant under the joint lattice gauge transformations, specified in (1.11) and (1.12). This gauge invariance persists in the formal continuum limit, obtained by inserting the formal expression (1.3) into (2.8), which leads to

$$G(-y, y) = \phi(-y) \exp\left[ie \int_{-y}^{y} A(l) dl\right] \bar{\phi}(y). \tag{2.9}$$

(2.9) is formally gauge-invariant under the combined continuum gauge-transformations, labeled by the real parameter M

$$A_\mu(x) \longrightarrow A_\mu(x) - M \partial_\mu \chi(x), \tag{2.10}$$

$$\phi(x) \longrightarrow \phi(x) \exp[ieM\chi(x)], \tag{2.11}$$

where the field $\chi(x)$ satisfies

$$\Box\chi(x) = 0. \tag{2.12}$$

A slightly different way to make the scalar two-point function gauge-invariant was adopted by Kennedy and King.[16] They discussed the latticized gauge-invariant Green's function

$$G_2(x,y) = \phi(x)\exp[-ie\sum_z A_\mu(z)h^\mu(z)]\bar{\phi}(y), \tag{2.13}$$

where h^μ is the electric field generated by charges $+1$ at x and -1 at y. h^μ can be written as a covariant derivative of the potential at the point Z:

$$h_\mu(z) = \partial_\mu V(z-x) - \partial_\mu V(z-y). \tag{2.14}$$

Under gauge transformations of the type discussed in (2.10) and (2.11), the exponent in (2.13) transforms formally as

$$e\sum_z A_\mu(z)h^\mu(z) \longrightarrow e\sum_z A_\mu(z)h^\mu(z) + eM\chi(x) - eM\chi(y), \tag{2.16}$$

which shows that $G_2(x,y)$ is gauge-invariant. Now one uses the fact that the situation simplifies in the Landau gauge, where

$$\partial_\mu A^\mu = 0 \tag{2.17}$$

implies

$$\sum_z A_\mu(z)h^\mu(z) = 0. \tag{2.18}$$

Consequently, the exponential in (2.13) becomes 1 and

$$G_2'(x,y) = \phi(x)\bar{\phi}(y). \tag{2.19}$$

Kennedy and King take this as the starting point for their proof that under suitable conditions the lattice two-point function satisfies

$$|\langle G_2'(x,y)\rangle| = |\langle\phi(x)\bar{\phi}(y)\rangle| \geq K > 0, \tag{2.20}$$

where K is independent of x and y. Furthermore they have a proof that

$$\lim_{x\to\infty} G_2'(0,x) = 0 \tag{2.21}$$

in all Gupta-Bleuler gauges.

Having presented explicit expressions for the two-point Green's functions G_1, G_2, we can now investigate whether the behaviour of $G(x,y)$ determines $\langle\phi\rangle$. As explained above, the large distance behaviour of $G(x,y)$ determines $\langle\phi\rangle$ only if the cluster decomposition property applies, i.e. if

$$\lim_{x\to\infty}|\langle G(0,x)\rangle - \langle\phi(0)\rangle\langle\phi(x)\rangle| = 0 \tag{2.22}$$

for spacelike x.

The two-point Green's function $G_1(x,y)$ in (2.9) is supported in a region connecting x and y. Because of the gauge string $\exp[ie\int A(l)dl]$ in (2.9), it does not cluster in arbitrary gauges. Hence we cannot conclude much about the vacuum expectation value $\langle\phi\rangle$ from the large distance behaviour of G_1.

For the two-point function $G'_2(x,y)$ in (2.19), the situation is more advantageous. Viewing the fields ϕ as local operators, the support regions of $\phi(x)$ and $\bar{\phi}(y)$ decompose for large spacelike separations $x-y$. Hence the preliminaries of the cluster theorem apply and we face the choice already mentioned before: either the vacuum expectation value is non-zero in the Landau gauge, or there is no clustering in this gauge.

II.2.3. The cluster theorem in covariant gauges

Here we argue in the case of free quantum electrodynamics that what is known is consistent with the non-validity of the cluster decomposition property in the Landau gauge, while it proves the cluster decomposition in the other covariant gauges. Though this proves nothing for the interacting quantum field theories which we discussed so far, it can be regarded as a useful guidance, hinting at the non-validity of the cluster decomposition property in the Landau gauge.

First of all, it is interesting to note that the gauge-field two-point function in free Landau gauge QED, given in equation (2.129) of reference [23], is too singular to define a measure. Hence it is not possible to derive the validity of the cluster decomposition property from criterion I.

Secondly, if the definition of the scalar product is altered in such a way that the two-point function defines a measure,* the translation operator $U(x)$ does not commute with the metric operator η:

$$[U(x), \eta] \neq 0. \tag{2.23}$$

This is the result of a straightforward calculation from the explicit formulations for $U(x)$ and η, given in the equations (2.135) and (2.136) of reference [23]. Hence it is not possible to derive the validity of the cluster decomposition property from criterion II.

From the same section of reference [23], it is apparent that

$$[U(x), \eta] = 0 \tag{2.24}$$

in all covariant gauges except of the Landau gauge. According to criterion II and (2.4), this proves that arbitrary clusters decompose for large spacelike separations in these covariant gauges. Hence we can conclude from (2.21) that the vacuum expectation value is zero in all covariant gauges except for the Landau gauge. For the Landau gauge, however, (2.2) cannot be excluded.

* For details of such a construction, see [23] and the Erratum ibid. Vol.17, p.1930.

III. THE ALGEBRAIC STRUCTURE OF THE HIGGS MECHANISM

III.1. The field algebra and its center

In this section, we assume that the cluster decomposition property is valid at least in its weakest form (2.4) with $N = 0$. From this we show that in a certain class of integral representations, the vacuum expectation value corresponds to an element of the center of the field algebra $\mathcal{A}$. With this approach we attempt to discuss the size of the vacuum expectation value in a setting which is not burdened with the difficulties of the lattice approach. From the notation introduced for the proof, we can indeed confirm the main results of section I under a different set of assumptions. This improves our argument that the cluster decomposition property does not hold in the Landau gauge. Furthermore, this additional characterization of the vacuum expectation value leads us to new conceptual questions about the nature of mass-generation through the Higgs mechanism.

The methods we employ are essentially algebraic. We start from the set of Wightman fields $\varphi(f)$, defined on a given Hilbert space $\mathcal{H}$ with a cyclic vacuum vector ψ. Here, $f \in \mathcal{S}(R^4)$, the set of infinitely often differentiable test functions which together with their derivatives approach zero faster than any power of the distance and φ is an operator-valued distribution on this test function space. Then we proceed formally by assuming that the Wightman fields $\varphi(f)$ correspond to elements $\phi(f)$ in the set of bounded operators $\mathcal{B}(\mathcal{H})$ acting on the Hilbert space $\mathcal{H}$,

$$\phi(f) \in \mathcal{B}(\mathcal{H}). \tag{3.1}$$

This assumption leads to a heuristic passage from the set of Wightman fields to a net of local field algebras, thereby sidestepping some deep-rooted technical difficulties.[26,27] Concretely, for all functions f with compact support in the space-time region $\mathcal{O}$, we consider the $\phi(f)$ to be elements of a local algebra $\mathcal{A}(\mathcal{O})$. The algebra $\mathcal{A}$ is regarded as the set-theoretic union $\bigcup_{\mathcal{O}} \mathcal{A}(\mathcal{O})$ of all $\mathcal{A}(\mathcal{O})$ together with its norm-closure. For the elements of $\mathcal{A}$, isotony, local commutativity and Lorentz covariance are assumed. Having obtained this net of local field algebras in a defining (vacuum) representation,

we might consider it to be an abstract algebra $\mathcal{A}$ which can be represented in other representations π too.

In what follows, we consider the complete algebra $\mathcal{A}_c$ containing the gauge-dependent matter fields $\phi(f)$ [cf. (1.2)] as well as all the other local bounded fields of the theory under consideration. This algebra corresponds in our approach to a local and covariant Wightman field theory connected with a nontrivial local gauge transformation. A theorem of Strocchi states that the metric in the state space of such a Wightman field theory cannot be positive definite.[17,23,28,34] For such theories, there exists a modification of the notion of state space, known as Strocchi-Wightman field theory and amounting to an indefinite metric formalism of the type introduced by Gupta and Bleuler.[23] An axiomatic setting of such a local and covariant gauge quantum field theory has been given by Strocchi.[17] One considers the fields as defined on a Hilbert space $\mathcal{H}$ with scalar product $(*,*)$. The physical expectation values are computed in terms of a bounded, hermitian product $\langle *,* \rangle = (*, \eta *)$ with metric operator η. This product is positive semidefinite on a nontrivial and maximal subspace $\mathcal{H}' \subset \mathcal{H}$. The space of physical states $\mathcal{H}_p$ is then defined as the quotient space $\mathcal{H}'/\mathcal{H}''$ where $\mathcal{H}''$ contains all elements of $\mathcal{H}'$ with vanishing $\langle *,* \rangle$-norm. There exists then a common dense domain $D' \subset \mathcal{H}'$, stable under the action of the Poincaré group, such that the translationally invariant state $\psi \in \mathcal{H}$ is a cyclic vector of the local field algebra and $\psi \in D'$. We shall adopt these concepts, proceeding formally by assuming that all local fields correspond to elements in the set of bounded operators $\mathcal{B}(\mathcal{H})$ acting on the Hilbert space $\mathcal{H}$. Having obtained this net of local field algebras in a defining representation, we might consider it to be an abstract algebra $\mathcal{A}_c$ which can be represented in other representations π too.

We are interested in integral representations of the field algebra over arbitrary compact spaces Λ. Therefore, we consider the direct integral representations $(\mathcal{H}, \pi)$ of the algebra $\mathcal{A}_c$ with[30]

$$\mathcal{H} = \int_\Lambda^{\oplus} d\mu(\lambda)\mathcal{H}_\lambda, \tag{3.2}$$

$$\pi = \int_\Lambda^{\oplus} d\mu(\lambda)\pi_\lambda, \tag{3.3}$$

obtained from the cyclic translationally invariant state

$$\psi = \int_\Lambda^{\oplus} d\mu(\lambda)\psi(\lambda). \tag{3.4}$$

Here Λ is a compact measure space with Borel measure μ, $\psi(\lambda)$ is a cyclic vacuum vector of the irreducible space $\mathcal{H}_\lambda$ for $\pi_\lambda(\mathcal{A}_c)$ and the scalar product on $\mathcal{H}$ is defined as*

$$(\psi_A, \psi_B) = \int_\Lambda d\mu(\lambda)(\psi_A(\lambda), \psi_B(\lambda)). \tag{3.5}$$

Hence

$$(\psi_A, \pi(D)\psi_B) = \int_\Lambda d\mu(\lambda)(\psi_A(\lambda), \pi_\lambda(D)\psi_B(\lambda)). \tag{3.6}$$

Furthermore, the metric operator η can be written as a direct integral over the η_λ which define the $\langle *,* \rangle$-norms on $\mathcal{H}'_\lambda$,

$$\eta = \int_\Lambda d\mu(\lambda)\eta_\lambda \tag{3.7}$$

and hence

$$\begin{aligned} \langle \psi_A, \pi(D)\psi_B \rangle &= (\psi_A, \eta\pi(D)\psi_B) \\ &= \int_\Lambda d\mu(\lambda)\langle \psi_A(\lambda), \pi_\lambda(D)\psi_B(\lambda)\rangle \\ &= \int_\Lambda d\mu(\lambda)(\psi_A(\lambda), \eta_\lambda\pi_\lambda(D)\psi_B(\lambda)). \end{aligned} \tag{3.8}$$

Our main result concerns the quasi-local quantity

$$\hat{\phi}(f) = \lim_{\mathcal{O}\to\infty}\frac{1}{|\mathcal{O}|}\int_{\mathcal{O}} d^4x\alpha_x(\phi(f)), \tag{3.9}$$

where α_x denotes the space-time translation automorphism and $|\mathcal{O}|$ is the volume of the finite space-time region $\mathcal{O}$. This object will be constructed from

$$\tilde{\phi}_V = \frac{1}{V}\int_V d^3\bar{x}\alpha_{\bar{x}}(\phi(f)). \tag{3.10}$$

* For notational convenience, we drop the direct sum symbol $\oplus$ in what follows.

where $\alpha_{\bar{x}}$ is the space translation automorphism and V is a finite space volume. To see that the integral (3.10) is well-defined, we mention that the operator norm $\| \alpha_{\bar{x}}(\phi(f)) \|$ is independent of $\bar{x}$ and that this norm is subadditive.[29] Hence the integrand can be bounded from above. We only have to check local integrability which is given in the sense of a Bochner integral (the generalization of the Lebesgue integral to integrands with values in a Banach space), if $\alpha_{\bar{x}}$ is strongly continuous. Hence (3.10) is well-defined.

These are the necessary preliminaries for the following proposition:

Proposition 3

Assume the cluster decomposition property in its weakest form (2.4) with $N = 0$. Given a direct integral representation $(\mathcal{H}, \pi)$ of $\mathcal{A}_c$ over a compact space Λ and a cyclic translation invariant vacuum state $\psi \in \mathcal{H}$.

3.1. $\hat{\phi}(f)$ exists as a weak limit in the locally convex topology defined by the $\langle *, * \rangle$-seminorms and is an element in $\mathcal{B}(\mathcal{H})$.

3.2. $\hat{\phi}(f) \in \mathcal{Z}(\pi(\mathcal{A}_c)'')$, the center of the bicommutant of $\pi(\mathcal{A}_c)$.

To prove Proposition 3, we require the following lemmata:

Lemma 1 Assume the cluster decomposition property in its weakest form (2.4) with $N = 0$. Given an irreducible representation $(\mathcal{H}, \pi)$ of $\mathcal{A}_c$ and a translation invariant cyclic vacuum state $\psi \in \mathcal{H}'$.

1.1. $\tilde{\phi}(f) = \lim_{V \to \infty} \tilde{\phi}_V(f)$ exists as a weak limit in the locally convex topology defined by the $\langle *, * \rangle$- seminorms and is an element in $\mathcal{B}(\mathcal{H})$.

1.2. $\tilde{\phi}(f)$ is a c-number.

Lemma 2 For all irreducible representations $(\mathcal{H}, \pi)$,

$$\tilde{\phi}(f) = \hat{\phi}(f). \tag{3.11}$$

Let us start by proving the lemmata.

Proof of Lemma 1 $(\mathcal{H}, \pi)$ is obtained from the cyclic state ψ, i.e. the set $\pi(\mathcal{A}_c)\psi$ is dense in $\mathcal{H}$. From this dense subset. we choose arbitrary vectors $\psi_A = \pi(A)\psi$,

$\psi_B = \pi(B)\psi$, where $A, B \in \mathcal{A}_c$ are observables with bounded localization in space-time. Then we consider the sesquilinear form

$$\begin{aligned}\langle \psi_A, \tilde{\phi}_V \psi_B \rangle &= \frac{1}{V} \int_V d^3\bar{x} \langle \psi_A, \pi(\alpha_{\bar{x}}(\phi(f))) \psi_B \rangle \\ &= \frac{1}{V} \int_V d^3\bar{x} \langle \psi, \pi(A^*)[\pi(\alpha_{\bar{x}}(\phi(f))), \pi(B)]\psi \rangle \\ &+ \frac{1}{V} \int_V d^3\bar{x} \langle \psi, \pi(A^*)\pi(B)\pi(\alpha_{\bar{x}}(\phi(f)))\psi \rangle. \end{aligned} \tag{3.12}$$

In the first term of (3.12), the commutator is only non-zero, if $\alpha_{\bar{x}}(\phi(f))$ lies in the causal shadow of B.* This is only the case for a finite volume V_f and hence we write

$$| \frac{1}{V} \int_V d^3\bar{x} \langle \psi, \pi(A^*)[\pi(\alpha_{\bar{x}}(\phi(f))), \pi(B)]\psi \rangle | \leq \frac{1}{V} c_1 V_f, \tag{3.13}$$

where c_1 is a finite real constant. To reformulate the second term in (3.12), we use the cluster decomposition property (2.4) for $N = 0$ in the following form: for $A_1, A_2 \in \mathcal{A}$ and ψ the cyclic vacuum,

$$| \langle \psi, \pi(A_1)\pi(\alpha_{\bar{x}}(A_2))\psi \rangle - \langle \psi, \pi(A_1)\psi \rangle \langle \psi, \pi(A_2)\psi \rangle | < C(\bar{x}), \tag{3.14}$$

where

$$| C(\bar{x}) | < c_2 \frac{1}{| \bar{x} |^2} \tag{3.15}$$

for $| \bar{x} | > 1$ and c_2 a finite constant. This allows us to write

$$\begin{aligned}&\frac{1}{V} \int_V d^3\bar{x} \langle \psi, \pi(A^*)\pi(B)\pi(\alpha_{\bar{x}}(\phi(f)))\psi \rangle \\ &= \frac{1}{V} \int_V d^3\bar{x} \langle \psi, \pi(A^*)\pi(B)\psi \rangle \langle \psi, \pi(\alpha_{\bar{x}}(\phi(f)))\psi \rangle + C_2(V), \end{aligned} \tag{3.16}$$

where $C_2(V)$ is a correction term. Using a set of increasing concentric spheres with radii R for the volumes V, equations (3.14) and (3.15) lead to an upper bound on C_2:

$$| C_2(V) | \leq c_3 \frac{1}{R^3} + c_4 \frac{1}{R^3} \int_1^R 4\pi r^2 \frac{1}{r^2} dr, \tag{3.17}$$

* The causal shadow of an element B of the local field algebra $\mathcal{A}$ is the unification of the forward and backward cones of those points in space-time in which B is localized.

where c_3 and c_4 are finite constants. Here, the first term comes from integrating in (3.16) over $|\bar{x}| < 1$, the second term from $|\bar{x}| > 1$. Furthermore, the vacuum is homogeneous (translationally invariant), which allows us to write

$$\frac{1}{V}\int_V d^3\bar{x}\langle\psi,\pi(A^*)\pi(B)\psi\rangle\langle\psi,\pi(\alpha_{\bar{x}}(\phi(f)))\psi\rangle$$
$$= \frac{1}{V}\int_V d^3\bar{x}\langle\psi,\pi(A^*)\pi(B)\psi\rangle\langle\psi,\pi(\phi(f))\psi\rangle$$
$$= \langle\psi,\pi(A^*)\pi(B)\psi\rangle\langle\psi,\pi(\phi(f))\psi\rangle. \tag{3.18}$$

Combining the equations (3.12)-(3.17), we can write

$$\langle\psi_A,\tilde{\phi}_V\psi_B\rangle = \langle\psi,\pi(A^*)\pi(B)\psi\rangle\langle\psi,\pi(\phi(f))\psi\rangle + C_1(V) + C_2(V), \tag{3.19}$$

where $C_1(V)$ denotes the left hand side of (3.13). From (3.14) and (3.17) it follows that

$$\lim_{V\to\infty} C_1(V) = 0; \tag{3.20}$$

$$\lim_{V\to\infty} C_2(V) = 0. \tag{3.21}$$

Since the first term in (3.19) has no volume dependence, it follows that

$$\lim_{V\to\infty}\langle\psi_A,\tilde{\phi}_V(f)\psi_B\rangle = \langle\psi_A,\tilde{\phi}(f)\psi_B\rangle$$
$$= \langle\psi,\pi(A^*)\pi(B)\psi\rangle\langle\psi,\pi(\phi(f))\psi\rangle$$
$$= \langle\psi_A,\psi_B\rangle\langle\psi,\pi(\phi(f))\psi\rangle, \tag{3.22}$$

for all ψ_A,ψ_B of a dense subset of $\mathcal{H}$. Hence, $\tilde{\phi}(f)$ exists as a weak limit of $\tilde{\phi}_V(f)$ and

$$\tilde{\phi}(f) = \langle\psi,\phi(f)\psi\rangle\cdot I, \tag{3.23}$$

Hence $\tilde{\phi}(f) \in \mathcal{B}(\mathcal{H})$ and $\tilde{\phi}(f)$ is a c-number in all irreducible representations. □

Proof of Lemma 2 With (3.18) and (3.22), it follows from the time translation invariance of the vacuum that

$$\langle\psi_A,\alpha_t(\tilde{\phi}(f))\psi_B\rangle = \langle\psi,\pi(A^*)\pi(B)\psi\rangle\langle\psi,\tilde{\phi}(f)\psi\rangle, \tag{3.24}$$

where α_t denotes the time translation operator. Hence, defining for finite T

$$\hat{\phi}_T(f) = \frac{1}{2T}\int_{-T}^{T} dt\alpha_t(\tilde{\phi}(f)), \tag{3.25}$$

we obtain

$$\langle\psi_A,\hat{\phi}_T(f)\psi_B\rangle = \langle\psi_A,\tilde{\phi}(f)\psi_B\rangle. \tag{3.26}$$

Hence, the left hand side is independent of T and it follows

$$\lim_{T\to\infty}\langle\psi_A,\hat{\phi}_T(f)\psi_B\rangle = \langle\psi_A,\tilde{\phi}(f)\psi_B\rangle. \tag{3.27}$$

□

Remark that the methods, used to prove the first lemma, do not apply to $\hat{\phi}(f)$. The causal shadow of a local observable B intersects with the support of $\hat{\phi}(f)$ in an infinite region of space-time. But it is for (3.13) that $\tilde{\phi}(f)$ intersects with the causal shadow of B in a finite region. This is the case since $\tilde{\phi}(f)$ is supported in a finite time slice only. Furthermore, it is interesting to note that there is a second way to prove Lemma 2, based on a theorem of Araki.[29] This theorem states that in all representations in which the space-time translation group is strongly continuously and unitarily represented, every element of the center commutes with all space-time translation operators. Lemma 2 can be seen as a direct consequence of this theorem. More generally, every quasi-local quantity, smeared out over space as in (3.10) and commuting with all local observables, is to the same quantity in all representations equal, smeared out in the time direction, too.

Lemma 1 and 2 provide a proof of Proposition 3 for irreducible representations. To complete the proof of Proposition 3, we extend these lemmata to integral representations over arbitrary compact spaces Λ.

Proof of Proposition 3 We start from (3.19) to reformulate the right hand side of (3.8)

$$\langle\psi_A,\tilde{\phi}_V(f)\psi_B\rangle$$
$$= \int_\Lambda d\mu(\lambda)\langle\psi(\lambda),\pi_\lambda(A^*)\pi_\lambda(B)\psi(\lambda)\rangle\langle\psi(\lambda),\pi_\lambda(\phi(f))\psi(\lambda)\rangle$$

$$+\int_\Lambda d\mu(\lambda)\langle\psi(\lambda),\psi(\lambda)\rangle(C_1(V)+C_2(V)) \tag{3.28}$$

Using (3.20) and (3.21), we conclude that the infinite volume limit exists:

$$\langle\psi_A,\tilde{\phi}(f)\psi_B\rangle = \lim_{V\to\infty}\langle\psi_A,\tilde{\phi}_V(f)\psi_B\rangle$$

$$= \int_\Lambda d\mu(\lambda)\langle\psi(\lambda),\pi_\lambda(A^*)\pi_\lambda(B)\psi(\lambda)\rangle\langle\psi(\lambda),\pi_\lambda(\phi(f))\psi(\lambda)\rangle. \tag{3.29}$$

Since all cyclic (vacuum) vectors $\psi(\lambda)$ are homogeneous by construction, the generalization of Lemma 2 to this case is straightforward and we obtain:

$$\langle\psi_A,\hat{\phi}(f)\psi_B\rangle = \langle\psi_A,\tilde{\phi}(f)\psi_B\rangle. \tag{3.30}$$

This establishes proposition 3.1. To establish 3.2, it is sufficient to decompose the integral representation of $\hat{\phi}(f)$,

$$\hat{\phi}(f) = \int_\Lambda d\mu(\lambda)\langle\psi(\lambda),\pi_\lambda(\phi(f))\psi(\lambda)\rangle I_\lambda, \tag{3.31}$$

where I_λ is the identity on the subspace $\mathcal{H}_\lambda$. (3.31) lies in the commutant of all decomposable operators. Hence it lies in the center of the bicommutant of $\pi(\mathcal{A}_c)$ by the von Neumann bicommutant theorem. □

III.2. Integral representations of the field algebra

We have seen in the last subsection that $\hat{\phi}(f)$, defined in (3.9), lies in the center of the von Neumann algebra $\pi(\mathcal{A}_c)''$ for all integrals of vacuum representations $(\mathcal{H},\pi)$ of $\mathcal{A}_c$ over a compact space Λ. From (3.23) and (3.30) one can see that $\hat{\phi}(f)$ is closely related to the vacuum expectation value of the Bose field. However, before discussing the consequences of this observation, we might use the notation introduced to discuss the size of the vacuum expectation value in a setting which is not burdened with the technical difficulties of the lattice approximation. Hence, we attempt to confirm the main results of section I about the size of the vacuum expectation value by specifying special integral representations in this subsection.

We consider integral representations of the form (3.2) - (3.4) in which the compact space Λ is the global gauge group G.

$$\Lambda = G, \tag{3.32}$$

i.e. we have a manifold isomorphic to G of "degenerate vacua" $\psi(g)$ for all $g \in G$.

In analogy to the definitions of section I, we define now two different Hilbert spaces

$$\mathcal{H} = \int_G d\mu(g)\mathcal{H}_g \tag{3.33}$$

by specifying two different measures:

Definition 3 $\langle\psi_A,\psi_B\rangle_1$ is called a *scalar product of the first kind*, if μ is a Haar measure supported by all $g \in G$, $(\mu(g) = dg)$, i.e.

$$\langle\psi_A,\psi_B\rangle_1 = \int_G dg\langle\psi(g),\pi_g(A^*)\pi_g(B)\psi(g)\rangle. \tag{3.34}$$

Definition 4 $\langle\psi_A,\psi_B\rangle_2$ is called a *scalar product of the second kind*, if μ is supported at one point $\bar{g} \in G$ only, i.e.

$$\langle\psi_A,\psi_B\rangle_2^{\bar{g}}$$

$$= \int_G dg\delta_{\bar{g}}(g)\langle\psi(g),\pi_g(A^*)\pi_g(B)\psi(g)\rangle$$

$$= \langle\psi(\bar{g}),\pi_{\bar{g}}(A^*)\pi_{\bar{g}}(B)\psi(\bar{g})\rangle. \tag{3.35}$$

While (3.34) is an inner product, defined on $\mathcal{H}$, (3.35) is an inner product on the irreducible Hilbert space $\mathcal{H}_{\bar{g}}$. Our motivation for defining (3.35) as a scalar product of the second kind is to choose our concepts as closely as possible to the path-integral formalism. In what follows, this path-integral formalism is considered to be a special method of calculating the scalar products $\langle\psi_A,\psi_B\rangle$. In section I, we have seen that the measure of the path-integral can be factorized into separate integrations over the local and global parts of the gauge group. Here, we decompose the Hilbert space $\mathcal{H}$ into irreducible representations, labeled by the elements of the global gauge group G. Hence, (3.34) corresponds to the definition of the Faddeev-Popov gauge fixing of the first kind (1.21), while (3.35) corresponds to (1.22).

Now it is interesting to obtain a relation between the different irreducible Hilbert spaces $\mathcal{H}_g$. Therefore we try to characterize the properties of the set of maps $\{U(g)\}$, $g \in G$,

$$U(g') : \mathcal{H}_{\overline{g}} \longrightarrow \mathcal{H}_{g'\overline{g}}, \tag{3.36a}$$

$$U(g') : \psi(\overline{g}) \longrightarrow \psi(g'\overline{g}) = U(g')\psi(\overline{g}), \tag{3.36b}$$

which establish a one-to-one correspondence between the different irreducible subspaces. One defining property of these maps is the fact that the π_g are unitarily equivalent if restricted to gauge invariant fields, i.e.

$$U(g') : \pi_{\overline{g}}(D) \longrightarrow \pi_{g'\overline{g}}(D) = U(g')\pi_{\overline{g}}(A)U(g'^{-1}) \tag{3.37}$$

for all gauge-invariant fields $A \in \mathcal{A}_c$. From (3.36b) and (3.37) it follows that

$$\langle \psi, \pi(A)\psi \rangle_2^{\overline{g}} = \langle \psi, \pi(A)\psi \rangle_2^{g'\overline{g}} \tag{3.38}$$

for all gauge-invariant fields A. On the other hand, (3.38) and (3.36b) imply (3.37). Hence, (3.38) does not hold for gauge-dependent fields.

To further characterize the relation between the vacuum expectation values on different Hilbert spaces, we consider the set of maps $\{\overline{U}(g)\}$, $g \in G$

$$\overline{U}(g') : \mathcal{V} \longrightarrow \mathcal{V} \tag{3.39a}$$

$$\overline{U}(g')f_D(g) = f_D(g'g). \tag{3.39b}$$

Here, $\mathcal{V}$ denotes a subset of the set of functions on G and inherits the vector space structure from it. $\mathcal{V}$ consists of functions of the type

$$f_D(g) = \langle \psi, \pi(D)\psi \rangle_2^{g}, \tag{3.40}$$

where $D \in \mathcal{A}_c$. We conclude that the map $\overline{U}(g')$ in (3.39b) has to be defined by

$$\overline{U}(g')(\langle \psi, \pi(D)\psi \rangle_2^{\overline{g}}) = \langle \psi, \pi(D)\psi \rangle_2^{g'\overline{g}} = r(D, g')\langle \psi, \pi(D)\psi \rangle_2^{\overline{g}}, \tag{3.41}$$

where $r(D, G)$ denotes a representation of G on the vector space $\mathcal{V}$. From the above remarks it follows that

i) $r(D, g) = 1_G$ for all $g \in G$, if $D \in \mathcal{A}_c$ is gauge-invariant under G. 1_G denotes the identity on $\mathcal{V}$.

ii) $r(D, G)$ is a faithful representation of a non-trivial subgroup of G if $D \in \mathcal{A}_c$ is gauge dependent.

The physical reason for these criteria is obvious: we require that the vacuum expectation values of gauge-independent observables of $\mathcal{A}_c$ do not depend on the chosen degenerate vacuum $\psi(g)$ while the expectation values of gauge-dependent fields do depend on this choice. From this the following corollary is easily obtained:

Corollary Assume that $\phi(f) \in \mathcal{A}_c$ is a local field, gauge dependent under the action of G.

C.1. If $\mathcal{A}_c$ is represented on $\mathcal{H}$, equipped with the scalar product (3.34), then

$$\hat{\phi}(f) = 0. \tag{3.42}$$

C.2. If $\mathcal{A}_c$ is represented on $\mathcal{H}_{\overline{g}}$, equipped with the scalar product (3.35), then $\hat{\phi}(f)$ is a c-number times 1 and can be non-zero.

Proof of the Corollary C.1. From (3.34) and (3.41) follows that

$$\begin{aligned} \langle \psi, \hat{\phi}(f)\psi \rangle_1 &= \int_G \langle \psi(g), \pi_g(\phi(f))\psi(g) \rangle_2^{g} dg \\ &= \int_G r(\phi, g)\langle \psi(g_0), \pi_{g_0}(\phi(f))\psi(g_0) \rangle_2^{g_0} dg. \end{aligned} \tag{3.43}$$

Since $\phi(f) \in \mathcal{A}_c$ is gauge-dependent, $r(D, G)$ represents a non-trivial subgroup of G faithfully and hence $\int_G r(\phi(f), g)dg = 0$. □

C.2. states nothing but the Lemma 1.2. for the special irreducible representation of $\mathcal{A}_c$ on $\mathcal{H}_{\overline{g}}$. Especially, it is consistent with the structure of the field algebra $\mathcal{A}_c$ to choose a representation such that $\langle \psi(g), \pi_g(\phi(f))\psi(g) \rangle \neq 0$. □

In this subsection, we have given a reformulation of the discussion carried out in section I. Clearly, there has been a conservation of technical difficulties. Our first attempt is based on the assumption that neither the notorious problems of the continuum limit, nor the effects of boundary conditions on the infinite volume limit can upset our argument. This second attempt relies on the assumptions that bounded localized objects

$\phi(f)$ can be made out of the fields $\varphi(f)$. Despite these technical caveats, it is interesting that in two conceptually different approaches we obtain essentially the same information: the vacuum expectation value is zero, unless the global symmetry is removed from the path integral (section I), or vacuum manifold (section III) respectively. This improves our argument about the additional problems in the Landau gauge: in this subsection, the Landau gauge is formulated with a set of degenerated vacua, isomorphic to G, i.e. it is an integral representation on a Hilbert space (3.31) with $d\mu(g) = dg$. This observation leads us to the same conclusion as at the end of section I: *If the result (2.20) of Kennedy and King, obtained for a lattizised abelian Higgs model, persists in the continuum limit, then the cluster decomposition property does not hold in the Landau gauge.*

CONCLUSION

In this work, we have introduced a simple classification of Faddeev-Popov gauge fixings into gauge fixings of the first and second kind. The Faddeev-Popov gauge fixings used in the literature are of the first kind and do not remove global gauge symmetries from the path integral expression. Faddeev-Popov gauge fixings of the second kind, however, break local and global gauge symmetry. In two different approaches, based on different assumptions, we have shown that the vacuum expectation value of the Higgs field (as of all gauge-dependent quantities) is zero in all path integral formulations with Faddeev-Popov gauge fixings of the first kind while path integral formulations with Faddeev-Popov gauge fixings of the second kind allow for non-zero vacuum expectation values. Furthermore, we have pointed out that these different path integral formulations correspond to different inequivalent representations of the field algebra. Faddeev-Popov gauge fixings of the second kind correspond to irreducible representations while Faddeev-Popov gauge fixings of the first kind correspond to direct integral ones.

One non-trivial observation stemming from this classification of Faddeev-Popov gauge fixings is the exceptional behaviour of the Landau gauge. Our arguments show that the generally used assumptions in lattice field theory and in continuum path integral formulations are consistent with the results in literature if and only if the cluster decomposition theorem fails to apply in the Landau gauge. In section II, we have traced back the root of this exceptional behaviour: in the Landau gauge, the matrix elements of the translation operators are too singular to define complex measures. This result which has been established rigorously for free quantum electrodynamics, upsets the proofs of the cluster decomposition theorem.

Furthermore, we have obtained an algebraic characterization of the vacuum expectation value: the vacuum expectation value can be understood as the expectation value of an element which lies in the center of the weakly closed field algebra in a suitable representation. Let us conclude this work by pointing out to the possible relevance of this observation. On an algebraic level, the time evolution of a physical system is

described by a one-parameter automorphism α_t, acting on the field algebra. In general however, α_t is not defined on the norm-closed algebra $\mathcal{A}$, but on its weak extension $\pi(\mathcal{A})''$ in a suitable class of representations π, i.e. the time evolution of the system does involve both local quantities, and quantities which are obtained as weak limits of such local quantities.[31–33] In the non-relativistic case, this property of the algebraic time evolution and its connection with spontaneous symmetry breaking has been clarified.[31] If the interactions are sufficiently long range then elements of the center of $\pi(\mathcal{A})''$ can get involved in the dynamical description. In simple lagrangian model theories, it has been shown that these elements of the center of the field algebra appear in the lagranian and/or in the equations of motion.[15] If these elements of the center are not left invariant under gauge transformations, then the gauge-invariance of the lagrangian and/or the equations of motion is broken by specifying an irreducible representation. This process shows all the characteristics of spontaneous symmetry breaking but it does neither involve an additional scalar field (as the Higgs mechanism), nor does it lead to Goldstone modes.[15,33] Now, we have seen in Proposition 3 that the vacuum expectation value of the Higgs field can be interpreted as an element of the center of the field algebra, restricted to a c-number in irreducible representation while being zero in direct integral ones. This vacuum expectation value appears in the lagrangian of the Glashow-Weinberg standard model to give masses to the fermions and gauge bosons. On the other hand, the phenomenological relevance of the dynamical field ϕ is unclear: the Higgs particle has not been observed. Hence, Proposition 3 motivates the search for other objects in the center of the field algebra $\pi(\mathcal{A})''$, which have the same properties as $\hat{\phi}$ but do not involve the dynamical field ϕ. In a companion paper, we shall report on such attempts.

ACKNOWLEDGEMENTS

The author wishes to thank Dr. N.P.Landsman for many discussions and invaluable support. This work was partly supported by a European Communities Research Training Fellowship under contract no. ERBCHBICT920018.

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