DAMTP 92-60

Bianchi IX Kahler surfaces

Andrew S. Dancer

Peterhouse, Cambridge, CB2 1RD

and

DAMTP, Silver Street, Cambridge, CB3 9EW

1. Introduction

The aim of this note is to study Kähler surfaces satisfying certain natural geometric conditions, and admitting an isometric action of $SU(2)$. We shall restrict ourselves to metrics of the form

$$g = (abc)^2 dt^2 + a^2\sigma_1^2 + b^2\sigma_2^2 + c^2\sigma_3^2 \tag{1}$$

where $\sigma_1, \sigma_2, \sigma_3$ are invariant one-forms, t is a coordinate orthogonal to the orbits of $SU(2)$, and a, b, c are functions of t only. It will be helpful to introduce the standard variables w_1, w_2, w_3 defined by

$$w_1 = bc, w_2 = ac, w_3 = ab \tag{2}$$

The three special kinds of geometry we shall consider on our Kähler surfaces are

(a) Zero scalar curvature

(b) The Einstein condition

(c) Constant scalar curvature (the Yamabe condition).

The first and second conditions are, of course, special cases of the third.

Pedersen and Poon [PP1] showed that if the equations

FERMILAB
JAN 27 1993
LIBRARY

DAMTP-92-60

Fermilab Library
0 1160 0021735 0

$$\begin{aligned} w_1' &= w_2 w_3 + \alpha w_1 \\ w_2' &= w_1 w_3 + \alpha w_2 \\ w_3' &= w_1 w_2 \end{aligned}$$

are satisfied, where α is a constant, then the metric (1) is Kählerian and of zero scalar curvature. If $w_1 = w_2$ the resulting metrics are those of LeBrun [L].

One purpose of this paper is to show that the Pedersen-Poon solutions are essentially the *only* scalar-flat Kähler but non-hyperkähler metrics which can be expressed in the form (1), that is, in diagonal Bianchi IX form. We remark that $SU(2)$-invariant hyperkähler metrics in complex dimension two have already been completely classified [AH],[GP].

We also derive a set of equations in w_1, w_2, w_3, expressing the condition that the metric be Kähler-Einstein with nonzero Einstein constant. In the case when two of the w_i are equal we can explicitly write down the metric, which is one found previously by Gibbons, Pope and Pedersen [GP],[Pd].

Finally, we produce a system of equations in the w_i which are essentially equivalent to the metric being Kähler-Yamabe, that is, Kähler and of constant scalar curvature. If two of the w_i are equal we obtain the metrics of [PP2].

2. Kahler forms

Suppose that we have a diagonal Bianchi IX metric expressed in the form (1)

$$g = (abc)^2 dt^2 + a^2\sigma_1^2 + b^2\sigma_2^2 + c^2\sigma_3^2$$

where a, b and c are functions of t, and σ_i are the invariant one-forms satisfying $d\sigma_1 = \sigma_2 \wedge \sigma_3$ etc. Explicitly

$$
\begin{aligned}
\sigma_1 &= -\cos\psi\, d\theta - \sin\psi \sin\theta\, d\phi \\
\sigma_2 &= \sin\psi\, d\theta - \cos\psi \sin\theta\, d\phi \\
\sigma_3 &= -d\psi - \cos\theta\, d\phi
\end{aligned}
$$

where θ, ϕ, ψ are Euler angles on the $SU(2)$ orbits.

The vector fields dual to $dt, \sigma_1, \sigma_2, \sigma_3$ are $\frac{\partial}{\partial t}, X_1, X_2, X_3$ where

$$
\begin{aligned}
X_1 &= -\cos\psi \frac{\partial}{\partial\theta} - \frac{\sin\psi}{\sin\theta}\frac{\partial}{\partial\phi} + \cot\theta \sin\psi \frac{\partial}{\partial\psi} \\
X_2 &= \sin\psi \frac{\partial}{\partial\theta} - \frac{\cos\psi}{\sin\theta}\frac{\partial}{\partial\phi} + \cot\theta \cos\psi \frac{\partial}{\partial\psi} \\
X_3 &= -\frac{\partial}{\partial\psi}.
\end{aligned}
$$

We have the relations

$$
\left[\frac{\partial}{\partial t}, X_i\right] = 0
$$

and

$$
[X_1, X_2] = -X_3 \quad \text{and cyclically.}
$$

Let us assume that the metric is Kählerian but *not* hyperkähler. Therefore the space of covariant constant two-forms will have dimension less than three. However this space is a real representation of the isometry group $SU(2)$ and hence will be acted on trivially by this group.

Now the Kähler form on a surface is always self-dual, so we deduce from the above remarks that our Kähler form Ω is given by

$$
\Omega = A(t)\Omega_1^+ + B(t)\Omega_2^+ + C(t)\Omega_3^+ \tag{3}
$$

where A, B, C are functions of t only, and

$$
\begin{aligned}
\Omega_1^+ &= e_0 \wedge e_1 + e_2 \wedge e_3 \\
\Omega_2^+ &= e_0 \wedge e_2 + e_3 \wedge e_1 \\
\Omega_3^+ &= e_0 \wedge e_3 + e_1 \wedge e_2.
\end{aligned}
$$

Here $e_0 = abc\, dt, e_1 = a\sigma_1, e_2 = b\sigma_2, e_3 = c\sigma_3$ defines an orthonormal coframe for g.

Now the condition $d\Omega = 0$ is equivalent to the equations

$$
(Abc)' = Aa^2bc \tag{4}
$$

$$
(Bac)' = Bab^2c \tag{5}
$$

$$
(Cab)' = Cabc^2. \tag{6}
$$

If we now introduce the standard variables $w_1 = bc, w_2 = ac, w_3 = ab$ and define functions α, β, γ by

$$
w_1' = w_2 w_3 + \alpha w_1 \tag{7}
$$

$$
w_2' = w_1 w_3 + \beta w_2 \tag{8}
$$

$$
w_3' = w_1 w_2 + \gamma w_3 \tag{9}
$$

then the equations (4-6) become

$$
A' = -\alpha A \tag{10}
$$

$$
B' = -\beta B \tag{11}
$$

$$
C' = -\gamma C. \tag{12}
$$

We see that for each metric g there is a 3-dimensional space of closed, selfdual, $SU(2)$-invariant two-forms, which are candidates for Kähler forms.

3. Integrability

Given such a form Ω we can use the metric to define an endomorphism I of the tangent bundle by

$$g(IX,Y) = \Omega(X,Y)$$

With Ω given by (3) the endomorphism is defined by

$$\begin{aligned}
I\frac{\partial}{\partial t} &= Aw_1X_1 + Bw_2X_2 + Cw_3X_3 && (13)\\
IX_1 &= -\frac{A}{w_1}\frac{\partial}{\partial t} + \frac{Cw_2}{w_1}X_2 - \frac{Bw_3}{w_1}X_3 && (14)\\
IX_2 &= -\frac{B}{w_2}\frac{\partial}{\partial t} - \frac{Cw_1}{w_2}X_1 + \frac{Aw_3}{w_2}X_3 && (15)\\
IX_3 &= -\frac{C}{w_3}\frac{\partial}{\partial t} + \frac{Bw_1}{w_3}X_1 - \frac{Aw_2}{w_3}X_2. && (16)
\end{aligned}$$

Moreover I is an almost complex structure if and only if $A^2 + B^2 + C^2 = 1$.

If the above constraints on A, B, C are satisfied then Ω is a Kähler form precisely when I is integrable. Let us now check when integrability holds.

Assume first that A is not identically ± 1.

Then we can take

$$\chi_1 = \frac{\partial}{\partial t} - iAw_1X_1 - iBw_2X_2 - iCw_3X_3$$

and

$$\chi_2 = \frac{iA}{w_1}\frac{\partial}{\partial t} + X_1 - \frac{iCw_2}{w_1}X_2 + \frac{iBw_3}{w_1}X_3$$

as independent (1,0) vector fields.

Their Lie bracket is given by

$$[\chi_1, \chi_2] = P\frac{\partial}{\partial t} + QX_1 + RX_2 + SX_3$$

where

$$\begin{aligned}
P &= i(\frac{A}{w_1})'\\
Q &= -\frac{A}{w_1}(Aw_1)' - (B^2 + C^2)\frac{w_2w_3}{w_1}\\
R &= -i(\frac{Cw_2}{w_1})' + (AB + iC)w_3 - \frac{A}{w_1}(Bw_2)'\\
S &= i(\frac{Bw_3}{w_1})' + (AC - iB)w_2 - \frac{A}{w_1}(Cw_3)'
\end{aligned}$$

Using the equations (10-12) we can simplify these expressions to

$$\begin{aligned}
P &= -\frac{iA}{w_1}(2\alpha + \frac{w_2w_3}{w_1})\\
Q &= -\frac{w_2w_3}{w_1}\\
R &= \frac{iCw_2}{w_1}(\gamma - \beta + \alpha + \frac{w_2w_3}{w_1})\\
S &= \frac{iBw_3}{w_1}(\gamma - \beta - \alpha - \frac{w_2w_3}{w_1}).
\end{aligned}$$

Now $[\chi_1, \chi_2]$ is a (1,0) vector field if and only if the following equations hold.

$$\begin{aligned}
-iP - \frac{A}{w_1}Q - \frac{B}{w_2}R - \frac{C}{w_3}S &= 0\\
Aw_1P - iQ - \frac{Cw_1}{w_2}R + \frac{Bw_1}{w_3}S &= 0\\
Bw_2P + \frac{Cw_2}{w_1}Q - iR - \frac{Aw_2}{w_3}S &= 0\\
Cw_3P - \frac{Bw_3}{w_1}Q + \frac{Aw_3}{w_2}R - iS &= 0.
\end{aligned}$$

On substituting in our expressions for P, Q, R, S these equations reduce to

$$\begin{aligned}
\alpha A &= -iBC(\gamma - \beta) && (17)\\
\alpha(1 + A^2) &= (B^2 - C^2)(\gamma - \beta) && (18)\\
(C - iAB)(\alpha - \beta + \gamma) &= 0 && (19)\\
(B + iAC)(\gamma - \beta - \alpha) &= 0 && (20)
\end{aligned}$$

From (19) we see that $\beta = \alpha + \gamma$ or $C = iAB$.

In the former case equation (20) implies that $\alpha = 0$ or $B = -iAC$. If $\alpha = 0$ then $\beta = \gamma$. If, on the other hand, $\alpha \neq 0$ and $B = -iAC$, then (17),(18) imply that $\alpha = C^2(\beta - \gamma)$ and hence $C^2 = 1$. It follows from (12) that $\gamma = 0$ and $\alpha = \beta$.

Similar manipulations in the case $C = iAB$ lead to the conclusion that either $\alpha = 0, \beta = \gamma$ or $\beta = 0, \alpha = \gamma$.

In the case $A \equiv 1$ we can take

$$\frac{\partial}{\partial t} - iw_1 X_1 \ , \ X_2 - i\frac{w_3}{w_2} X_3$$

as independent (1,0) vector fields. Imposing the integrability condition leads to the equation

$$w_1 - (\frac{w_3}{w_2})' = \frac{w_1 w_3^2}{w_2^2}$$

which implies that $\beta = \gamma$. We also have $\alpha = 0$ because $A \equiv 1$. The case $A \equiv -1$ is similar.

We have shown, therefore, that our metric is only Kählerian in the following three cases

(i) $\alpha = 0, \beta = \gamma$

(ii) $\beta = 0, \alpha = \gamma$

(iii) $\gamma = 0, \alpha = \beta$.

In fact, it is straightforward to show that in case (i) we must have either $B \equiv C \equiv 0$ or else $\alpha = \beta = \gamma = 0$ (in which case our metric is hyperkähler). Similar statements hold, with appropriate permutations, in cases (ii) and (iii).

4. The scalar-flat condition

Let us now require that the scalar curvature R_{scalar} is zero.

Using the expressions of Pedersen-Poon [PP1] for the connexion forms of the metric we can calculate the Riemann curvature tensor and hence the scalar curvature. We find that

$$R_{scalar} = \frac{-1}{4w_1 w_2 w_3}(2\alpha' + 2\beta' + 2\gamma' + \alpha^2 + \beta^2 + \gamma^2 - 2\alpha\beta - 2\beta\gamma - 2\alpha\gamma) \qquad (21)$$

Comparing with the expressions of (i),(ii),(iii), we find that the scalar-flat condition forces α, β, γ to be constant.

We summarise our results as follows.

Theorem 1

The only scalar-flat Kähler metrics given by (1) which are not hyperkähler are (up to permutations) those given in [PP1]. □

5. Kahler-Einstein metrics

We can also use the techniques of this paper to study Kähler-Einstein metrics of the form (1). We assume that the Einstein constant Λ is nonzero, so the metric is not hyperkähler.

The Einstein equations in terms of a, b, c are

$$\begin{aligned}
2(\frac{a'}{a})' + 2\Lambda(abc)^2 &= a^4 - (b^2 - c^2)^2 \\
2(\frac{b'}{b})' + 2\Lambda(abc)^2 &= b^4 - (c^2 - a^2)^2 \\
2(\frac{c'}{c})' + 2\Lambda(abc)^2 &= c^4 - (b^2 - a^2)^2 \\
4(\frac{a'b'}{ab} + \frac{b'c'}{bc} + \frac{c'a'}{ca}) + 4\Lambda(abc)^2 &= -a^4 - b^4 - c^4 + 2a^2b^2 + 2b^2c^2 + 2c^2a^2
\end{aligned}$$

Let us assume that our metric is Kähler. From the discussion in sections 1-3 we can without real loss of generality take $\alpha = \beta$ and $\gamma = 0$. Using (2) and (7-9) the Einstein equations reduce to the constancy of Λ and the relation

$$\alpha = -\Lambda w_3^2$$

so we obtain the following result.

Theorem 2

The Kähler-Einstein metrics of form (1) which are not hyperkähler are given, up to permutations, by solutions of the equations

$$\begin{aligned} w_1' &= w_1 w_2 - \Lambda w_3^2 w_1 \\ w_2' &= w_1 w_3 - \Lambda w_3^2 w_2 \\ w_3' &= w_1 w_2 \end{aligned}$$

□

If $w_1 = w_2$ then the solution to these equations is

$$w_1 = w_2 = \sqrt{(w_3')}$$

where

$$w_3' = w_3^2 - \frac{2}{3}\Lambda w_3^3 + \nu$$

and ν is a constant.

After making the substitution $w_3 = \rho^2/4$ we arrive at the Kähler-Einstein metric

$$g = (1 - \frac{\Lambda}{6}\rho^2 + \frac{16\nu}{\rho^4})^{-1} d\rho^2 + \frac{1}{4}\rho^2(\sigma_1^2 + \sigma_2^2) + \frac{1}{4}\rho^2(1 - \frac{\Lambda}{6}\rho^2 + \frac{16\nu}{\rho^4})\sigma_3^2$$

which was obtained in [GP] and [Pd].

6. Kahler-Yamabe metrics

Finally, we consider Kähler metrics satisfying the Yamabe condition, that is, constancy of the scalar curvature. Taking $\alpha = \beta$ and $\gamma = 0$ as before, we see from (21) that the scalar curvature is given by

$$R_{scalar} = -\frac{\alpha'}{w_1 w_2 w_3}$$

so the Yamabe condition is equivalent to

$$\alpha' = 2\kappa w_1 w_2 w_3$$

for some constant κ.

As $\gamma = 0$ this is equivalent to

$$\alpha' = 2\kappa w_3' w_3$$

and hence to

$$\alpha = \kappa w_3^2 + \lambda$$

where λ is a constant.

From Theorems 1 and 2 we see that $\kappa = 0$ is equivalent to scalar-flatness, while $\lambda = 0$ is equivalent to the Einstein condition.

Theorem 3

Kähler-Yamabe metrics of the form (1) which are not hyperkähler are given, up to permutations, by solutions of the equations

$$\begin{aligned} w_1' &= w_2 w_3 + w_1(\kappa w_3^2 + \lambda) \\ w_2' &= w_1 w_3 + w_2(\kappa w_3^2 + \lambda) \\ w_3' &= w_1 w_2 \end{aligned}$$

□

As in section 5, we can write down the metric explicitly in the case $w_1 = w_2$ (that is, when the isometry group is $U(2)$). It is

$$g = \Delta^{-1} dr^2 + \frac{1}{4} r^2(\sigma_1^2 + \sigma_2^2 + \Delta\sigma_3^2)$$

where

$$\Delta = 1 + \frac{\kappa}{6}r^2 + \frac{8\lambda}{r^2} + \frac{16\mu}{r^4}$$

and κ, λ, μ are constants.

This is the family of Kähler-Yamabe metrics discovered by Pedersen and Poon in [PP2].

Acknowledgements

I am grateful to Gary Gibbons, Henrik Pedersen and Paul Tod for discussions, and to Peterhouse, Cambridge, for a Stone Research Fellowship.

References

[AH] M.F. Atiyah and N.J. Hitchin. The geometry and dynamics of magnetic monopoles. Princeton University Press (Princeton 1988).

[GP] G.W. Gibbons and C.N. Pope. The positive action conjecture and asymptotically Euclidean metrics in quantum gravity Commun. Math. Phys. 66 (1979) 267-290.

[L] C. LeBrun. Counter-examples to the generalised positive action conjecture. Commun. Math. Phys. 118 (1988) 591-6.

[Pd] H. Pedersen. Eguchi-Hanson metrics with cosmological constant. Classical and Quantum Gravity 2 (1985) 579-587

[PP1] H. Pedersen and Y.S. Poon. Kähler surfaces with zero scalar curvature. Classical and Quantum Gravity 7 (1990) 1707-1719.

[PP2] H. Pedersen and Y.S. Poon. Hamiltonian construction of Kähler-Einstein metrics and Kähler metrics of constant scalar curvature. Commun. Math. Phys. 136 (1991) 309-26.