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Quantum group gauge theory and q-monopoles

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ABSTRACT

We describe the main formulae in a general quantum group gauge theory on quantum spaces developed recently by the authors, and compute its universal Dirac monopole on the quantum sphere for the case of Woronowicz's 4D calculus.

1. In [1] we have introduced a general formalism for quantum group gauge theory on quantum spaces. Both the base space (spacetime) and the fiber are allowed to be quantum spaces described by non-commutative algebras. Principal bundles, connections on them, associated vector bundles etc, are defined and much of the standard classical theory goes through. We also gave a construction for the canonical connection on a quantum homogeneous space in the case of the universal quantum differential calculus. We report on some of this work here. After outlining the general theory of [1] we compute our universal monopole construction for the case of Woronowicz's 4D differential calculus[6] on the total space. The base space is the quantum sphere of Podleś[5]. One of the unusual features of quantum geometry, as is well-known, is that there can be available rather more than one quantum differential structure. Even the Dirac monopole looks different in each of them. Such flexibility may be significant at the Planck scale, where it

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is intended to apply our formalism in a q-gravity context. The need for such a generalized geometry is not firmly established but at the very least, it can provide a new parameter q to regularise infinities encountered there. In addition, it may actually be that $q \neq 1$ as a model of feedback from quantum effects to the structure of the background geometry. There are many other applications that can be envisaged also, such as the formulation of quantum inverse scattering in terms of quantum Lax pairs as quantum flat connections.

2. Before describing the general theory, we summarise the case of quantum trivial bundles and the gauge theory related to them[1, Sec. 3]. We need the following data: a quantum group or a Hopf algebra A with comultiplication Δ counit ϵ and antipode S, an associative unital algebra B and an algebra V equipped with a corepresentation $\rho_R : V \to V \otimes A$. The quantum group A should be thought of as like $C^\infty(G)$, B like $C^\infty(X)$ and V like $C^\infty(\mathbf{C}^n)$, where G is a Lie group, X is a spacetime manifold and $\mathbf{C}^n$ is a vector space on which G is represented. Assume for a simplicity that A is a matrix quantum group generated by the matrix $\mathbf{t} = (t^i{}_j)$ and V is generated by a vector $\mathbf{v} = (v^i)$. Thus A has coproduct $\Delta(t^i{}_j) = t^i{}_k \otimes t^k{}_j$ and $\rho_R(v_i) = v_j \otimes t^j{}_i$. As matter fields we consider maps $\sigma : V \to B$. A gauge transformation is an identity-preserving map $\gamma : A \to B$. We assume that γ is convolution invertible, i.e. there is an inverse to γ with respect to a product $*$ defined as

$$\gamma * \gamma' = \cdot_B(\gamma \otimes \gamma')\Delta, \tag{1}$$

$\gamma, \gamma' : A \to B$. This product is a quantum group generalisation of the usual pointwise multiplication. If we adopt a shorthand $\gamma(t^i{}_j) = \gamma^i{}_j$ etc than (1) becomes

$$(\gamma * \gamma')(t^i{}_j) = \gamma^i{}_k \gamma'^k{}_j. \tag{2}$$

We define a gauge transformation of σ by

$$\sigma^\gamma = \sigma * \gamma = \cdot_B(\sigma \otimes \gamma)\rho_R \tag{3}$$

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or in matrix form $\sigma_i^\gamma = \sigma_j \gamma^j{}_i$, where $\sigma_i = \sigma(v_i)$. A gauge potential is a linear map $\beta : A \to \Omega^1(B)$, where $\Omega^1(B)$ is a space of one-forms on B. We write $\beta(t^i{}_j) = \beta^i{}_j$. If β transforms according to

$$\beta^{\gamma i}{}_j = (\gamma^{-1})^i{}_k \beta^k{}_l \gamma^l{}_j + (\gamma^{-1})^i{}_k d\gamma^k{}_j \tag{4}$$

then the operation $\nabla : \{V \to B\} \to \Omega^1(B)$,

$$\nabla\sigma_i = d\sigma_i - \sigma_j \beta^j{}_i \tag{5}$$

is covariant under the gauge transformation γ. This is a covariant exterior derivative. Similarly to the classical case we can define a field-strength $F : A \to \Omega^2(B)$ by

$$F = d\beta + \beta * \beta \tag{6}$$

or $F^i{}_j = d\beta^i{}_j + \beta^i{}_k \beta^k{}_j$, which is also covariant under the gauge transformation γ.

3. Working in this algebraic way (with function algebras) we can then proceed[1, Sec. 4] to generalize to the quantum group case the usual construction of non-trivial principal and vector bundles [4]. We define a nontrivial principal bundle P with a structure quantum group A and the base space B as follows: P is a right A-comodule algebra (this means that there is an algebra map $\Delta_R : P \to P \otimes A$ such that $(\Delta_R \otimes \mathrm{id})\Delta_R = (\mathrm{id} \otimes \Delta)\Delta_R$) and $B = P^A = \{u \in P : \Delta_R u = u \otimes 1\}$ (thus B is a subalgebra of P invariant under the coaction of A). We define the bundle to be trivial if there is a convolution invertible inclusion map $\Phi : A \hookrightarrow P$ such that $\Delta_R \circ \Phi = (\Phi \otimes \mathrm{id}) \circ \Delta$ and $\Phi(1_A) = 1_P$.

To construct a gauge field on a principal bundle we have to consider a differential structure on P given by the algebra $\Omega(P)$ of differential forms on P. The algebra $\Omega(P)$ has to be compatible with the structure of the bundle. ¿From the space $\Gamma_P = \Omega^1(P)$ of 1-forms on P we can extract the subspace $\Gamma_{hor} = P\Gamma_B P$ of horizontal 1-forms. A connection Π on P is an assignment of a left P-submodule $\Gamma_{ver} \subseteq \Gamma_P$ such that $\Gamma_P = \Gamma_{hor} \oplus \Gamma_{ver}$ and the projection $\Pi : \Gamma_P \to \Gamma_{ver}$ is right-invariant in the sense $\Delta_R \Pi = (\Pi \otimes \mathrm{id})\Delta_R$.

As is well known, for any quantum group A there is a right adjoint coaction $\mathrm{Ad} : A \to A \otimes A$ given as

$$\mathrm{Ad}(a) = \sum a_{(2)} \otimes (S a_{(1)}) a_{(3)} \tag{7}$$

where $(\mathrm{id} \otimes \Delta)\Delta(a) = \sum a_{(1)} \otimes a_{(2)} \otimes a_{(3)}$. Assume now that there exists a map $\omega : A \to \Gamma_P$ intertwining Δ_R with Ad such that

$$\Pi(udv) = \cdot_{\Gamma_P}(u \otimes \omega)\Delta_R v \tag{8}$$

for any $udv \in \Gamma_P$. We call the map ω a connection 1-form or gauge field. Finally, we define the exterior covariant derivative as the projection of d on the subspace of horizontal forms.

Having constructed a principal bundle P we can define a quantum vector bundle E associated to it. Let V be a vector space equipped with a corepresentation $\rho_R : V \to V \otimes A$. The tensor product space $P \otimes V$ becomes a comodule of A with the tensor product coaction $\Delta_E : P \otimes V \to P \otimes V \otimes A$, $\Delta_E = (\mathrm{id} \otimes \cdot_A) \circ (\mathrm{id} \otimes \tau_{23} \otimes \mathrm{id}) \circ (\Delta_R \otimes \rho_R)$, where τ_{ij} transposes the i-th and j-th factors in a tensor product. The vector bundle E associated to P is then defined as $E = (P \otimes V)^A$. There is a natural inclusion map $j_E : E \hookrightarrow P \otimes V$. Using that map we define a cross-section $s : E \to B$ by $s \circ j_E = \mathrm{id}$.

It is shown in [1] that a trivialization Φ of P induces a map $\Phi_E : V \hookrightarrow E$, $\Phi_E = (\Phi \circ S^{-1} \otimes \mathrm{id}) \circ \tau_{12} \circ \rho_R$. We say that E is trivial if E is isomorphic to $B \otimes V$ via the maps Φ_E and j_E. As shown in [1], in this case we obtain the theory described in Section 2, where s is identified with σ, ω with β and D with ∇.

4. An important example of the construction described above is the q-deformed or quantum Dirac monopole bundle. The bundle itself was described in [1] but we will compute its monopole for the 4D differential calculus. We put $P = SO_q(3)$ and $A = SO(2) = \mathbb{C}[Z, Z^{-1}]$. The group $SO_q(3)$ is defined as the even elements of $SU_q(2)$. The latter is generated by a matrix $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ with commutation relations

$$\alpha\beta = q\beta\alpha, \quad \alpha\gamma = q\gamma\alpha, \quad \alpha\delta = \delta\alpha + \lambda\beta\gamma$$

$$\beta\gamma = \gamma\beta, \quad \beta\delta = q\delta\beta, \quad \gamma\delta = q\delta\gamma \tag{9}$$

where $\lambda = q - q^{-1}$, and a determinant relation $\alpha\delta - q\beta\gamma = 1$. The base manifold B is a quantum sphere $S^2_q = SO_q(3)/SO(2)$ of Podleś [5] generated by $1, b_- = \alpha\beta$, $b_+ = \gamma\delta$, $b_3 = \alpha\delta$ with commutation relations

$$\begin{aligned} b_3 b_- &= (1-q^{-2})b_- + q^{-2}b_-b_3, \quad b_3 b_+ = b_+(1-q^2) + q^2 b_+ b_3 \\ b_3^2 &= b_3 + q^{-1}b_-b_+, \quad q^{-2}b_-b_+ = q^2 b_+ b_- - \lambda(b_3 - 1). \end{aligned} \tag{10}$$

The monopole bundle $SO(2) \leftarrow SO_q(3) \leftarrow S^2_q$ has two localizations $P_0 = SO_q(3)[\delta^{-1}\alpha, \alpha^{-1}\delta]$ over $B_0 = S^2_q[b_3^{-1}]$ and $P_1 = SO_q(3)[\gamma\beta^{-1}, \beta\gamma^{-1}]$ over $B_1 = S^2_q[(1-b_3)^{-1}]$ corresponding to a double covering of S^2_q. They are trivial principal bundles with trivializations

$$\begin{aligned} \Phi_0(Z) &= \delta^{-1}\alpha, \quad \Phi_0(Z^{-1}) = \alpha^{-1}\delta \\ \Phi_1(Z) &= -\gamma\beta^{-1}, \quad \Phi_1(Z^{-1}) = -\beta\gamma^{-1} \end{aligned} \tag{11}$$

respectively. These are pasted together on the overlapping bundle P_{01} over B_{01} by the gauge transformation

$$\gamma_{01}(Z) = \Phi_0(Z)\Phi_1(Z)^{-1} = -q^2 b_3^{-1} b_-^2 (b_3 - 1)^{-1}. \tag{12}$$

We have also an Ad-invariant inclusion $i : \mathbb{C}[Z, Z^{-1}] \to SO_q(2)$, $i(Z) = \delta^{-1}\alpha$, which works locally in P_0. Formally it can be used to construct a canonical connection ω in the quantum monopole bundle, namely

$$\omega = \cdot_{\Gamma_P}(S \otimes d) \circ \Delta \circ i. \tag{13}$$

Now we can compute a form of the canonical connection in a particular differential calculus, namely the 4D calculus of Woronowicz [6] (see also [2], [3], [7]). The 4D calculus is given by a bicovariant bimodule Γ, spanned by the left-invariant generators $\omega^1, \omega^2, \omega^+, \omega^-$ which have the following commutation relations with generators of $SU_q(2)$,

$$\omega^1\alpha = q^{-1}\alpha\omega^1, \qquad \omega^1\beta = q\beta\omega^1$$

$$\begin{aligned} &\omega^-\alpha = \alpha\omega^- - q\lambda\beta\omega^1, \qquad \omega^-\beta = \beta\omega^- \\ &\omega^+\alpha = \alpha\omega^+, \qquad \omega^+\beta = \beta\omega^+ - q\lambda\alpha\omega^1 \\ &\omega^2\alpha = q\alpha\omega^2 - \lambda\beta\omega^+ + q\lambda^2\alpha\omega^1, \qquad \omega^2\beta = q^{-1}\beta\omega^2 - \lambda\alpha\omega^-. \end{aligned}$$

The remaining relations can be obtained by the replacement $\alpha \to \gamma$, $\beta \to \delta$. Bicovariancy provides the existence of an exterior product of one-forms, we refer the reader to [7] for detailed formulae. The exterior derivative d can be written in terms of the forms ω^i as

$$\begin{aligned} d\alpha &= \frac{\lambda - q^{-2}}{q+1}\alpha\omega^1 - q^{-2}\beta\omega^+ + \frac{q^{-1}}{q+1}\alpha\omega^2 \\ d\beta &= \frac{q}{q+1}\beta\omega^1 - q^{-2}\alpha\omega^- - \frac{q^{-2}}{q+1}\beta\omega^2. \end{aligned} \tag{14}$$

Applying these formulae to the definition of the gauge potential (13) and using bicovariancy of (Γ, d) we obtain

$$\omega(Z) = q^{-1}\omega^2 - (q^{-1} - \lambda)\omega^1. \tag{15}$$

Notice that $\omega(Z)$ is globally defined. Using the definition of the curvature $F(Z)$ and the Maurer-Cartan formulae as listed in [7] we can easily write the monopole field strength,

$$F(Z) = -2q^{-2}\omega^- \wedge \omega^+. \tag{16}$$

To obtain the local trivialization of $\omega(Z)$ on B_0 say, we have to subtract the vertical part of $\omega(Z)$ and to project the remaining part down to the base space. A little algebra gives,

$$\beta_0(Z) = q^{-1}b_3^{-1}b_-\nu^+ - \lambda(1 - b_3^{-1})\nu^1 - b_3^{-1}b_+\nu^- \tag{17}$$

where $\nu^+ = \alpha^{-1}\delta\omega^+$, $\nu^1 = \omega^1$, and $\nu^- = \delta^{-1}\alpha\omega^-$ are elements of $\Omega^1(B_0)$ provided $0 <| q - 1 |$ is sufficientely small. The field strength $F(Z)$ projected down to B_0 takes the form

$$F_0(Z) = -2q^{-2}\nu^- \wedge \nu^+ + 2\lambda b_3^{-1}b_+\nu^- \wedge \nu^1. \tag{18}$$

Similarly we can write down formulae for the trivialization B_1. In that case

$$\beta_1(Z) = -(b_3 - 1)^{-1}b_+\sigma^+ - \lambda(q^2(b_3 - 1)^{-1} + 1)\sigma^1 + q^{-1}(b_3 - 1)^{-1}b_-\sigma^- \tag{19}$$

where $\sigma^+ = -\beta\gamma^{-1}\omega^+$, $\sigma^1 = \omega^1$, and $\sigma^- = -\gamma\beta^{-1}\omega^-$ are elements of $\Omega^1(B_1)$. The localized field strength reads

$$F_1(Z) = -2q^{-2}\sigma^- \wedge \sigma^+ - 2q^{-1}\lambda(b_3 - 1)^{-1}b_-\sigma^- \wedge \sigma^1. \tag{20}$$

The two are related by the gauge transformation γ_{01}.

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