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Integrable Generalisations of Quantum-Mechanical Models Integrable via Separation of Variables.

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Abstract

Given a simple quantum mechanical model whose integrability is achieved by separation of variables, it is shown how to create a wide range of generalisations of its potential terms without loss of integrability, and then how to construct explicitly a suitable set of commuting integrals, whose existence proves the integrability in question. This is illustrated for a simple model of a two oscillator system, with results of some interest, and applied to the Neumann model, giving a variety of new potential terms for it, and exhibiting a new set of integrals, alternative to the Uhlenbeck set available for the basic models. The new set are given explicitly and shown to be commutative in the generalised situation.

1. Introduction

Suppose we are given a quantum mechanical problem with n degrees of freedom for which we can establish integrability by separation of variables in its Schrodinger equation. Then the purpose of this paper can be stated as follows. It is to describe a systematic procedure for generalising the interaction or potential terms of the model in a manner that leaves its integrability properties unaltered. Indeed the process of separation of variables leads directly, not only for the original simple version of the model, but also for the generalisations we have in mind, to a set of n integrals, including the Hamiltonian of the model,

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such that any two of them commute. That, of course, is what we mean by integrability.

The paper begins (section two) by illustrating the above programme for a simple model whose Hamiltonian is

$$H_0 = \tfrac{1}{2}(p_1^2 + 4x_1^2 + p_2^2 + x_2^2), \tag{1.1}$$

for which it is known [1] that solution by separation of variables in parabolic coordinates is of considerable interest. We discuss this in a way that makes clear that the model is ripe for a wide range of generalisations, whose integrability may not be a priori at all obvious, although that of (1.1) itself is. At this point one such generalisation may be cited. It has the Hamiltonian

$$H = \tfrac{1}{2}(p_1^2 + 4x_1^2 + 16fx_1^4) + \tfrac{1}{2}(p_2^2 + x_2^2 + fx_2^4) + 6fx_1^2x_2^2. \tag{1.2}$$

Details for it are given below.

The simple example sets the scene for the development of a more substantial study. In section three, we apply our approach to the Neumann model [2], whose integrability at the classical level is discussed in [3,4]. For the simple Neumann model describing the motion of a particle on the sphere S^{N-1} in the presence of forces corresponding to the potential

$$U = \tfrac{1}{2}\sum_{k=1}^{N} \varepsilon_k x_k^2, \tag{1.3}$$

Uhlenbeck [3] proved integrability at the classical level by exhibiting a set of $N-1$ independent conserved quantities, which include the Hamiltonian, and such that any two have zero Poisson bracket. A recent paper [5] discussed the model in detail and demonstrated, amongst other things, integrability by separation of variables. We extended this recently [6] to allow for the addition to (1.3) of the potential of Rosochatius [7]. Here we reformulate the process of separation of variables for the quantum Neumann model, following the approach employed in section two. In this way we show that the potential term (1.3) allows for a wide range of generalisations, some well-known like the Braden potential [8,9], others new here, all of which leave the separability properties of the model intact. In addition, we show how the process of separation of variables explicitly leads (by a procedure of rather general applicability) to a set of $N-1$ integrals, which includes the Hamiltonian, and such

that any two of them commute. These integrals are an alternative set to the Uhlenbeck integrals.

Our work on the Neumann model is divided into seven subsections: introduction, notation, determinantal identities, separation of variables, generalisation of the model, construction of integrals, and the proof that any two of them commute.

2. A Two Oscillator System.

It is easy to write the Schrodinger equation corresponding to

$$H_0 = \tfrac{1}{2}(p_1^2 + 4x_1^2 + p_2^2 + x_2^2) \tag{2.1}$$

in terms of the parabolic coordinates y_1 and y_2 defined by

$$2x_1 = y_1^2 - y_2^2, \quad x_2 = y_1 y_2. \tag{2.2}$$

To do this we use

$$(\partial^2/\partial x_1^2) + (\partial^2/\partial x_2^2) = (y_1^2 + y_2^2)^{-1}[(\partial^2/\partial y_1^2) + (\partial^2/\partial y_2^2)], \tag{2.3}$$

and

$$V_0 = \tfrac{1}{2}(4x_1^2 + x_2^2) = \tfrac{1}{2}\tfrac{(y_1^6+y_2^6)}{(y_1^2+y_2^2)}. \tag{2.4}$$

However we can immediately generalise, in the spirit of chapter 5 of [10], to

$$H = \tfrac{1}{2}(p_1^2 + p_2^2) + V, \tag{2.5}$$

where V is any potential which can be written in the form

$$V = \frac{u(y_1^2) + w(y_2^2)}{y_1^2 + y_2^2}, \tag{2.6}$$

involving arbitrary functions u an w. Clearly (2.4) is a simple special case of (2.6). For H of (2.5) we have

$$H = (y_1^2 + y_2^2)^{-1}(L_1 + L_2), \tag{2.7}$$

where

$$L_1 = -\tfrac{1}{2}(\partial/\partial y_1)^2 + u(y_1), \tag{2.8}$$

$$L_2 = -\tfrac{1}{2}(\partial/\partial y_2)^2 + w(y_2), \tag{2.9}$$

and the Schrodinger equation for H is

$$(L_1 + L_2)\Phi = E(y_1^2 + y_2^2)\Phi. \tag{2.10}$$

Writing $\Phi = Y(y_1)Y(y_2)$, we solve (2.10) by separation of variables, finding, for g a separation constant, that Y_1 and Y_2 obey

$$L_1 Y_1 - E y_1^2 Y_1 = +g Y_1, \tag{2.11}$$

$$L_2 Y_2 - E y_2^2 Y_2 = -g Y_2. \tag{2.12}$$

In the special case V_0 of V there is a great deal of interesting material in [1] on the solution of (2.11) and (2.12) for Y_1 and Y_2. To demonstrate the integrability of the system described by (2.7), which is well known in the case of V_0, and then trivially seen in (2.1), we must exhibit a quantity G which commutes with H. In fact, the process of separation of variables automatically provides one. While L_1 and L_2 are a pair of commuting operators, they do not commute with H as given by (2.7). Rather, we find G by eliminating E from (2.11) and (2.12), and writing the result in the form

$$G\Phi = g\Phi. \tag{2.13}$$

This yields

$$G = \frac{y_2^2}{(y_1^2 + y_2^2)} L_1 - \frac{y_1^2}{(y_1^2 + y_2^2)} L_2. \tag{2.14}$$

One checks directly that $[H, G] = 0$, using the obvious result $[L_1, L_2] = 0$, but without the need to evaluate explicitly the commutator of L_1 or L_2 with $(y_1^2 + y_2^2)^{-1}$. Thus $[H, G] = 0$ holds independently of the forms of the functions u and w which determine V via (2.6). In the special case

$$u = \tfrac{1}{2}y_1^6, \quad w = \tfrac{1}{2}y_2^6 \tag{2.15}$$

that corresponds to V_0, eq. (2.14) agrees via change of variables with the result quoted in [1]

$$G_0 = \{x_2p_1 - x_1p_2, p_1\} + x_1x_2^2 \tag{2.16}$$

for the quantity G_0 which commutes with H_0 .We remark that this result was there [1] given at the outset in Cartesians x_1 and x_2, etc., rather than emerging from the separation process. The work of the next section indicates that such emergence of the required commuting quantities is a general feature of the separation of variables.

We next ask what choices of u and w in (2.6) give rise to interesting generalisations (2.5) of (2.1), possibly involving interactions between the oscillators, and what then is the operator G which commutes with H, and thereby confirms that (2.5) remains, as the above analysis implies, integrable. We have

$$G = \{x_2p_1 - x_1p_2, p_1\} + \Gamma, \tag{2.17}$$

where Γ is obtained from (2.14) in the form

$$(y_1^2 + y_2^2)\Gamma = y_2^2u(y_1^2) - y_1^2w(y_2^2). \tag{2.18}$$

One way to get results of simple form sets

$$u(y_1^2) = \lambda(y_1^2), \quad w(y_2^2) = -\lambda(-y_2^2). \tag{2.19}$$

Then for

$$\lambda = a + by^2 + cy^4 + dy^6 + ey^8, \tag{2.20}$$

we get

$$V = b + 2cx_1 + d(4x_1^2 + x_2^2) + 2ex_1(4x_1^2 + 2x_2^2), \tag{2.21}$$

$$\Gamma = a + cx_2^2 + 2dx_1x_2^2 + ex_2^2(4x_1^2 + x_2^2). \tag{2.22}$$

Also from $\lambda = y^2$ we find

$$V = -x_2^{-2}, \quad \Gamma = 2x_1x_2^{-2},$$

and from $\lambda = -(a+y)^{-1}$ we find

$$V = N^{-1}, \quad \Gamma = (a + 2x_1)N^{-1}, \quad N = a^2 + 2ax_1 + x_2^2.$$

The choice

$$\lambda = \tfrac{1}{2}y^6 + fy^{10}, \tag{2.23}$$

gives rise to a rather interesting model. This has

$$H = \tfrac{1}{2}(p_1^2 + 4x_1^2 + 16fx_1^4) + \tfrac{1}{2}(p_2^2 + x_2^2 + fx_2^4) + 6fx_1^2x_2^2, \tag{2.24}$$

which describes an integrable model of two interacting oscillators, for which the operator G, such that $[G, H] = 0$, is given by (2.17) and

$$\Gamma = x_1x_2^2[1 + f(2x_1^2 + x_2^2)]. \tag{2.25}$$

Other choices, not necessarily of the type (2.14), may be of interest.For example, $u = w = 2f$ yields

$$V = f(x_1^2 + x_2^2)^{-1/2}, \quad \Gamma = -2fx_1(x_1^2 + x_2^2)^{-1/2}. \tag{2.26}$$

The fact that this potential term can be added to (2.1) without disturbing separability in parabolic coordinates is a non trivial result.

3. The Quantum Neumann Model.

3A. Introduction

The quantum Neumann Model describes the quantum mechanics of a particle confined the the sphere S^{N-1} in N dimensional Euclidean space, with coordinates $x_k, 1 \le k \le N$, subject to harmonic forces. These arise from the potential

$$U_1 = \tfrac{1}{2}\textstyle\sum_k \varepsilon_k x_k^2, \tag{3A.1}$$

where the real quantities $\varepsilon_k, 1 \le k \le N$, obey

$$\varepsilon_1 < \varepsilon_2 ... < \varepsilon_N. \tag{3A.2}$$

The model was shown to be integrable at the classical level by Uhlenbeck [3] who constructed conserved quantities F_k such that

$$\sum_k F_k = 1, \quad \sum_k \varepsilon_k F_k = 2H, \tag{3A.3}$$

where H is the Hamiltonian. The commutativity of the F_k was proved directly in [5], where the process of separation of variables was developed elegantly. Here, using as far as is convenient the notation of [5], we recast the separation procedure in the spirit of section two, in a way that allows us to realise a wide range of (integrable) generalisations of the Neumann model. We emphasise also that the separation procedure yields explicit expressions for the correct number of commuting quanitities required for the integrability of any such generalisation.

3B. Notation.

Neumann coordinates $t_\alpha, 1 \leq \alpha \leq N-1$, are defined as the roots $t_1 < t_2 ... < t_{N-1}$ of

$$u(t) \equiv \sum_k x_k^2/(t-\varepsilon_k) = 0. \qquad (3B.1)$$

We use lower case Latin and Greek letters for indices that range from 1 to N and from 1 to $N-1$ respectively. Writing

$$u(t) = Q(t)/\Delta(t), \qquad (3B.2)$$

where

$$Q(t) = \prod_\alpha (t-t_\alpha), \quad \Delta(t) = \prod_k (t-\varepsilon_k),$$

we use partial fractions to identify

$$x_k^2 = \prod_\alpha (\varepsilon_k - t_\alpha) \prod_{l \neq k} (\varepsilon_k - \varepsilon_l)^{-1}.$$

Also rearranging (3B.1) and (3B.2) gives

$$\prod_\alpha (t-t_\alpha) = \sum_k x_k^2 \prod_{l \neq k} (t-\varepsilon_k). \qquad (3B.3)$$

The $O(t^{N-1})$ and $O(t^{N-2})$ terms of (3B.3) yield $\sum_k x_k^2 = 1$, and

$$\sum_\alpha t_\alpha = \sum_k \varepsilon_k - \sum_k \varepsilon_k x_k^2. \qquad (3B.4)$$

We also define the standard quantities, cf. [10],

$$\begin{aligned} a_1 &= \sum t_1, \\ a_2 &= \sum t_1 t_2, \\ a_3 &= \sum t_1 t_2 t_3, \\ &\vdots \\ a_{n-1} &= t_1 t_2 ... t_{N-1}, \end{aligned} \qquad (3B.5)$$

where, for example, the sum in third line is over all products of three distinct t's. So (3B.4) involves a_1. Looking at coefficients of other powers of t in (3B.3) yields easily

$$\begin{aligned} a_2 &= (\sum \varepsilon_1 \varepsilon_2) - (\sum_l \varepsilon_l)(\sum_k \varepsilon_k x_k^2) + \sum_k \varepsilon_k^2 x_k^2, \\ &\vdots \\ a_{N-1} &= (\prod_l \varepsilon_l)(\sum_k (x_k^2/\varepsilon_k)). \end{aligned} \qquad (3B.6)$$

To simplify our later results, without loss of generality, we add to the potential U_1 of (3A.1) a constant term $c = c\sum_k x_k^2$, and redefine the ε_k in such a way that

$$\sum_k \varepsilon_k = 0. \qquad (3B.7)$$

3C. Determinantal Identities.

We review from [5],[6] identities related to the Vandermonde determinant and assemble those generalisations that are needed in our later work on the generalised Neumann models. Let $D \equiv D(t_1, ..., t_{N-1})$ be the Vandermonde determinant. It has the entry t_α^{m-1} in the α-th place of its m-th row, for $1 \leq \alpha, m \leq N-1$.Then

$$D = \prod_{\beta > \alpha} (t_\beta - t_\alpha). \qquad (3C.1)$$

Let D_α be the determinant that arises by deletion of the last row and α-th column of D. Then

$$D/D_\alpha = (-)^{N-\alpha-1} \prod_{\beta \neq \alpha} (t_\alpha - t_\beta). \tag{3C.2}$$

Next, let $D(\theta)$ be the determinant that arises from D by replacing the entries t_α^{N-2} of its last row by θ_α for $1 \leq \alpha \leq N-1$. By expansion on the last row, we get

$$D(\theta) = \sum_\alpha (-)^{N-\alpha-1} D_\alpha \theta_\alpha. \tag{3C.3}$$

In particular

$$D(t^{N-2}) = \sum_\alpha (-)^{N-\alpha-1} D_\alpha t_\alpha^{N-2} = D, \tag{3C.4}$$

$$D(t^m) = \sum_\alpha (-)^{N-\alpha-1} D_\alpha t_\alpha^m = 0, \tag{3C.5}$$

for $m = 0, 1 ... N-3$. Below, we shall deal with potentials, cf. eq. (3E.1), of the form

$$V(\lambda(t)) = D(\lambda(t))/D, \tag{3C.6}$$

needing explicit expressions in terms of the t_α in the cases $\lambda(t) = t^m, m = N-1, N, N+1, ...$ and $m = -1$ and -2. The answers can be given in terms of the $a_1, ... a_{N-1}$ of (3B.5). We have

$$V(t^{N-1}) = \sum t_1 = a_1, \tag{3C.7}$$

$$V(t^N) = a_1^2 - a_2, \tag{3C.8}$$

$$V(t^{N+1}) = a_1^3 - 2a_1 a_2 + a_3, \tag{3C.9}$$

$$V(t^{-1}) = (a_{N-1})^{-1}, \tag{3C.10}$$

$$V(t^{-2}) = (a_{N-2})(a_{N-1})^{-2}. \tag{3C.11}$$

The last two results are true only to within an unimportant sign. One can prove such results by methods explained in chapter six of [11], or else by direct methods.

3D. Separation of Variables.

As in [5]. we write the Schrodinger equation of the Neumann model as

$$(-\tfrac{1}{2}\Delta + U_1 - E)\Phi = 0, \tag{3D.1}$$

where Δ is the Laplacian on S^{N-1} in Neumann coordinates. For it, we use the result derived in detail in [5] in the form

$$\Delta = -4 \sum_\alpha (-)^{N-\alpha-1} (D_\alpha/D) \Delta(t_\alpha) \mathcal{L}_\alpha,$$

where

$$\mathcal{L}_\alpha = \frac{\partial^2}{\partial t_\alpha^2} + \frac{1}{2} \sum_k \frac{1}{t_\alpha - \varepsilon_k} \frac{\partial}{\partial t_\alpha}, \tag{3D.2}$$

and D/D_α is given by (3C.2). Using (3C.4) for the E term of (3D.1), and using (3B.4), (3B.7) and (3C.7) for the U_1 term, we write these terms as in [5]

$$\tfrac{1}{2} \textstyle\sum_\alpha (-)^{N-\alpha-1} (D_\alpha/D)(-t_\alpha^{N-1} - 2E t_\alpha^{N-2}). \tag{3D.3}$$

Then, we write

$$\Phi = \prod_\alpha \psi_\alpha(t_\alpha), \tag{3D.4}$$

and recall (3C.5).Now we can effect the separation of variables for (3D.1). We find [5] that each $\psi_\alpha(t_\alpha)$ satisfies an ordinary differential equation of the same form

$$4\Delta(t)\mathcal{L}\psi = (t^{N-1} + 2Et^{N-2} + 2 \sum_{i=0}^{N-3} \mu_{i+1} t^i)\psi. \tag{3D.5}$$

Herein an index $\alpha, 1 \leq \alpha \leq N-1$, on each ψ and each t has been supressed. Also (but with notation that differs slightly from that used in [6]), the μ_i are a set of separation constants, arising, as noted, from our use of (3C.5). We shall return to them in subsection 3F below.

3E. Generalisation of the Neumann Potential.

Inspection of (3D.5) indicates that the separation of variables works, not only for the potential U_1, but also for any potential of the form

$$V = \sum_\alpha (-)^{N-\alpha-1}(D_\alpha/D)\lambda_\alpha(t_\alpha), \qquad (3E.1)$$

involving $N-1$ arbitrary functions $\lambda_\alpha(t_\alpha)$ of a single variable. We shall consider here only the simple situation in which all the functions λ_α are the same function λ. We then consider various choices of this function and determine which potentials as a function of the x_k arise. Consider the cases

$$\lambda(t) = t^N, \quad t^{N+1}, \ldots \qquad (3E.2)$$

$$\lambda(t) = t^{-1}, \qquad (3E.3)$$

$$\lambda(t) = t^{-2}, \qquad (3E.4)$$

for illustrative purposes. Other choices of interest can probably be thought of and linear superpositions of allowed potentials can be used. The results of subsection 3C allow the potentials (3E.1) that correspond to (3E.2 to 4) to be computed. Using (3C.10) and (3B.6), we first note that (3E.3) yields a potential term proportional to

$$(\sum_k x_k^2/\varepsilon_k)^{-1}. \qquad (3E.5)$$

That such a term can be added to the Neumann model without disturbing its integrability properties is a well known result due to Braden [8,9]. The other choices give rise to new potentials. Using (3E.2), (3C.8), (3B.6) and (3B.7), we find that λ^N leads to the potential

$$(\sum_k \varepsilon_k x_k^2)^2 - \sum_k \varepsilon_k^2 x_k^2. \qquad (3E.6)$$

While the sextic potential that is found in the $\lambda = t^{N+1}$ case of (3E.2) can readily be computed, we do not display it here. Finally we turn to (3E.4). This yields a potential that can be calculated using (3C.11) along with (3B.6,7). We do not present the clumsy

expression that can be written for general N as a function of the x_k, but quote only results valid for $N = 3, 4$. In these cases, the potentials are, respectively

$$\frac{\sum_k \varepsilon_k x_k^2}{(\sum_l x_l^2/\varepsilon_l)^2}, \qquad (3E.7)$$

$$\frac{C - \sum_k \varepsilon_k^2 x_k^2}{(\sum_l x_l^2/\varepsilon_l)^2}, \qquad (3E.8)$$

where C is a constant. Choices $\lambda(t)$ of the form t^{-m} for $m > 2$ are not considered because they yield terms in (3D.5) such that $t = 0$ is not a regular singular point of the differential equation.

3F. Integrability.

We have seen that any potential term of the form (3E.1) can be included in the Neumann model without causing failure of the separation of variables procedure. Here we show that no failure of integrability occurs either. This in itself is no surprise. But we do the job of proving it by constructing explicitly $N-1$ operators of which one is the Hamiltonian H, which commute pairwise for any choice of the $\lambda_\alpha(t_\alpha)$ in (3E.1). The construction is easy and generally applicable. We return to (3D.5) with $-\frac{1}{2}t_\alpha^{N-1}$ corresponding to U_1 replaced by $\lambda_\alpha(t_\alpha)$ corresponding to the general potential (3E.1). Write this now as

$$K_\alpha \psi_\alpha(t_\alpha) = (E t_\alpha^{N-2} + \sum_{i=0}^{N-3} \mu_{i+1} t_\alpha^i)\psi_\alpha(t_\alpha), \qquad (3F.1)$$

where

$$K_\alpha = \mathcal{L}_\alpha + \lambda_\alpha(t_\alpha). \qquad 83F.2)$$

As in section two, we now view (3F.1) roughly as a set of linear equations in $N-1$ unknowns

$$\mu_1, \mu_2, \ldots \mu_{N-2}, \mu_{N-1} = E. \qquad (3F.3)$$

More precisely, we seek to view $M_\alpha, 1 \le \alpha \le N-1$, as a set of operators, with these eigenvalues μ_α, linearly related to the K_α via the set of equations

$$K_\alpha = \sum_\beta D_{\beta\alpha} M_\beta, \qquad (3F.4)$$

where $D_{\beta\alpha} = (t_\alpha)^{\beta-1}$ is the $\beta\alpha$ element of a matrix whose determinant is the Vandermonde determinant of subsection 3C.It follows that

$$M_\alpha = \sum_\beta (D^{-1})_{\beta\alpha} K_\beta, \tag{3F.5}$$

. a result which can be nicely made explicit using Kramer's rule. Further, the M_α clearly act on $\Phi = \prod_\alpha \psi_\alpha(t_\alpha)$ according to

$$M_\alpha \Phi = \mu_\alpha \Phi, \tag{3F.6}$$

for $\alpha = 1, ..., N-1$, with $M_{N-1} = H$. The procedure followed suggests that the M_α are the required integrals for the generalised Neumann models. To complete the proof, we must show the the M_α commute pairwise. This is done by direct calculation in subsection 3G.

3G. The integrals M_α commute.

It is actually easy to establish the result. The proof for the generalised models seems easier than its counterpart [5] for the Uhlenbeck integrals for the basic Neumann model.

Suppose that $\alpha \neq \beta$. Set $D^{-1} = B^{tr}$ for convenience.

$$\begin{aligned}
[M_\alpha, M_\beta] &= [B_{\alpha\gamma} K_\gamma, B_{\beta\delta} K_\delta] \\
&= B_{\alpha\gamma}[K_\gamma, B_{\beta\delta}]K_\delta + B_{\beta\delta}[B_{\alpha\gamma}, K_\delta]K_\gamma \\
&= B_{\alpha\delta}[K_\delta, B_{\beta\gamma}]K_\gamma - B_{\beta\delta}[K_\delta, B_{\alpha\gamma}]K_\gamma.
\end{aligned}$$

Here we have used the fact that K_α and K_β commute because the former depends only on t_α and the latter only on t_β. Also elements of B commute with each others as they do not involve any derivatives. We have omitted obvious summations over repeated indices. Next we use

$$[K_\delta, B_{\sigma\gamma}] = -B_{\sigma\rho}[K_\delta, D_{\tau\rho}]B_{\tau\gamma}.$$

Hence

$$[M_\alpha, M_\beta] = -B_{\alpha\delta}B_{\beta\rho}[K_\delta, D_{\sigma\rho}]B_{\sigma\gamma}K_\gamma + B_{\beta\delta}B_{\alpha\rho}[K_\delta, D_{\sigma\rho}]B_{\sigma\gamma}K_\gamma.$$

This now simplifies because $D_{\sigma\rho}$, for each value of σ, is some power of t_ρ, and hence $[K_\delta, D_{\sigma\rho}]$ can receive non zero contributions only if $\delta = \rho$. This is sufficient to complete the proof.

The proof just given not depend in any way on the choice of $\lambda_\alpha(t_\alpha)$ in (3E.1) or (3F.5). So we have constructed a set of $N-1$ pairwise commutative integrals M_α for the generalised Neumann model.This construction appears to be new.

This concludes our use of the Neumann model to illustrate a systematic approach to some aspects of the theory of integrable quantum mechanical systems. It is available whenever the Schrodinger equation of the theory can be solved by separation of variables: if one seeks the operators whose eigenvalues are the separation constants, then one finds a suitable set of commutative integrals for the theory. For the Neumann model these serve as an alternative to the Uhlenbeck set of integrals. We shall discuss the relationship between the two sets elsewhere.

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