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Quantization on Riemannian spaces from groupoïd C^*-algebras*

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Preprint DAMTP-92-46

July 21, 1992

Abstract

Let (P, Q, H) be a principal H-bundle over a Riemannian space Q. It is shown that the Poisson algebra $C_c^\infty((T^*P)/H)$ is quantized by the C^*-algebra of the gauge groupoïd of P. Mackey's quantization on $Q = G/H$ is a special case, the groupoïd C^*-algebra in that case reducing to the crossed product $C^*(G, G/H)$. For H trivial one obtains a generalization of Weyl quantization to any Riemannian space. Physical applications include the motion of a particle on Q coupling to a Yang-Mills field with gauge group H, and quantization on multiply connected spaces.

*Invited paper at the XXI Conference on Differential Geometric Methods in Physics, Nankai University, Tianjin, China, June 5-9, 1992

†Supported by an S.E.R.C. Advanced Research Fellowship

1 Deformation quantization and operator algebras

Quantization may be understood as some (functorial) correspondence between Poisson algebras and operator algebras. We here consider Poisson algebras of the type $C_c^\infty(P)$, to be completed into the commutative C^*-algebra $C_0(P)$, which is topologized by the sup-norm. Its 'quantization' $\mathcal{A}$ is taken to be a C^*-algebra, which is noncommutative in general. A further essential ingredient is a family of maps $\{Q_\hbar\}_{\hbar\in I}$ from a dense subalgebra $\mathcal{A}_0$ of $C_0(P)$, on which the Poisson bracket is defined, to $\mathcal{A}$; here I is an interval containing 0. This family has to be real (i.e., $Q_\hbar(\overline{f}) = Q_\hbar(f)^*$), and should satisfy the usual asymptotic correspondence between the algebraic structure of classical and quantum mechanics. That is, $\lim_{\hbar\to 0}(1/2[Q_\hbar(f), Q_\hbar(g)]_+ - Q_\hbar(fg)) = 0$ and $\lim_{\hbar\to 0}(i/\hbar[Q_\hbar(f), Q_\hbar(g)]_- - Q_\hbar(\{f,g\})) = 0$ in norm. Continuity in $\hbar$ may be imposed by demanding that either the maps $\hbar \to Q_\hbar(f)$ from I to $\mathcal{A}$, or else the functions $\hbar \to \| Q_\hbar(f) \|$, are continuous for all $f \in \mathcal{A}_0$, cf. [9, 1].

This notion of quantization fits in well with both the modern geometric theory of classical mechanics [7] and the algebraic theory of superselection sectors in quantum mechanics [3, 5], and, in fact, attempts to connect the two [6]. It has been tested on a system of physical relevance, namely a particle with a nonabelian charge moving on a homogeneous configuration space $Q = G/H$; the charge couples to an external Yang-Mills field with gauge group H. The classical mechanics of this system is described by the Poisson manifold $P = (T^*G)/H$, whose symplectic leaves correspond to coadjoint orbits in $\mathcal{H}^*$ (the dual of the Lie algebra of H), which may be identified with classical charges the particle may have (see [7], and refs. therein to the original work of Sternberg, Weinstein, and others). Its quantization is the crossed product $\mathcal{A} = C^*(G, Q)$, whose irreducible representations correspond to irreducible unitary representations of H [5], which are the quantum charges the particle may have. The deformation of $C^*(G, Q)$ into $C_0((T^*G)/H)$ (that is, the maps $Q_\hbar$) was constructed in [6].

That $C^*(G, Q)$ is indeed the correct quantization of $C_0((T^*G)/H)$ could only be

ascertained with hindsight; a similar comment applies to the special case $H = G$, previously considered by Rieffel [9]; here the Poisson algebra $C_c^\infty(\mathcal{G}^*)$ (with the Lie-Kirillov Poisson bracket) is quantized by the group algebra $C^*(G)$. This case is of special interest in connection with quantum groups, which may be arrived at by deforming the Poisson structure on $\mathcal{G}^*$, and subsequently quantizing in the way sketched above.

The purpose of the present note is to generalize these results to arbitrary Riemannian configuration spaces Q (and principal H-bundles over Q, where H is taken to be an amenable Lie group, for simplicity), and this is achieved by reformulating them in terms of the so-called gauge groupoïd of the bundle (P, Q, H).

2 From groupoïds to quantization

A groupoïd Γ [8, 2] is a set op points (the base space or unit space B of Γ), plus a collection of arrows between some of these points. These arrows are invertible, and may be composed if their endpoints match. Γ is fibered over B, the fiber F_b at a point $b \in B$ being the set of arrows in Γ that end at b. An element $x \in \Gamma$ is both an arrow from $s(x)$ to $t(x)$, and an isomorphism I_x between $F_{s(x)}$ and $F_{t(x)}$. We assume that Γ is a manifold. Given a measure on each fiber, such that the family of measures thus obtained is invariant under all isomorphisms I_x, one can construct a C^*-algebra $C^*(\Gamma)$, whose representations correspond to the representations of Γ itself [8]. $C^*(\Gamma)$ is the norm-closure of the function space $C_c^\infty(\Gamma)$, equipped with a generalized convolution product, and a natural norm and involution.

Now regard $\Gamma_0 = (T^*P)/H$ as a trivial groupoïd (that is, $B = \Gamma_0$ and every point is only connected to itself by a single arrow), with each point having measure 1. Then $C^*(\Gamma_0) = C_0(\Gamma_0)$, the commutative algebra of classical observables of a charged particle moving on Q. This system is quantized by replacing Γ_c by the groupoïd $\Gamma_q = P \times_H P \equiv \Gamma_H(P)$, the gauge groupoïd of the principal bundle (P, Q, H) . This has base space Q, and its elements are equivalence classes $[p_1, p_2] \equiv [p_1 h, p_2 h]$, interpreted as classes of H-equivariant paths in P [2]. If $\pi : P \to Q$ is the bundle projection on P, the groupoïd projection $\pi_g : \Gamma_H(P) \to Q$ is given by $\pi_g([p_1, p_2]) = \pi(p_1)$. The arrow $x = [p_1, p_2]$ starts at $s(x) = \pi(p_2)$ and ends at $t(x) = \pi(p_1)$, from which the composition rule follows.

The reason that the passage $\Gamma_c \to \Gamma_q$ quantizes the system is that $\mathcal{A} = C^*(\Gamma_q)$ turns out to be exactly the right quantum algebra of observables of the given physical system. It may be shown that $\mathcal{A} \simeq \mathcal{K}(L^2(P))^H \simeq \mathcal{K}(L^2(Q)) \otimes C^*(H)$, where $\mathcal{K}(\mathcal{H})$ is the C^*-algebra of compact operators on a Hilbert space $\mathcal{H}$, $L^2(M)$ is the Hilbert space of square-integrable half-densities on a manifold M, and $\mathcal{K}(L^2(P))^H$ are the compact operators which are invariant under the natural unitary representation of H on $L^2(P)$. Hence $\mathcal{A}$ is Morita-equivalent to $C^*(H)$, and its irreducible representations yield the correct superselection sectors of the system, cf. [5] for the case $P = G$. In that case, $C^*(\Gamma) = C^*(G, Q)$, which has been exhaustively analyzed as a quantum algebra of observables of a particle moving on Q [5, 6].

With hindsight, the passage from Γ_c to Γ_q has hereby been justified, but it may be motivated in itself in various ways (none of which is compelling). Firstly, there is a canonical construction due to Connes (unpublished, but reported in [4]), which, given an injection $i : \Gamma_1 \hookrightarrow \Gamma_2$ of groupoïds (which are manifolds) canonically produces a continuous field of C^*-algebras over the interval [0, 1], that is, a deformation quantization (the C^*-algebra defined by this field is the C^*-algebra of the so-called normal groupoïd corresponding to the injection i). If we apply this to the injection $i : Q \hookrightarrow \Gamma_H(P)$ (with Q regarded as a trivial groupoïd), defined by $i(q) = [s(q), s(q)]$ (with $s : Q \to P$ an arbitrary section), we obtain the C^*-algebras Γ_c and Γ_q above, as well as a full family of cross-sections interpolating between them. Hence apart from the construction of the quantization maps $Q_\hbar$, which is a separate problem discussed below, deformation quantization on Riemannian manifolds is equivalent to Connes' construction of a certain normal groupoïd. *(I am very grateful to R. Nest and G. Skandalis, who pointed this out in the special case $P = G$ in connection with the deformation given in [6]).*

Alternatively, the fact that $\Gamma_c = (T^*P)/H$ is 'quantized' by $\Gamma_q = P \times_H P$ appears natural if one actually constructs the quantization maps $Q_\hbar$. To make this more transparent, we put $H = \{e\}$ for the moment, and give the argument for

$\Gamma_c = T^*P$ and $\Gamma_q = P \times P$; the result will quantize H-invariant functions on T^*P by H-invariant compact operators on $L^2(P)$, so that one can simply quotient by H to cover the general case.

Hence consider the diagonal embedding $i : P \to P \times P$. The corresponding normal bundle is the tangent bundle TP, and by the tubular neighbourhood theorem there exists a diffeomorphism φ from a neighbourhood $U \subset TP$ containing P (identified with the zero section) to a neighbourhood $V \subset P \times P$ containing P (identified with $i(P)$). Using the Riemannian metric on P, we define φ by $\varphi^{-1}(X) = (\exp(X/2), \exp(-X/2)$. Now choose a Lebesgue measure on each of the fibers of TP, so that the resulting collection is invariant under parallel transport. This defines a bijection between $C^*(T^*P) = C_0(T^*P)$ and $C^*(TP)$ (the C^*-algebra of the groupoïd TP with base space P, and all arrows closed, composition being addition in a given fiber) by a fiberwise Fourier transform $\mathcal{F}$. For $\hbar > 0$ fixed, one thus obtains a diffeomorphism $\varphi_\hbar$ between $U/\hbar$ and V given by $\varphi_\hbar^{-1}(X) = \varphi(\hbar X)$. As will become clear shortly, this diffeomorphism is essentially quantization, which is thus seen to consist of 'stretching out' the tangent bundle TP into the path space $P \times P$, the stretching factor being $\hbar$. Other diffeomorphisms would correspond to different operator ordering prescriptions.

The dense subalgebra $\mathcal{A}_0 \subset C_0(T^*P)$ of quantizable functions consists of those functions f whose 'Fourier transform' $\mathcal{F}f$ is in $C_c^\infty(TP)$. Such f has a quantization $Q_\hbar(f)$ for those $\hbar$ for which $\hbar(\mathrm{supp} f) \subset U$. It is given by $Q_\hbar(f) = \Delta\, \varphi_\hbar^* f$, where Δ is an f-independent function on $P \times P$ which depends on the precise choice of Lebesgue measures mentioned above (for example, if $P = G$ and these measures are related by the action of the group on itself then Δ is constant). $Q_\hbar(f)$ is in $C_c(P \times P)$, and should be identified with the Hilbert-Schmidt operator defined by this kernel. An easy argument using the majorization of the uniform operator norm by the Hilbert-Schmidt norm shows that this quantization satisfies the conditions stated in sect. 1. Application of this scheme to $P = \mathbf{R}^n$ reproduces the Weyl quantization of functions on $T^*\mathbf{R}^n$, and the multiplication of Hilbert-Schmidt kernels is equivalent to the Weyl calculus of pseudo-differential operators, restricted to the compacts.

The connection between e.g. traditional geometric quantization and our method is not straightforward, as we do not quantize symplectic spaces as such, but entire Poisson manifolds. The algebraic correspondence between $C_0((T^*P)/H)$ and its quantization $C^*(\Gamma_H(P))$ is clear (and given by deformation), but their respective representation theories (on symplectic leaves and Hilbert spaces, respectively [6]) are not so clearly related. Recent techniques to quantize Poisson algebras using symplectic groupoïds [2] take a quite different route, too (for example, the groupoïd structure on T^*P is not taken to be the trivial one, as in our method, but the one we use on TP).

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