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A family of gravitational instantons

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A gravitational instanton is a four-manifold with a nonsingular Riemannian metric satisfying Einstein's equations. It is thus a critical point for the Einstein-Hilbert action, and so is of interest in the path integral approach to quantum gravity [1].

A special class of gravitational instantons is that of hyperkähler four-manifolds. The hyperkähler condition means that there are three covariant constant complex structures I, J and K, (with respect to which the metric is hermitian), which satisfy the quaternionic multiplication relations. This implies the existence of a whole two-sphere's worth of Kähler structures on the manifold. Hyperkähler manifolds are Ricci-flat, so in particular are Einstein.

In this note we shall present a three-parameter family of manifolds admitting hyperkähler structures, exhibit them as hypersurfaces in $\mathbb{C}^3$, and find their topology. They will, in fact, be deformations of the singular hypersurface in $\mathbb{C}^3$ with equation

$$x^2 - zy^2 = z \tag{1}$$

This singular space may be identified with the quotient of $\mathbb{C}^2$ by $\mathbf{Z}$ (acting by translations $(u,v) \mapsto (u+n, v)$) and the involution $(u,v) \mapsto (-u,-v)$.

Hitchin [2] argued using twistor methods that the space with equation (1) should admit deformations carrying nonsingular hyperkähler structures (this was also noted on heuristic grounds by Page [3]). We shall prove this result directly using the hyperkähler quotient construction, which we now describe.

If M is a hyperkähler manifold with metric h and complex structures I, J and K, then it carries three Kähler forms $\omega_1, \omega_2, \omega_3$ defined by

$$\omega_1(X,Y) = h(IX, Y)$$

etc. Suppose now that a group G acts on M preserving h, I, J, K. If G is compact we can define three moment maps μ_1, μ_2, μ_3 from M to the dual of the Lie algebra of G by

$$< \mu_i(x), \xi >= f_i^\xi(x) \quad (x \in M, \xi \in \mathfrak{g})$$

where df_i^ξ is the contraction of ω_i with ξ, and we identify an element of $\mathfrak{g}$ with the Killing field it generates on M.

It is a well known result [4] that if $\lambda_1, \lambda_2, \lambda_3$ lie in the centre of $\mathfrak{g}^*$, and if $\mu^{-1}(\lambda_1, \lambda_2, \lambda_3)/G$ is nonsingular, then it is hyperkähler. We refer to this manifold as the hyperkähler quotient of M by G.

It has been observed [5],[6] that if M is taken to be the infinite-dimensional quaternionic affine space of points $T_0 + iT_1 + jT_2 + kT_3$ where T_i are analytic $\mathfrak{u}(2)$-valued functions on $[0,1]$, then M is hyperkähler. Moreover the group G of $U(2)$-valued functions on $[0,1]$ which are the identity at $t = 0, 1$ acts on M by

$$\begin{aligned} T_0 &\mapsto gT_0g^{-1} - \dot{g}g^{-1} \\ T_i &\mapsto gT_ig^{-1} \quad (i = 1,2,3) \end{aligned}$$

preserving the hyperkähler structure. The moment map for this action is

$$F : (T_0, T_1, T_2, T_3) \mapsto \begin{pmatrix} \dot{T}_1 + [T_0, T_1] & - & [T_2, T_3] \\ \dot{T}_2 + [T_0, T_2] & - & [T_3, T_1] \\ \dot{T}_3 + [T_0, T_3] & - & [T_1, T_2] \end{pmatrix} \tag{2}$$

We deduce that $N = F^{-1}(0,0,0)/G$ is hyperkähler. This is just the moduli space of solutions to Nahm's equations for 2×2 analytic matrices.

Nahm's equations are equivalent to

$$\dot{\beta} + 2[\alpha, \beta] = 0 \tag{3}$$

$$\dot{\alpha} + \dot{\alpha}^* + 2([\alpha, \alpha^*] + [\beta, \beta^*]) = 0 \tag{4}$$

where $\alpha = \frac{1}{2}(T_0 + iT_1)$ and $\beta = \frac{1}{2}(T_2 + iT_3)$. As pointed out in [6], the moduli space N can be identified with the quotient by the complexification of G of the space of solutions to equation (3) alone.

Following Kronheimer [6], we can gauge α to zero by solving the equation $\dot{g} = gT_0$ and then β is a constant matrix in $\mathfrak{gl}(2, \mathbb{C})$. The map $(\alpha, \beta) \mapsto (g(1), g(1)\beta(1)g(1)^{-1})$ is a bijection between N and $GL(2, \mathbb{C}) \times \mathfrak{gl}(2, \mathbb{C})$. The latter space is just the cotangent bundle of $GL(2, \mathbb{C})$.

We can now perform two further hyperkähler quotients to obtain from N a family of four-dimensional hyperkähler manifolds.

First of all, observe that the group of functions from $[0, 1]$ to the centre of $U(2)$ acts on N preserving the metric and complex structures. It is straightforward to check that the associated moment map is

$$m : (T_0, T_1, T_2, T_3) \mapsto (\mathrm{Tr}T_1, \mathrm{Tr}T_2, \mathrm{Tr}T_3) \tag{5}$$

The quotient $N^0 = m^{-1}(0, 0, 0)/U(1)$ may be identified with the 12-dimensional submanifold of N whose points consist of tracefree Nahm matrices. As a complex manifold it is $SL(2, \mathbb{C}) \times \mathfrak{sl}(2, \mathbb{C})$. N^0 is also hyperkähler and admits an action of $SU(2) \times SU(2)$ given by gauge transformations taking values in $SU(2)$. This action preserves the metric and the complex structures on N^0. Furthermore, N^0 has an action of $SO(3)$, defined by

$$\begin{aligned} T_0 &\mapsto T_0 \\ T_i &\mapsto \sum a_{ij} T_j \quad (i = 1, 2, 3) \end{aligned}$$

where $(a_{ij}) \in SO(3)$. The action of $SO(3)$ is isometric, but it permutes the two-sphere of complex structures rather than fixing them.

Let us now consider the diagonal $U(1) \times U(1)$ subgroup of $SU(2) \times SU(2)$ associated to the $U(1)$ subgroup of $SU(2)$ which stabilises the Pauli spin matrix $\sigma_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$ in the adjoint representation. As in [7], we can calculate the moment map for this action, and it is given by

$$\mu : (T_0, T_1, T_2, T_3) \mapsto \begin{pmatrix} (< T_1(0), \sigma_1 >, & < T_1(1), \sigma_1 >) \\ (< T_2(0), \sigma_1 >, & < T_2(1), \sigma_1 >) \\ (< T_3(0), \sigma_1 >, & < T_3(1), \sigma_1 >) \end{pmatrix} \tag{6}$$

We deduce that if $\mathbf{k}, \mathbf{l} \in \mathbb{R}^3$ then $\mu^{-1}(\mathbf{k}, \mathbf{l})/U(1) \times U(1)$ is a hyperkähler four-manifold when it is nonsingular. Of course, we could perform a similar construction for any other $U(1) \times U(1)$ subgroup of $SU(2) \times SU(2)$. However the resulting hyperkähler four-manifolds are in fact isometric to $\mu^{-1}(\mathbf{k}, \mathbf{l})/U(1) \times U(1)$ for some $\mathbf{k}, \mathbf{l}$ via the $SU(2) \times SU(2)$ action on N^0. Similarly, the $SO(3)$ action on N^0 gives an isometry between $\mu^{-1}(\mathbf{k}, \mathbf{l})/U(1) \times U(1)$ where $\mathbf{k} = (k_1, k_2, k_3), \mathbf{l} = (l_1, l_2, l_3)$ and $\mu^{-1}(\mathbf{k}, \mathbf{l})/U(1) \times U(1)$ where $\mathbf{k} = (0, \| k \|, 0), \mathbf{l} = (0, l_2, \sqrt{(l_1^2 + l_3^2)})$. We shall therefore restrict ourselves to the three-parameter family of spaces $M(k, l)$ $(k \in \mathbb{R}, l \in \mathbb{C})$ defined by

$$M(k, l) = \mu^{-1}((0, 2k, 0), (0, 2\mathrm{Re}l, 2\mathrm{Im}l))/U(1) \times U(1)$$

The factor of 2 is for convenience.

It easily follows from the isomorphism between N and $T^*GL(2, \mathbb{C})$ that $M(k, l)$ is isomorphic as a complex manifold to the quotient by $\mathbb{C}^* \times \mathbb{C}^*$ of

$$\{(A, B) \in SL(2, \mathbb{C}) \times \mathfrak{sl}(2, \mathbb{C}) : str(B) = 2k, str(A^{-1}BA) = 2l\} \tag{7}$$

where $str \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = a_{11} - a_{22}$

The $\mathbb{C}^* \times \mathbb{C}^*$ action is given by

$$B \mapsto \phi B \phi^{-1}, \quad A \mapsto \phi A \psi^{-1}$$

where $\phi = diag(\tau, \tau^{-1}), \psi = diag(\rho, \rho^{-1})$ and $\tau, \rho \in \mathbb{C}^*$.

It is easily checked that fixed points occur only if B is diagonal and either $A = \begin{pmatrix} p & 0 \\ 0 & p^{-1} \end{pmatrix}$ with $\phi = \psi$ or $A = \begin{pmatrix} 0 & q \\ -q^{-1} & 0 \end{pmatrix}$ with $\phi = -\psi$. In these cases, we find that $str(B) = \pm str(A^{-1}BA)$ respectively. We see that $M(k,l)$ is nonsingular precisely when $k \neq \pm l$.

Our next aim is to identify the gravitational instantons $M(k,l)$ with hypersurfaces in $\mathbb{C}^3$. Let us put

$$A = \begin{pmatrix} p & q \\ r & s \end{pmatrix}, B = \begin{pmatrix} k & a \\ b & -k \end{pmatrix}$$

The map

$$(A, B) \mapsto (X_0, X_1, X_2, X_3, X_4) = (a, b, ps, pq, rs)$$

defines an isomorphism between $M(k,l)$ and the quotient of the space

$$\{(X_0, X_1, X_2, X_3, X_4) : X_2^2 - X_2 = X_3X_4, (2X_2 - 1)k - l = X_1X_3 - X_0X_4\} \quad (8)$$

by the $\mathbb{C}^*$ action

$$(X_0, X_1, X_2, X_3, X_4) \mapsto (\tau^2 X_0, \tau^{-2} X_1, X_2, \tau^2 X_3, \tau^{-2} X_4)$$

Note that this action is free provided that $k \neq \pm l$.
Now consider the map

$$(X_0, X_1, X_2, X_3, X_4) \mapsto (Y_0, Y_1, Y_2) = (X_2, X_0X_4, X_0X_1)$$

This map gives an isomorphism of $M(k,l)$ with the variety

$$\{(Y_0, Y_1, Y_2) : Y_2(Y_0^2 - Y_0) = ((2Y_0 - 1)k - l + Y_1)Y_1\}$$

Putting $x = Y_1, y = i(2Y_0 - 1), z = -\frac{1}{4}Y_2$ we get the hypersurface in $\mathbb{C}^3$ with equation

$$x^2 - zy^2 = z + ikxy + lx \quad (9)$$

It is readily seen that this equation defines a nonsingular affine surface if and only if $k \neq \pm l$, in accordance with our findings above.

We can use this description of $M(k,l)$ to find its homotopy type. First observe that the map $(x, y, z) \mapsto (y, z)$ exhibits $M(k,l)$ as a double cover of $\mathbb{C}^2$ with branch locus isomorphic to $\mathbb{C}$ with two points removed. Hence $M(k,l)$ has Euler characteristic equal to three. Our description of $M(k,l)$ as a hypersurface in $\mathbb{C}^3$ shows that it is Stein, and hence has the homotopy type of a CW-complex with no cells of dimension higher than two. Moreover an application of van Kampen's theorem shows that the manifold is simply connected. We deduce that $M(k,l)$ is homotopy equivalent to a bouquet of two 2-spheres.

We remark that it follows from the definition of $M(k,l)$ as a quotient of a level set of μ, that $M(k,l)$ admits an isometric action of a circle subgroup of $SO(3)$ if $\mathrm{Im}\, l = 0$, but otherwise has no apparent isometries.

Our gravitational instantons are deformations of the singular hypersurface in $\mathbb{C}^3$ whose equation is

$$x^2 - zy^2 = z$$

Now there is a whole series of singular hypersurfaces, with equations

$$x^2 - zy^2 = z^{j-1}$$

which are associated to the dihedral groups D_j. It is known [8] that for $j \geq 4$ these admit asymptotically locally Euclidean (ALE) hyperkähler deformations. We conjecture that there exists a construction generalising the one in this paper which produces hyperkähler deformations of the dihedral singular surfaces as quotients of moduli spaces of Nahm data, and that these deformations are asymptotically locally flat (ALF). We would then have a series of ALF spaces associated to the dihedral groups, comparable to the multi-Taub-NUT series associated to the cyclic groups. Work on these questions is in progress.

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