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BRAIDED GROUPS AND DUALS OF MONOIDAL CATEGORIES[1]

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ABSTRACT Let $F : \mathcal{C} \to \mathcal{V}$ be a functor between monoidal categories. Previously we have shown that there is a dual monoidal category $\mathcal{C}^*$ of representations of $\mathcal{C}$ in $\mathcal{V}$ and a functor $\mathcal{C} \to \mathcal{C}^{**}$ generalizing the Pontryagin duality of Abelian groups. If $\mathcal{V}$ is braided then there are also such notions as a coadjoint action of $\mathcal{C}$ on $\mathcal{C}^*$ and cross products by it. In another context we have shown in this situation (in the rigid case) that there is a Hopf algebra $\mathrm{Aut}(\mathcal{C}, F, \mathcal{V})$ in a cocompletion of $\mathcal{V}$ together with a functor $\mathcal{C} \to \mathrm{Aut}(\mathcal{C}, F, \mathcal{V})$-comodules (the category of comodules in $\mathcal{V}$). We report on these results and explore their relationship. We show that $\mathcal{C}^* \cong \mathrm{Aut}(\mathcal{C}, F, \mathcal{V})$-modules (modules in $\mathcal{V}$) and explain how these categorical duality constructions can be applied in an algebraic context to obtain new results about ordinary quasitriangular Hopf algebras (quantum groups), as well as their significance for physics in the context of quantum-gravity.

1 INTRODUCTION

This paper recalls a construction of the author in [16] of the dual monoidal category of a monoidal category, and develops it further. We begin in Section 2 by recalling this construction and giving a number of details (including diagrammatic proofs) that were omitted in [16] for reasons of space. We then proceed in later sections to connect this construction with the author's recent work on braided groups and quantum braided groups[18], in the case when the monoidal category is braided. The braided group $\mathrm{Aut}(\mathcal{C})$ of a braided monoidal category

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$\mathcal{C}$ is a group-like object and expresses in its multiplication the monoidal product of $\mathcal{C}$. Section 3 begins by summarizing this construction (with full details to be found in the preprint [18]), and proceeds to show that in this case $\mathcal{C}^*$ can be identified with the category of $\mathrm{Aut}(\mathcal{C})$-modules. This thereby develops our duality work of [16] in the braided group context, and is the modest new result of the paper. The results hold generally over a base category $\mathcal{V}$.

These categorical constructions are all fairly straightforward once the right definitions are found. Yet they have important and non-trivial applications. We focus on the applications to Drinfeld's theory of quasitriangular Hopf algebras (quantum groups). Section 4 begins by recalling these and then proceeds to describe some non-trivial corollaries of our general constructions from Sections 2 and 3. The main result is a new category-theoretic proof of the result of [21] that Drinfeld's quantum double $D(H)$ of a quantum group H is a semidirect product. This was proven by algebraic means, using an involved theorem of Radford [25]; we recover it now as a simple corollary of the main result of Section 3 applied in the case where $\mathcal{C}$ is the category of H-modules. The semidirect product explicitly takes the form $\mathrm{Aut}(\mathcal{C}) \rtimes H$.

We note that a number of other applications of braided groups in this same context of quantum groups have recently been given in [19][20](with further applications in physics[14][15]). These applications use the general construction $\mathrm{Aut}(\mathcal{C}, F, \mathcal{V})$ (i.e., over base $\mathcal{V}$) as a way to shift the category in which an algebraic structure is defined. We begin with an algebraic structure of one sort, and use it to generate a category $\mathcal{C}$. By targeting $\mathcal{C}$ in another category $\mathcal{V}$ we can then transfer our original structure to one of the form $\mathrm{Aut}(\mathcal{C}, F, \mathcal{V})$ in $\mathcal{V}$. This process of 'transmutation' can be very useful in physics. For example, we may be presented with a group or quantum group in an ordinary sense but which lives more naturally (and looks simpler) when transmuted to the category of super-spaces, or vice-versa[14]. For a braided example, we have shown in [15] how the Weyl algebra of quantum mechanics of a one-dimensional particle lives more naturally in a braided category, where it corresponds simply to the braided line. These applications are all connected with braided monoidal categories and Yang-Baxter equations etc, which are quite a popular topic in

physics and certain areas of mathematics. However, from our point of view they are a special case of more general duality considerations of quite wide validity.

It is not within our scope to discuss these detailed applications here. Instead, the paper concludes in Section 5 with a brief outline for category theorists of a vision of physics based on these duality considerations and category theory. This section is strictly optional but we hope it will be interesting for a reader who wants to know why the author, with a background in physics, should be led into category theory. Roughly speaking, one can argue[23] that complete theories of physics have two aspects, a combinatorial (quantum) and a geometrical, and these are in duality. During this conference I benefited from discussions with F.W. Lawvere and A. Kock who convinced me that there were some connections between these duality ideas and Lawvere's notions of 'extrinsic' and 'intrinsic' in topos theory. To explore such a connection would certainly be an interesting line for further work.

2 DUALS OF MONOIDAL CATEGORIES

In this section we recall the notion of duals of braided monoidal categories, with details of some of the proofs omitted in our report [16], and some further results. We use the standard definitions of a monoidal category and (weak) monoidal functor as in the text of Mac Lane[11]. Such a functor $F : \mathcal{C} \to \mathcal{V}$ means $F(X)\otimes F(Y)\cong F(X\otimes Y)$ by functorial isomorphisms which we denote $c_{X,Y}$, compatible with the associativity isomorphisms Φ in the two categories. These functorial associativity isomorphisms $\Phi_{X,Y,Z} : X\otimes(Y\otimes Z) \to (X\otimes Y)\otimes Z$ and $\Phi_{U,V,W} : U\otimes(V\otimes W) \to (U\otimes V)\otimes W$ will generally be suppressed for clarity, as also the isomorphisms $X\otimes\underline{1}\cong X$, $\underline{1}\otimes X\cong X$ for the unit object $\underline{1}$ (and similarly for the unit in $\mathcal{V}$). Throughout the paper, X, Y, Z etc denote general objects of $\mathcal{C}$ and U, V, W etc general objects of $\mathcal{V}$.

Definition 2.1 *[16] Let $F : \mathcal{C} \to \mathcal{V}$ be a monoidal functor between monoidal categories. We define a right $(\mathcal{C}, F, \mathcal{V})$-module in $\mathcal{V}$ to be a pair (V, λ_V) where V is an object of $\mathcal{V}$ and $\lambda_V \in \mathrm{Nat}(V\otimes F, F\otimes V)$ is an invertible natural transformation $\lambda_{V,X} : V\otimes F(X) \to F(X)\otimes V$ obeying*

$$\lambda_{V,\underline{1}} = \mathrm{id}, \qquad (\mathrm{id}\otimes\lambda_{V,Y})(\lambda_{V,X}\otimes\mathrm{id}) = (c^{-1}_{X,Y}\otimes\mathrm{id})\lambda_{V,X\otimes Y}(\mathrm{id}\otimes c_{X,Y}).$$

This clearly 'represents' the monoidal product in a natural way, and generalizes the definition of representations of groups and Hopf algebras as we will see in the next section. The main results are

Theorem 2.2 *[16] The collection of right $(\mathcal{C}, F, \mathcal{V})$-modules in Definition 2.1 form a monoidal category, denoted $\mathcal{C}^\bullet$. The morphisms $(V, \lambda_V) \to (W, \lambda_W)$ are morphisms $\phi : V \to W$ in $\mathcal{V}$ such that the modules are intertwined in the form*

$$(\mathrm{id}\otimes\phi)\lambda_{V,X} = \lambda_{W,X}(\phi\otimes\mathrm{id}), \qquad \forall X \text{ in } \mathcal{C}.$$

The monoidal structure is $(\underline{1}, \lambda_{\underline{1}})$, $\lambda_{\underline{1},X} = \mathrm{id}$ and

$$(V, \lambda_V)\otimes(W, \lambda_W) = (V\otimes W, \lambda_{V\otimes W}), \ \lambda_{V\otimes W,X} = (\lambda_{V,X}\otimes\mathrm{id})(\mathrm{id}\otimes\lambda_{W,X}).$$

Proposition 2.3 *[16] In the setting of the preceding theorem, there is a monoidal functor $\psi : \mathcal{C} \to \mathcal{C}^{\bullet\bullet}$ defined tautologically by*

$$X \mapsto (F(X), \lambda_{\psi(X)}), \qquad \lambda_{\psi(X),(V,\lambda_V)} = \lambda^{-1}_{V,X}.$$

There is also a notion of left $(\mathcal{C}, F, \mathcal{V})$-module, the collection of which is denoted ${}^\bullet\mathcal{C}$ and a functor $\mathcal{C} \to {}^\bullet(\mathcal{C}^\bullet)$ given by $\lambda_{V,X}$ without inversion in Proposition 2.3. In these results it is clear that there are also monoidal functors $F^\bullet : \mathcal{C}^\bullet \to \mathcal{V}$ etc (given by the forgetful functor) so that $(\mathcal{C}^\bullet, F^\bullet, \mathcal{V})$ and $({}^\bullet\mathcal{C}, {}^\bullet F, \mathcal{V})$ are dual systems of the same type as $(\mathcal{C}, F, \mathcal{V})$. We can think of $\mathcal{V}$ as a fixed category over which $\mathcal{C}, \mathcal{C}^\bullet, {}^\bullet\mathcal{C}, \mathcal{C}^{\bullet\bullet}$ etc are functored. We consider a morphism between such triples as a monoidal functor between the first members that is compatible with the functors to $\mathcal{V}$. This is the setting in which we work. For example, the functor $\psi : \mathcal{C} \to \mathcal{C}^{\bullet\bullet}$ is such a morphism.

The proof of these results is in [16]. In fact, once the right statements are known, it is not hard to verify them by elementary and standard diagram filling. The results show clearly the similarity between the construction of $\mathcal{C}^\bullet$ and the classical duality of Abelian groups, with their canonical inclusion $G \subseteq \hat{\hat{G}}$. However, by generalizing this Pontryagin duality to monoidal categories

we extend it to non-Abelian groups also (as well as beyond these to include Hopf algebra duality). Many constructions for groups and Hopf algebras can be extended in the same way to the setting of monoidal categories over a base. An example of such a result, when the base is braided, is

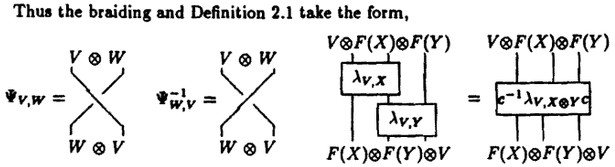
Theorem 2.4 *[16] Given $(\mathcal{C}, F, \mathcal{V})$ with $\mathcal{V}$ braided monoidal in the sense of [7], there is a functor $\alpha : \mathcal{C} \times \mathcal{C}^\circ \to \mathcal{C}^\circ$ (compatible with F, F°) with isomorphisms of the form $\alpha(X, \alpha(Y, \ )) \cong \alpha(X \otimes Y, \ )$ for all X, Y in $\mathcal{C}$, i.e. the two functors*

$$\alpha(\mathrm{id} \times \alpha), \alpha(\otimes \times \mathrm{id}) : \mathcal{C} \times \mathcal{C} \times \mathcal{C}^\circ \to \mathcal{C}^\circ$$

are naturally equivalent. We call α the left coadjoint action *of $\mathcal{C}$ on $\mathcal{C}^\circ$. It is given tautologically by $\alpha(X, (V, \lambda_V)) = (F(X) \otimes V, \lambda_{\alpha(X,(V,\lambda_V))})$ with*

$$\lambda_{\alpha(X,(V,\lambda_V)),Y} = (\mathrm{id} \otimes \lambda_{V,X})(\lambda_{V,Z} \otimes \mathrm{id})(\mathrm{id} \otimes \Psi_{V,F(Y)})(\lambda^{-1}_{V,X} \otimes \mathrm{id}).$$

Proof This (and similar proofs) were omitted in [16] for lack of space. We give it now in detail as a demonstration of the new techniques needed when working with monoidal categories in this way. Recall first that a braided monoidal category $\mathcal{V}$ is a monoidal one equipped with a braiding or 'quasisymmetry' of $\Psi_{V,W} : V \otimes W \to W \otimes V$ obeying two hexagon conditions and compatible with $\underline{1}$. We do not assume $\Psi_{W,V}\Psi_{V,W} = \mathrm{id}$. Omitting the associativity, the hexagons take the form $\Psi_{V,W\otimes U} = \Psi_{V,U}\Psi_{V,W}$ and $\Psi_{V\otimes W,U} = \Psi_{V,U}\Psi_{W,U}$. This is well known by now, as is the resulting coherence theorem for braided categories[7]. A useful technique for our proofs is a diagrammatic one. In this, all morphisms are written pointing downwards, Ψ and Ψ^{-1} are written as braid crossings and other morphisms are written in boxes or as nodes. Functoriality of Ψ, Ψ^{-1} says that such boxes can be translated through braid crossings. Recently, such short-hand notation has been given an ambient topological interpretation in [8]. Thus the braiding and Definition 2.1 take the form,



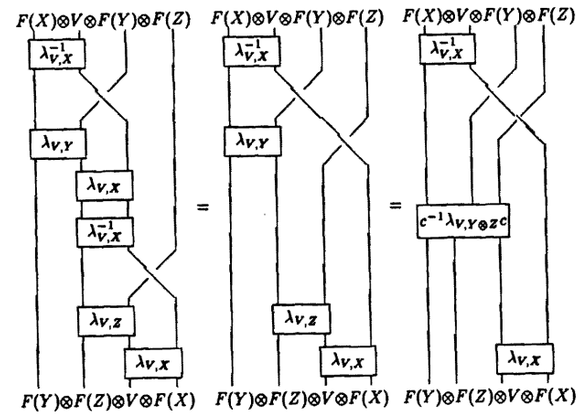


Figure 1: Proof that coadjoint action $\mathcal{C} \times \mathcal{C}^\circ \to \mathcal{C}^\circ$ is defined

The proof that $\alpha(X, (V, \lambda_V))$ obeys Definition 2.1 is then easily seen from the diagram in Figure 1. The upper three boxes on the left are the definition of $\lambda_{\alpha(X,(V,\lambda_V)),Y}$ and the lower three of $\lambda_{\alpha(X,(V,\lambda_V)),Z}$. After cancelling inverses and using the braid relation for Ψ we obtain $c^{-1}\lambda_{\alpha(X,(V,\lambda_V)),Y\otimes Z}c$ as required. The action of α on morphisms is the obvious one given by $F \otimes \mathrm{id}$, with its result viewed as a morphism in $\mathcal{C}^\circ$. The proof that this functor α is a left action in the form stated is more complicated and shown in Figure 2. On the left is the definition of

$$\lambda_{\alpha(X,\alpha(Y,(V,\lambda_V))),Z} = \lambda_{\alpha(Y,(V,\lambda_V)),X}\lambda_{\alpha(Y,(V,\lambda_V)),Z}\Psi_{F(X),F(Z)}\lambda^{-1}_{\alpha(Y,(V,\lambda_V)),X}$$

which, after cancellation, the braid relations and Definition 2.1 for $\lambda_{V,X\otimes Y}$ computes to $c^{-1}\lambda_{\alpha(X\otimes Y,(V,\lambda_V)),Z}c$. Thus the natural equivalence required in the theorem is induced by c. Functoriality follows at once from that of c. This completes the proof.

The ordinary coadjoint action plays a fundamental role in many applications of groups in physics (in the realm of quantization), so it is interesting that it generalizes to our setting. An application in category theory was given in [16] where we used it to construct a double cross product $\mathcal{C} \bowtie \mathcal{C}^\circ$ in the case

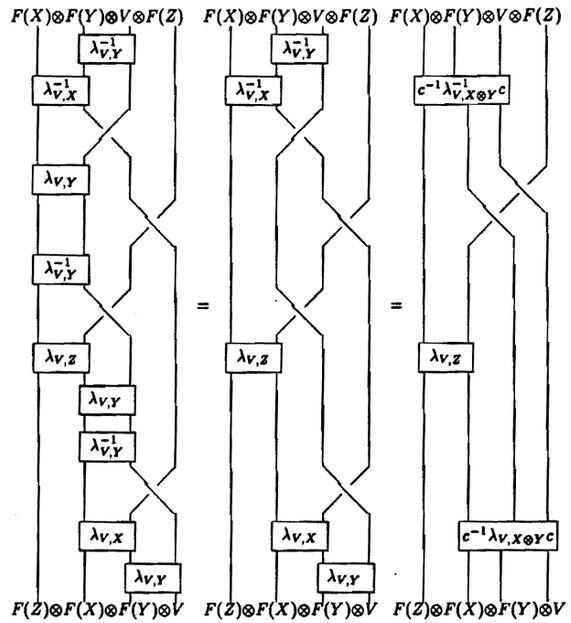


Figure 2: Proof that coadjoint action $\mathcal{C} \times \mathcal{C}^\circ \to \mathcal{C}^\circ$ is an action

when $\mathcal{C} \cong \mathcal{C}^{\circ\circ}$. Here there is an action $\beta : \mathcal{C} \times \mathcal{C}^\circ \to \mathcal{C}$ analogous to the above and $\mathcal{C}_\beta \bowtie_\alpha \mathcal{C}^\circ$ is a double semidirect product (along the lines of [22, Sec. 3.2] for groups and Hopf algebras) by these two mutual coadjoint actions simultaneously. When it exists, it is a monoidal category that non-trivially factorizes into the monoidal categories $\mathcal{C}$ and $\mathcal{C}^\circ$, and can be called the *outer quantum double* $D(\mathcal{C})$ of $\mathcal{C}$.

Returning now to the general situation, let us describe some results that do not have familiar group analogues, i.e. some genuinely new phenomena. One such canonical fact is that the dual in the special case $Z(\mathcal{C}) = (\mathcal{C}, \mathrm{id}, \mathcal{C})^\circ$ is tautologically braided (with $\Psi_{(V,\lambda_V),(W,\lambda_W)} = \lambda_{V,W}$). This has been observed

by Drinfeld[3] in another context, as well as independently in [9]. This $Z(\mathcal{C})$ can be called the *inner quantum double* and provides a way to generate braided monoidal categories from monoidal ones. We note also that our general $\mathcal{C}^\circ$ construction has subsequently been generalized still further in [10] in the setting of 2-categories. For a different type of result we observe also

Proposition 2.5 *Given $(\mathcal{C}, F, \mathcal{V})$ with $\mathcal{C}$ braided there is a monoidal functor $R : \mathcal{C} \to \mathcal{C}^\circ$ defined tautologically by*

$$R(X) = (F(X), \lambda_{R(X)}), \quad \lambda_{R(X),Y} = c^{-1}_{X,Y} F(\Psi_{X,Y}) c_{X,Y}, \qquad \forall X, Y \text{ in } \mathcal{C}.$$

Proof The output of R lies in $\mathcal{C}^\circ$ due to F applied to one of the hexagon identities for the braiding Ψ of $\mathcal{C}$ and is a monoidal functor due to F applied to the other. The other facts are equally easy.

3 BRAIDED GROUPS

We have said that the duality constructions above have been modeled on Hopf algebras, with $\mathcal{C}^\circ$ considered heuristically as the modules of some Hopf algebra whose multiplication corresponds to the monoidal product of $\mathcal{C}$. In nice cases this can be made precise, i.e. the underlying Hopf algebra for the data $F : \mathcal{C} \to \mathcal{V}$ actually exists and $\mathcal{C}^\circ$ can be identified with its modules. This is the main result of the present section. The necessary Hopf algebra (in $\mathcal{V}$) that we need here has already been introduced in [18], cf[17] and we begin by briefly recalling it.

Recall that an algebra (or monoid) in a monoidal category is an object A and a morphism $A \otimes A \to A$ which is associative, and has a suitable unit morphism $\mathbb{1} \to A$. A coalgebra (comonoid) is the same thing with the arrows reversed. It has a comultiplication $\Delta : A \to A \otimes A$ and counit $\epsilon : A \to \mathbb{1}$. A bialgebra (bimonoid) is usually defined in a symmetric monoidal category and has both of these structures with the additional condition that Δ, ϵ are algebra homomorphisms. Here $A \otimes A$ has the tensor product algebra structure $(A \otimes A) \otimes (A \otimes A) \to A \otimes A$ defined by applying first the symmetry $\Psi_{A,A}$ in the middle two factors and then multiplications for A. This definition can be adopted also in the case when Ψ is a braiding not a symmetry, i.e. $A \otimes A$ is still

associative. Bialgebras in braided monoidal categories were introduced in this way, and studied, in [17]. A Hopf algebra (Hopf monoid) is a bialgebra with an additional antipode morphism $S : A \to A$ playing the role of inversion. Finally, a (right) module of A is an object V and a morphism $V \otimes A \to V$ obeying the obvious axioms. A (right) comodule is an object V and a morphism $V \to V \otimes A$ obeying analogous axioms with the arrows reversed. It is clear that when the category is rigid, i.e. there are suitable dual objects, then A^* is also a Hopf algebra and comodules of A^* correspond to modules of A of appropriate (left or right) form.

Theorem 3.1 *[18], cf[17] Let $F : \mathcal{C} \to \mathcal{V}$ be as in Section 2, with $\mathcal{V}$ braided, rigid and cocomplete. Then there is a bialgebra $\mathrm{Aut}(\mathcal{C}, F, \mathcal{V})$ in $\mathcal{V}$, characterized as universal with the property that F factors through its category of right comodules in $\mathcal{V}$ via the forgetful functor. If $\mathcal{C}$ is also rigid then $\mathrm{Aut}(\mathcal{C}, F, \mathcal{V})$ is a Hopf algebra in $\mathcal{V}$.*

This $\mathrm{Aut}(\mathcal{C}, F, \mathcal{V})$ can be realized explicitly (cf [27][1]) as a representing object for a certain functor to Sets or, in the language of [11], the coend

$$\mathrm{Aut}(\mathcal{C}, F, \mathcal{V}) = \int^X F^*(X) \otimes F(X).$$

Here F^* is a contravariant functor provided by the rigidity in $\mathcal{V}$ and cocompleteness in needed for the existence of the coend to be assured. In another variant we can take $\mathcal{C}$ rigid instead of $\mathcal{V}$. Details are in [18].

The factorization of F in the theorem is via a monoidal functor $\mathcal{C} \to \mathcal{V}^{\mathrm{Aut}(\mathcal{C},F,\mathcal{V})}$ where $\mathcal{V}^{\mathrm{Aut}(\mathcal{C},F,\mathcal{V})}$ denotes the category of right $\mathrm{Aut}(\mathcal{C}, F, \mathcal{V})$ - comodules in $\mathcal{V}$. Thus $\mathcal{C}$ is mapped to the comodules of this underlying Hopf algebra. However, except in special cases (such as $\mathcal{V}$=Vec), this functor is not an equivalence. Thus we are not dealing here with a (braided version of) a full Tannaka-Krein reconstruction theorem as in [27][1][24]. Nevertheless, $\mathrm{Aut}(\mathcal{C}, F, \mathcal{V})$ exists and contains a lot of information about the system $(\mathcal{C}, F, \mathcal{V})$. Indeed, we now show

Theorem 3.2 *In the situation of Theorem 3.1, there is an isomorphism of monoidal categories*

$$\mathcal{C}^* \cong \mathcal{V}_{\mathrm{Aut}(\mathcal{C},F,\mathcal{V})}$$

where $\mathcal{V}_{\mathrm{Aut}(\mathcal{C},F,\mathcal{V})}$ denotes the category of right $\mathrm{Aut}(\mathcal{C}, F, \mathcal{V})$-modules in $\mathcal{V}$.

Proof Firstly, rigidity of $\mathcal{V}$ means more precisely that for every object V there is a dual V^* and morphisms $\pi_V : \underline{1} \to V \otimes V^*$, $\mathrm{ev}_V : V^* \otimes V \to \underline{1}$ obeying some axioms. Omitting the associativity morphisms etc as usual, we require that $(\mathrm{id} \otimes \mathrm{ev}_V)(\pi_V \otimes \mathrm{id})$ and $(\mathrm{ev}_V \otimes \mathrm{id})(\mathrm{id} \otimes \pi_V)$ compose to the identity on V and V^* respectively. Using these maps it is not hard to see that there is a bijection

$$\hom(W \otimes U^* \otimes T, V) \cong \hom(W \otimes T, U \otimes V) \qquad (1)$$

for any objects T, U, V, W in $\mathcal{V}$. The proof is similar to Lemma 2.1 of [18], for example if $f : W \otimes U^* \otimes T \to V$ is given, the corresponding morphism on the right is

$$W \otimes T \xrightarrow{\pi_U} W \otimes U \otimes U^* \otimes T \xrightarrow{\Psi_{W,U}} U \otimes W \otimes U^* \otimes T \xrightarrow{f} U \otimes V.$$

The automorphism bialgebra $A = \mathrm{Aut}(\mathcal{C}, F, \mathcal{V})$ was defined in [18], cf[17] as the representing object of a functor $V \mapsto \hom(F, F \otimes V)$, i.e. there is a bijection $\hom(\mathrm{Aut}(\mathcal{C}, F, \mathcal{V}), V) \cong \mathrm{Nat}(F, F \otimes V)$. We then used rigidity of $\mathcal{V}$ in [18] and an observation like (1) (with W trivial) to give the coend realization mentioned above. We now use (1) in a similar way to conclude that

$$\hom(V \otimes \mathrm{Aut}(\mathcal{C}, F, \mathcal{V}), V) \cong \mathrm{Nat}(V \otimes F, F \otimes V)$$

for all objects V. This gives the correspondence that we need for the theorem. Explicitly, if $\alpha : V \otimes A \to V$ is a morphism the corresponding natural transformation is $\lambda_{V,X} = (\mathrm{id} \otimes \alpha)\Psi_{V,F(X)}(\mathrm{id} \otimes \beta_X)$ where $\beta_X : F(X) \to F(X) \otimes A$ is the natural transformation in $\mathrm{Nat}(F, F \otimes A)$ corresponding to the identity morphism $A \to A$ (it is the A-comodule associated to X in Theorem 3.1). Figure 3 then shows that α a right module corresponds to λ_V a right $(\mathcal{C}, F, \mathcal{V})$-module in the sense of Definition 2.1. The left hand side is the definition of $c^{-1}\lambda_{V,X \otimes Y}c$ after using $c^{-1}\beta_{X \otimes Y}c = (\mathrm{id} \otimes \cdot)\Psi_{A,F(Y)}(\beta_X \otimes \beta_Y)$ (this is the definition of the multiplication in A from [18, Lemma 2.4]). The right hand side is $\lambda_{V,Y}\lambda_{V,X}$ and the two are equal as shown if (and only if) α is a right A-module. The proof can also be done by more conventional diagram-filling. This completes the proof of the theorem.

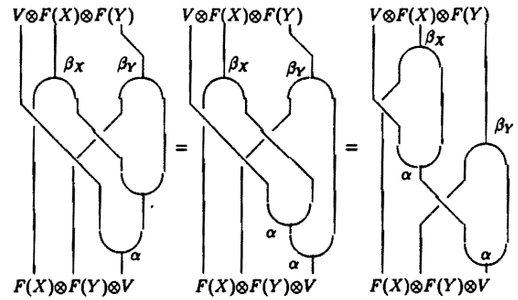


Figure 3: Diagram in proof of Theorem 3.2

This representation of $\mathcal{C}^*$ in terms of $\mathrm{Aut}(\mathcal{C}, F, \mathcal{V})$, has a number of corollaries for the work of Section 2, at least in these nice cases where $\mathrm{Aut}(\mathcal{C}, F, \mathcal{V})$ exists. For example,

Corollary 3.3 *Let $F : \mathcal{C} \to \mathcal{V}$ as in Section 2, with both $\mathcal{C}, \mathcal{V}$ rigid and $\mathcal{V}$ braided. Then $\mathcal{C}^*$ is rigid.*

Proof If $\mathrm{Aut}(\mathcal{C}, F, \mathcal{V})$ actually lives in $\mathcal{V}$ (rather than a cocompletion), it is a Hopf algebra in $\mathcal{V}$ and hence its category of modules in $\mathcal{V}$ is also rigid. This is an elementary feature of the category of modules, or of comodules, of a Hopf algebra in a rigid braided monoidal category. The case of comodules (in the braided setting) was spelled out in [18, Lemma 1.1]. More generally we can argue by continuity. In fact the necessary maps π and ev are induced by those in $\mathcal{V}$ (viewed now as morphisms in $\mathcal{C}^*$) and hence serve quite generally.

In addition, if $\mathcal{C}$ is also braided then $\mathrm{Aut}(\mathcal{C}, F, \mathcal{V})$ is a (dual) quantum group, i.e. a dual quasitriangular bialgebra or Hopf algebra in $\mathcal{V}$, with various consequences for the structure of $\mathcal{C}^*$. We will discuss quantum groups in detail in the next section, in the simplest algebraic setting, and refer to [18] for the general case. Suffice it to say that 'dual quasitriangular' means that there is a morphism $\mathcal{R} : \mathrm{Aut}(\mathcal{C}, F, \mathcal{V}) \otimes \mathrm{Aut}(\mathcal{C}, F, \mathcal{V}) \to \underline{1}$ and $\mathrm{Aut}(\mathcal{C}, F, \mathcal{V})$ is 'commutative' in a certain sense (in the braided category) up to conjugation by this in the relevant convolution algebra. In addition, the morphism $\mathcal{R}$ in suitable

conventions converts comodules to modules, along the lines of an extension of Proposition 2.5.

We mention that in general terms, the dual quasitriangular structure $\mathcal{R}$ of $\mathrm{Aut}(\mathcal{C}, F, \mathcal{V})$ measures the failure of F to respect the braidings: it is given by the ratio of the braidings in $\mathcal{C}$ and $\mathcal{V}$ and is trivial in the case $\mathrm{Aut}(\mathcal{C})$ where $\mathcal{V}$ is a cocompletion of $\mathcal{C}$ and F is the identity. This $\mathrm{Aut}(\mathcal{C})$ is therefore exactly 'commutative' in a certain braided sense and hence can be called a 'braided group' (in analogy with affine group schemes), while the general case can be called a 'quantum braided group'. See [18].

4 APPLICATION TO QUANTUM GROUPS

In this section we apply the category theoretic constructions above to obtain a result in the theory of quantum groups. Various applications of $\mathrm{Aut}(\mathcal{C}, F, \mathcal{V})$ and $\mathrm{Aut}(\mathcal{C})$ have already been made in this way in [19][20] as well as in [13][6]. Our new application is based on the connection with the duality of Section 2 as found in Theorem 3.2.

Briefly, an ordinary quantum group for us is a pair $(H, \mathcal{R})$ where H is a Hopf algebra over a field or commutative ring k, and $\mathcal{R} \in H \otimes H$ (the quasitriangular element) is invertible and obeys the axioms of Drinfeld[4]

$$\Delta^{op} = \mathcal{R}(\Delta\ )\mathcal{R}^{-1}, \quad (\mathrm{id}\otimes\Delta)\mathcal{R} = \mathcal{R}_{13}\mathcal{R}_{12}, \ (\Delta\otimes\mathrm{id})\mathcal{R} = \mathcal{R}_{13}\mathcal{R}_{23}.$$

Here Δ^{op} denotes Δ followed by the usual permutation. If G is a group, its group algebra kG is trivially a quantum group with $\mathcal{R} = 1\otimes 1$ (it is cocommutative). Quantum groups are important in mathematics and physics because they provide, in their representation theory, a source of braided monoidal categories. The monoidal product (tensor product) of two H-modules is obtained by pulling back along the comultiplication Δ, in the usual way. The braiding Ψ is provided by the action of $\mathcal{R}$ followed by usual permutation of the underlying vector spaces.

If H is any Hopf algebra, say finite dimensional for simplicity, Drinfeld showed how to construct from it a quantum group $D(H)$. Most other quantum groups are quotients of such quantum doubles, so this is one of the key con-

structions of the theory. In the conventions of [21] it coincides as a linear space and coalgebra with $H^*\otimes H$, but has a modified algebra structure

$$(a\otimes h)(b\otimes g) = \sum b_{(2)}a\otimes h_{(2)}g < Sh_{(1)}, b_{(1)} >< h_{(3)}, b_{(3)} >,\ h, g \in H,\ a, b \in H^*$$

where $\Delta h = \sum h_{(1)}\otimes h_{(2)}$ etc [29] and $<\ ,\ >$ denotes evaluation. The quasitriangular structure is $\mathcal{R} = \sum_a (f^a\otimes 1)\otimes(1\otimes e_a)$ where $\{e_a\}$ is a basis of H and $\{f^a\}$ a dual one. The structure of $D(H)$ is a double cross product $H^{*op}\bowtie H$ [22, Sec. 4]. This is an example of the factorization of a Hopf algebra into two factors H^{*op} and H. It is clear that $D(H)$-modules are vector spaces on which both H and H^{*op} act, in a compatible way, (or on which H acts and also coacts, in a compatible way, i.e. crossed H-modules as in [30]). Moreover, if $\mathcal{C}$ is the category of H-modules then $Z(\mathcal{C}) = (\mathcal{C}, \mathrm{id}, \mathcal{C})^\circ$ in Section 2 is the category of $D(H)$-modules[3].

Knowing this, a corollary of Proposition 2.5 for example is that if H is itself a quantum group, then there is a Hopf algebra projection $p : D(H) \to H$ inducing the functor $\mathcal{C} \to \mathcal{C}^\circ$. It can be easily be computed in terms of $\mathcal{R}$, and was the starting point of [21]. Now, as a corollary of Theorem 3.2 we have,

Corollary 4.1 *Let H be a finite dimensional quasitriangular Hopf algebra (quantum group) and let $\mathcal{C}$ be the category of finite-dimensional H-modules. Then*

$$D(H)\cong \mathrm{Aut}(\mathcal{C})\rtimes H$$

as a semidirect product algebra and semidirect product coalgebra by an action and coaction respectively of H.

Proof Identifying $\mathcal{C}^\circ$ as the category of $D(H)$-modules as explained, Theorem 3.2 now says that these correspond to $\mathrm{Aut}(\mathcal{C})$-modules in $\mathcal{C}$. This means vector spaces V on which $\mathrm{Aut}(\mathcal{C})$ acts H-equivariantly. Here H acts on both V and $\mathrm{Aut}(\mathcal{C})$ as objects in the category. It is clear that such V are the same thing as $\mathrm{Aut}(\mathcal{C})\rtimes H$-modules where the semidirect product is by the canonical action of H on $\mathrm{Aut}(\mathcal{C})$. This establishes a correspondence between $D(H)$-modules and $\mathrm{Aut}(\mathcal{C})\rtimes H$-modules, which one can easily see to be monoidal and compatible with the forgetful functors to Vec (i.e. it does not change the underlying vector space). As a result, by standard Tannaka-Krein ideas $\mathrm{Aut}(\mathcal{C})\rtimes H$ is a Hopf

algebra and isomorphic to $D(H)$. The coalgebra structure here comes out as a semidirect coproduct by the H-comodule structure induced by a functor from H-modules to H^*-modules (given by pull back along $\mathcal{R}$ as a map $H^* \to H$).

Let us note that the explicit form of $\mathrm{Aut}(\mathcal{C})$ in this example is known from [18]. Writing $A = H^*$ so that $\mathcal{C}$ is the category of A-comodules, we have from [18] that $\mathrm{Aut}(\mathcal{C}) = \underline{A}$, realised on the same linear space and coalgebra as H^* but with a modified multiplication and a non-trivial H-module structure. Explicitly,

$$(\Delta\phi)(h\otimes g) = \phi(hg), \quad (g.\phi)(h) = \sum \phi((Sg_{(1)})hg_{(2)})$$

$$(\phi\psi)(h) = \sum (\mathcal{R}^{(2)}.\psi)(h_{(1)})\,\phi(h_{(2)}\mathcal{R}^{(1)})$$

for $h, g \in H$ and $\phi, \psi \in H^*$ and $\mathcal{R} = \sum \mathcal{R}^{(1)}\otimes\mathcal{R}^{(2)}$. This gives an algebra and coalgebra but not a Hopf algebra in the ordinary sense. This explains the structure of $D(H)$ found by algebraic means in [21]. There, using a theorem of Radford[25] it was shown that $D(H)\cong B\rtimes H$ where B was a mysterious algebra-coalgebra. We recognize it now as a Hopf algebra in a braided category. After this we can go further and express Radford's theorem itself in this language. It then reads: if $H_1 \to H$ is a Hopf algebra projection between ordinary Hopf algebras then there exists a bialgebra B in the category of $D(H)$-modules such that $H_1\cong B\rtimes H$ by the corresponding action and coaction of H.

Our Corollary 4.1 can also be viewed as an example of a general 'bosonization' theorem in [20]. Using ideas of transmutation mentioned in the introduction, this asserts that any Hopf algebra B in the braided monoidal category of H-modules of a quantum group H, can be converted by a tautological semidirect product to an ordinary Hopf algebra $B\rtimes H$. In these terms, Corollary 4.1 says that Drinfeld's celebrated quantum double $D(H)$ is the 'bosonization' of the braided group $\underline{A} = \mathrm{Aut}(\mathcal{C})$.

5 A VISION OF QUANTUM-GRAVITY

This section aims to indicate how the above duality considerations fit into an approach to the unification of quantum theory and gravity in [22][23]. Such

a unification is of course a fundamental problem in theoretical physics and I believe that category theory provides some fundamental insight into it.

The idea behind this application is very simple. Firstly, it should be stressed that theoretical physics is not so much concerned with 'what' is observed but with the question 'why is it observed', i.e. always seeks to explain structures in terms of still more fundamental structures or principles. This reductionist programme takes the naive view that there are indeed some fundamental laws of nature of which our experiments and observations are representations. Thus it is supposed that something is absolutely true, and that something else measures or observes it.

However, every mathematician knows that such evaluations should generally be thought of more symmetrically as a 'pairing' of one structure with another. An evaluation $f(x)$ can also be read $x(f)$ where f is an element of a dual structure. Since theoretical physics adopts the language of mathematics, such an 'observer-observed' reversed interpretation of the mathematical structure can always be forced, but will the dual interpretation also describe physics? The idea behind [22][23] was that the answer should be 'Yes', i.e. that a fundamental theory of physics is incomplete unless such a role-reversal is possible. We can go further and hope to fully determine the (supposed) structure of fundamental laws of nature among all mathematical stuctures by this self-duality condition. See [23] and the introduction of [22].

Certainly, such duality considerations in some form are evident in the context of quantum theory and gravity (and may also be relevant to recent developments of duality in string theory). The situation is summarized to the left in Figure 4. For example, Lie groups provide the simplest examples of Riemannian geometry, while the representations of related Lie groups provide the quantum numbers of elementary particles in quantum theory. Thus both quantum theory and non-Euclidean geometry are needed for a self-dual picture. Now, Hopf algebras (in the form of Hopf-von Neumann or Kac algebras) precisely serve to unify these mutually dual structures. Motivated by this we described in [23] systems of particles on homogeneous spaces where this was achieved. The algebra of observables was indeed a Hopf (von Neumann) algebra of self-dual

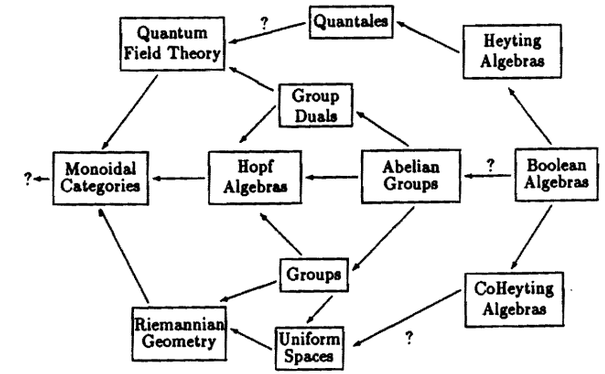


Figure 4: Representation-theoretic approach to quantum-gravity

type provided the homogeneous space and its dual one were suitably matched. The dual von Neumann algebra is built on the Banach space predual, so that roughly speaking, the role of obervables and states in the quantum theory can be interchanged. For example, the quantum (von Neumann) algebra of observables of a $1+1$-dimensional differentiable dynamical system was shown to be a self-dual Hopf algebra if and only if the effective dynamics was that of a particle moving in the background of a 2-dimensional black hole-type metric.

The duality considerations of Section 2 and [16] are then motivated by the question of what is the right mathematical structure to unify more complex quantum and gravitational systems. We have established that (small) monoidal categories over a base make up a self-dual category, with the duality $\mathcal{C} \mapsto \mathcal{C}^*$ covering Hopf algebra duality (which in turn covers the duality operation on Abelian groups), and that one can begin to do physics in this setting (for example, in the form of the coadjoint action in Section 2). It should be mentioned here that the functors in Figure 4 are intended to be of the nature of embeddings and to respect the relevant duality functors. Moreover, the collection of monoidal categories over a base is large enough to receive various functors from both quantum systems (as in [2], as well as the modular functor in confor-

mal field theory) and from Riemannian geometry. Precisely which functors we should take here and the search for self-dual objects are topics for ongoing research. If they can be found then such self-dual objects would be interpretable as (possibly generalized) quantum theories and (possibly generalized) geometries, simultaneously. This is our approach to quantum-gravity.

Finally, we note that in addition to the 'end' of theoretical physics in the form of quantum-gravity, we can ask also about its 'birth'. We take the view that the simplest theories of physics are based on classical logic or, roughly speaking, Boolean algebras. Some physical (not mathematical) arguments in the last of [23] suggest that the relevant duality here is provided by complementation. The situation is summarized on the right in Figure 4. Boolean algebras are considered self-dual according to De Morgan's theorem. Going above the axis to Heyting algebras and beyond takes us into intuitionistic logic and presumabley embeds (ultimately) into an axiomatic framework of quantum field theory. Dual to this, below the axis, F.W. Lawvere and his school have shown that coHeyting algebras can be viewed as the 'birth' of geoemtry[12]. Specifically, it has been observed that in a coHeyting algebra one can view $\partial a = a \wedge \tilde{\ } a$ as the 'boundary' of a, with the relevant derivation property[12]

$$\partial(a \wedge b) = (\partial a) \wedge b \vee a \wedge (\partial b).$$

Here $\tilde{\ } a$ is the smallest element such that $a \vee \tilde{\ } a = 1$. To develop this geometrical interpretation of cointuitionistic logic to embed (ultimately) in metric spaces and Riemannian (or Lorentzian) geometry is a task for further work.

Finally, we can ask if such problems as the unification of quantum theory and gravity have a useful parallel for the unification of (generalized) intuitionistic and (generalized) cointuitionistic logical systems. For example, this might be achieved using suitable forms of Hopf algebras or in the more general setting of monoidal categories. We note in this context that monoidal structures are now used routinely in various models of logic since the fundamental work of [5], see [28]. This unification is an interesting question suggested by the above. I would like to note the recent work [26] as a concrete step in this direction.

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