Completing Einstein's Proof of $E = Mc^2$

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Abstract

It was believed that Einstein has provided a theoretical proof for the formula $E = mc^2$. It is shown that Einstein's proof is actually incomplete and therefore is not yet valid. A crucial step is his implicit assumption of treating light as a bundle of massless particles. However, the energy-stress tensor of massless particles may not be compatible with an electromagnetic energy-stress tensor. Thus, to complete Einstein's proof, it is necessary to show that the total energy of light includes also non-electromagnetic energy such that the notion of photons as massless particles is compatible with electromagnetism. Moreover, observation requires that the non-electromagnetic energy must be much smaller than the electromagnetic energy. This can be accomplished with the energy-stress tensor of the gravitational wave component that is accompanying the electromagnetic wave component. In conclusion, his implicit assumption is valid although it must go beyond the current theoretical framework of general relativity.

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1. Introduction.

In physics, the most famous formula is probably $E = mc^2$. Einstein himself has made clear that this formula must be understood in terms of energy conservation [1]. This formula means that there is energy related to a mass, but it does not means that, for any type of energy, there is a related mass [2]. Moreover, general relativity also makes it explicit that the gravity generated by mass and that by the electromagnetic energy are different as shown by the Riessner-Nordstrom metric [3]. Ironically, it is also this formula that many physicists do not understand properly [2, 4]. Since not every type of energy is equivalent to mass as should be expected from general relativity, the relationship between mass and energy is actually far more complicated than as commonly believed.

Perhaps, a root of such misunderstanding is related to the fact that the derivation of this formula [5] has not been fully understood. In Einstein's derivation, a crucial step is his implicit assumption of treating light as a bundle of massless particles. However, it was not clear that an electromagnetic energy-stress tensor is compatible with the energy-stress tensor of massless particles, although it is also traceless. Such an issue is valid since the divergence of an electromagnetic energy-stress tensor $\nabla^c T(E)_{cb}$ generates only the Lorentz force, whereas the divergence of a massive energy-stress tensor $\nabla^c T(m)_{cb}$ would generate the geodesic equation [6].

Thus, it is expected that the energy-stress of photons $T(L)_{ab}$ is

$$T(L)_{ab} = T(E)_{ab} + T(N)_{ab} \qquad \text{or} \qquad T(N)_{ab} = T(L)_{ab} - T(E)_{ab} \tag{1}$$

where $T(E)_{ab}$ and $T(N)_{ab}$ are respectively the electromagnetic energy-stress tensor and the non-electromagnetic energy-stress tensor. Based on the fact that the electromagnetic energy is dominating experimentally, the non-electromagnetic energy must be comparatively much smaller. Therefore, it is natural to assume as shown later that $T(N)_{ab}$ is in fact the gravitational energy-stress $T(g)_{ab}$.

2. A Field Equation for the Gravity of Light and the Accompanying Gravitational Wave.

Equation (1) suggest an equation,

$$G_{ab} \equiv R_{ab} - \frac{1}{2} g_{ab} R = KT(g)_{ab} = - K[T(E)_{ab} - T(L)_{ab}], \tag{2}$$

which is different from Einstein equation with an extra term $T(L)_{ab}$ with an anti-gravity coupling. However, equation (2) is similar to the modified Einstein equation,

$$G_{ab} \equiv R_{ab} - \frac{1}{2} g_{ab} R = - K[T(m)_{ab} - T(g)_{ab}], \tag{3}$$

which is necessitated by the Hulse-Taylor experiment [7, 8]. Both equations (2) and (3) have an anti-gravity coupling for the gravitational energy-stress $T(g)_{ab}$. Moreover, because of equation (3), the anti-gravity coupling for $T(g)_{ab}$ is necessary. It should be noted also that, for the energy-stress tensor $T(E)_{ab}$ of an electromagnetic wave, there is no physical solution, which satisfies the symmetry consideration, for an Einstein equation with a source term $T(E)_{ab}$ alone [9]. Thus, there is no alternative for equation (2).

Now the remaining question is whether (2) would produce a gravitational wave. However, we should address first whether an electromagnetic wave has an accompanying gravitational wave. The answer is affirmative because the electromagnetic energy is propagating in the allowed maximum speed in special relativity. Thus, the gravity due to the light energy should be distinct from that generated by massive matter [10, 11].

Since a field emitted from an energy density unit means a non-zero velocity relative to that unit, it is instructive to study the velocity addition. According to special relativity, the addition of velocities is as follows [5] :

$$u_x = \frac{\sqrt{1-v^2/c^2}}{1+u'_z v/c^2} u'_x, \quad u_y = \frac{\sqrt{1-v^2/c^2}}{1+u'_z v/c^2} u'_y, \quad \text{and} \quad u_z = \frac{u'_z + v}{1+u'_z v/c^2}, \tag{4}$$

where velocity v is in the z-direction, (u'_x, u'_y, u'_z) is a velocity in a system moving with velocity v, c is the light speed, $u_x = dx/dt$, $u_y = dy/dt$, and $u_z = dz/dt$. When v = c, independent of (u'_x, u'_y, u'_z) one has

$$u_x = 0, \quad u_y = 0, \quad \text{and} \quad u_z = c. \tag{5}$$

Thus, neither the direction nor the magnitude of the velocity c have been changed. This implies that, since c is the light speed in vacuum, nothing can be emitted from a light ray, and therefore no field can be generated outside the light ray. To be more specific, from a light ray, no gravitational field can be generated outside the ray although, accompanying the light ray, a gravitational field g_{ab} ($\neq \eta_{ab}$ the flat metric) is allowed within the ray. According to the principle of causality [11], this accompanying gravity g_{ab} should be a gravitational wave since an electromagnetic wave is the physical cause. This would put general relativity into a severe test for theoretical consistency. But, this examination would also have the benefit of knowing whether Einstein's implicit assumption in his proof for $E = mc^2$ is valid.

3. Verification of the Rectified Einstein Equation.

Now consider the case of plane wave $A_k(ct - z)$ as Einstein [5] did. First, consider the energy-stress tensor $T(L)_{ab}$ for photons. If a geodesic equation must be produced, for a monochromatic wave with frequency ω, the form of a photonic energy tensor should be similar to that of massive matter. Observationally, there is very little interaction, if any, among photons of the same ray. Theoretically, since photons travel in the velocity of light, there should not be any interaction (other than collision) among them. Therefore, the photonic energy tensor should be dust-like as follows:

$$T^{ab}(L) = \rho\, P^a P^b, \tag{6a}$$

where ρ is a scalar and is a function of u (= ct - z). In the units $c = \hbar = 1$, $P_t = \omega$. It has been obtained [10] that

$$\rho\,(u) = - A_m g^{mn} A_n \;\; \geq 0. \tag{6b}$$

Here, ρ (u) is related to gravity through g^{mn} . Since light intensity is proportional to the square of the wave amplitude, ρ which is Lorentz gauge invariant, can be considered as the density function of photons. Then

$$T_{ab} = -T(g)_{ab} = T(E)_{ab} - T(L)_{ab} = T(E)_{ab} + A_m g^{mn} A_n\, P_a\, P_b\,. \tag{6c}$$

Thus, $T_{ab}(P)$ has been derived completely from the electromagnetic wave A_k and metric g_{ab} .

Now, consider an electromagnetic plane-waves of circular polarization, propagating to the z-direction

$$A_x = \frac{1}{\sqrt{2}} A_0 \cos \omega u\,, \text{ and } \; A_y = \frac{1}{\sqrt{2}} A_0 \sin \omega u\,, \tag{7}$$

where A_0 is a constant. The rotational invariants with respect to the z-axis are constants. These invariants are: G_{tt} , R_{tt} , $T(E)_{tt}$, G, $(g_{xx} + g_{yy})$, g_{tz} , g_{tt} , g , and etc. It follows that [10, 11]

$$g_{xx} = -1 - C + B_\alpha \cos(\,\omega_1 u + \alpha\,), \qquad g_{yy} = -1 - C - B_\alpha \cos(\,\omega_1 u + \alpha\,), \tag{8}$$

and

$$g_{xy} = \pm B_\alpha \sin\,(\,\omega_1 u + \alpha\,),$$

where C and B_α are small constants, and $\omega_1 = 2\omega$. Thus, metric (8) is a circularly polarized wave with the same direction of polarization as the electromagnetic wave (7). On the other hand, one also has

$$G_{tt} = 2\omega^2 B_\alpha{}^2/G \geq \;0\,, \quad \text{where} \quad G = (1+C)^2 - B_\alpha{}^2 \;\; \geq \;\; 0\,, \tag{9}$$

but

$$T(E)_{tt} = \frac{1}{2G}\, \omega^2 A_0{}^2 (1 + C - B_\alpha \cos \alpha\,) > 0.$$

Thus, it is not possible to satisfy Einstein's equation because $T(E)_{tt}$ and G_{tt} have the same sign [6] . Thus, it is necessary to have a photonic energy-stress tensor.

As expected, this tensor $T_{ab}(L)$ enables a gravity solution for wave (7). According to eq. (2) and formula (6),

$$T_{tt} = -\frac{1}{G}\, \omega^2 A_0{}^2 B_\alpha \cos \alpha < 0\,, \qquad \text{since} \quad B_\alpha = \frac{K}{2} A_0{}^2 \cos \alpha\,. \tag{10}$$

$T(g)_{tt}$ is of order K. Note that, pure electromagnetic waves can exist since cos α = 0 is also possible.

To confirm the general validity of (2) further, consider a wave linearly polarized in the x-direction,

$$A_x = A_0 \cos \omega(ct - z) . \tag{11}$$

Then, one has

$$T_{tt} = \frac{g_{yy}}{G} \omega^2 A_0^2 \cos [2\omega (ct - z)] . \tag{12}$$

Since the gravitational component is not an independent wave, $T(g)_{tt}$ is allowed to be negative or positive. Eq. (12) implies $(g_{xx} + g_{yy})'$ to be of first order [10, 11] , and therefore its polarization has to be different.

It turns out that the solution is a linearly polarized gravitational wave and that, as expected, the time-average of $T(g)_{tt}$ is positive of order K [11]. From the viewpoint of physics, for an x-directional polarization, gravitational components related to the y-direction, remains the same. In other words,

$$g_{xy} = 0 , \quad \text{and} \quad g_{yy} = -1 . \tag{13a}$$

It follows [10, 11] that the general solution wave (12) is:

$$g_{xx} = 1 + C_1 - (K/2) A_0^2 \cos [2\omega (ct - z)] , \tag{13b}$$

and

$$g_{tt} = - g_{zz} = \sqrt{g / g_{xx}} , \tag{13c}$$

where C_1 is a constant and g is the determinant of the matrix g_{ab}. Note that he frequency ratio is the same as that of a circular polarization. However, there is no phase difference to control the amplitude of the gravitational wave.

However, if the term $T(L)_{ab}$ were absent, one would have a solution,

$$g_{xx} = 1 + C_1 - (K/4) A_0^2 \{2\omega^2 (ct - z)^2 + \cos [2\omega(ct - z)]\} + C_2 (ct - z) , \tag{14}$$

where C_1 and C_2 are constants. But solution (14) is invalid in physics since $(ct - z)^2$ grows very large as time goes by. This would "represent" the effects if special relativity were invalid, and the wave energy were equivalent to mass. This illustrates also that Einstein's notion of weak gravity, which is the theoretical basis for his calculation on the bending of light, may not be compatible with an Einstein equation if its source term is inadequate.

4. Conclusions and Discussions

Now, a photonic energy-stress tensor has been obtained to satisfy the demanding physical requirements. The energy and momentum of a photon is proportional to its frequency although, as expected from a classical theory, their relationship with the Planck constant h is not yet clear; and the photonic energy-stress tensor is a necessary source term in the

Einstein equation. As expected from special relativity, indeed, the gravity of an electromagnetic wave is an accompanying gravitational wave propagating with the same speed.[1)]

One might object on the ground that the coordinates in special relativity may not be valid as coordinates when gravity is considered. However, such an objection is not valid because it has been proven [12, 13] that the spatial coordinates are the same in special and general relativity because the space coordinates must have the Euclidean-like structure.[2)] For this case, even the time coordinate is the same since the plane wave satisfied the Maxwell equation in terms of both special and general relativity [14], and the light speed remains the same. Thus, as expected, special relativity and general relativity are consistent with each other.

In this analysis, it has been shown further from another viewpoint that the electromagnetic energy is distinct from the energy of a rest mass. Interestingly, *it is precisely because of this non-equivalence of mass and energy that photonic energy tensor (6a) is valid, and the formula $E = mc^2$ can be proven.* One may argue that experiment shows the notion of massless photons is valid. However, it is necessary to reconcile the incompatibility between the energy-stress of massless particles and the energy-stress tensor of electromagnetism. The addition of two massless particles may end up with a rest mass, but the energy-stress tensor of electromagnetism cannot represent a rest mass [2].

Both quantum theory and relativity are based on the phenomena of light. The gravity of photons finally shows that there is a link between them. It is gravity that makes the notion of photons compatible with electromagnetic waves. Now, it is clear that gravity is no longer just a macroscopic phenomena as many believed, but also a microscopic phenomena of crucial importance to the formula $E = mc^2$. Moreover, it is also clear that the gravity due to the light is negligible in calculating the light bending. In this paper, the main issue is the crucial role of the gravity of an electromagnetic wave in the proof of $E = mc^2$, and Einstein's proof of $E = mc^2$ is completed.

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Endnotes

1) Although some parts of this paper have been presented in the literature [6], they are included here for the convenience of the readers.
2) Einstein called this structure as "in the sense of Euclidean geometry, but failed to understand its physical meaning in terms of measurements.

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