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An economical means of generating the mass ratios of the quarks and leptons (Rev.)

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Abstract

A mass formula that generates the mass ratios of the quarks and leptons is described. The formula succeeds by exploiting a symmetry among the Fibonacci numbers that helps explain why there are no more than three particle families.

I. A Formula for the Fine Structure Constant Inverse

The fine structure constant inverse $\frac{1}{\alpha}$ can be approximated closely with the aid of the constants 10 and 3

$$\frac{1}{\alpha} \approx \frac{10^3}{3^3} + 10^2 \quad ,$$

$$\approx 137.037037... \quad , \tag{1}$$

where the 2002 CODATA value for $\frac{1}{\alpha}$ equals 137.03599911 (46) [1]. The effectiveness of this approximation lends support to the conjecture that the constants 10 and 3, which will be employed later in a mass formula, are fundamental constants of nature.

Of course, one could plausibly object that the above approximation achieves its close fit of $\frac{1}{\alpha}$ by coincidence, and that other approximations of the same form might achieve a better fit while employing even smaller integers.

To resolve this issue, a computer searched for a better approximation of $\frac{1}{\alpha}$ in the form

$$\frac{A^a}{B^b}+C^c \ ,$$

where the exponents *a*, *b*, and *c* were integers arbitrarily allowed to range from 0 to 5, inclusive, and *A*, *B*, and *C* were integers allowed to range from 1 to 10, inclusive. Across these ranges no better approximation was found.

As it is, to find a better approximation requires that *A*, *B*, and *C* be allowed to range up to **37**, as follows

$$28^3 \ / \ \mathbf{37}^2 + 11^2 = 137.0350620... \ , \qquad (2)$$

with, once again, *a*, *b*, and *c* limited to between 0 and 5, inclusive. Accordingly, *for values of A, B, and C less than 37*, the best fit is achieved by the unusually small integers

$$A = C = 10 \ , \qquad (3a)$$

$$B = 3 \ , \qquad (3b)$$

which will appear as the key values in the mass formula to be introduced.

Additional suggestive results can be obtained by carrying out a search for a refined version of the approximation $\frac{10^3}{3^3}+10^2$, specifically one in the form

$$\frac{10^3 - D^d}{3^3} + 10^2 - E^e ,$$

where the exponents d and e are integers arbitrarily allowed to range from 0 to −3, inclusive, and D and E are integers arbitrarily allowed to range from 1 to 30, inclusive (a total of 4 × 4 × 30 × 30 = 14,400 possibilities). Within these restrictions the best fit of the experimental value of the fine structure constant inverse is provided when

$$D = E = 10$$

and

$$d = e = -3 \ ,$$

so that

$$D^d = E^e = 10^{-3} \ ,$$

and

$$\frac{1}{\alpha} \approx \frac{10^3 - 10^{-3}}{3^3} + 10^2 - 10^{-3} \ .$$

$$\approx \frac{999.999}{3^3} + 99.999 ,$$

$$\approx 137.036 . \tag{4}$$

Remarkably, the integer 10 now occurs no less than four times, while reproducing exactly the celebrated 137.036, which fits the fine structure constant inverse to within 6.5 parts per billion [1]. This four-fold repetition of 10 is further evidence that the constants 10 and 3 are fundamental constants of nature.

II. A Mass Formula for Heavy Quarks and Leptons

We begin by exploiting the terms A and B of Eqs. (3a) and (3b) to define the mass formula

$$R(j,k) = \left(1 + \frac{1}{A} + B\right)^j \times A^k \times B$$

$$= 4.1^j \times 10^k \times 3 . \tag{5}$$

Note that the absence of a physical understanding of the nature of mass precludes explaining why this equation can generate key mass ratios. For the same reason, it cannot now be put in its canonical form. As it is, it follows the general method the author has used elsewhere [2,3,4,5,6].

The above formula now allows the mass ratios

$\frac{M_{tau}}{M_{electron}}$	$\frac{M_{top}}{M_{charmed}}$
$\frac{M_{muon}}{M_{electron}}$	$\frac{M_{bottom}}{M_{charmed}}$

to be generated as follows

$R(F_5,0)=3475.68603$	$R(F_1,1)=123$
$R(F_4,0)=206.763$	$R(F_0,0)=3$

Note that the Fibonacci sequence extends in both directions and includes the terms

... -3 2 -1 1 **<u>0</u>** **<u>1</u>** **1** **2** **3** **5** ...

where each term equals the sum of the preceding two, and where 0 and 1 are the sequence initiators (underlined above). In the equations above, the values for F_n are taken from the Fibonacci sequence's first six terms (in boldface). Accordingly,

$$F_0 = 0 \ ,$$

$$F_1 = 1 \ ,$$

$$F_4 = 3 \ ,$$

and,

$$F_5 = 5 \ .$$

III. Analysis of Results

The calculated values for $\dfrac{M_{top}}{M_{charmed}}$ and $\dfrac{M_{bottom}}{M_{charmed}}$ fit the roughly-known experimental quark mass ratios within, or close to, their broad limits of error. These experimental mass ratios are calculated below by choosing from the experimental values' upper or lower bounds in an effort to fit the calculated values. Experimentally, the t-quark's mass equals 172,700 ± 2,900 MeV [7]; while the b-quark's mass ranges from 4,100 to 4,400 MeV [8]; and the c-quark's mass ranges from 1,150 to 1,350 MeV [8]. It follows that

$$\frac{172{,}700\ \text{MeV} - 2{,}900\ \text{MeV}}{1350\ \text{MeV}} = 125.77...$$

and

$$\frac{4{,}100\ \text{MeV}}{1{,}350\ \text{MeV}} = 3.037... \ ,$$

which are in rough accord with their calculated values of 123.04... and 3.000... .

Furthermore, the calculated values for $\dfrac{M_{tau}}{M_{electron}}$ and $\dfrac{M_{muon}}{M_{electron}}$ fit their corresponding experimental values to roughly 1 part in 2,000, and 1 part in 40,000, respectively [8].

The above assignments now make it possible to produce mass ratios $\frac{M_p}{M_q}$ for particles p and q while using as a parameter, either $-(F_p - F_q)$,

$R(-(F_\tau - F_e),0) = 3475.68603$	$R(-(F_t - F_c),1) = 123$
$R(-(F_\mu - F_e),0) = 206.763$	$R(-(F_b - F_c),0) = 3$

or, with identical results, $F'_p - F'_q$:

$R(F'_\tau - F'_e,0) = 3475.68603$	$R(F'_t - F'_c,1) = 123$
$R(F'_\mu - F'_e,0) = 206.763$	$R(F'_b - F'_c,0) = 3$

That this dual solution exists is remarkable. It is also noteworthy that the quarks and leptons, which appear segregated in the upper table of parameter assignments (LLLQQQ), symmetrically interleave in the lower table (LQLQLQ). But, once we decide to exploit the integer sequences ***F*** and ***F'*** when making our parameter assignments, if the mass formula is to fit the mass data, then the sequences LLLQQQ and LQLQLQ are forced—no other order will yield parameters that fit the mass data. Accordingly, the $\frac{\text{LLLQQQ}}{\text{LQLQLQ}}$ symmetry of the above parameter assignments is not the result of choice, but a consequence of the experimental data.

Another important aspect of the above assignments is that they see to it that there is no "first", or "ground state", particle. The upper table (using sequence $\boldsymbol{F}$), and the lower table (using sequence $\boldsymbol{F}'$), can equally lay claim to being the best way to assign the mass formula parameters, and so the particles occur in no particular order. It would be undesirable to single out one particle as a "ground state", because the quark and lepton mass spectrum does not appear to furnish the indefinite number of "excited states" that should go with it.

Furthermore, because the parameters exploited by the mass formula

n	-4	-3	-2	-1	0	1
F_n	-3	2	-1	1	0	1
$F_n' \equiv F_n \pmod 6$	3	2	5	1	0	1

are symmetrical in ways that cannot be extended to encompass more than the six Fibonacci terms of sequence $\boldsymbol{F}$, one can readily exploit the symmetries of $\boldsymbol{F}$ and $\boldsymbol{F}'$ to formulate conditions that automatically limit the number of heavy quarks and leptons. In fact, this was done in another article by the author, although the key symmetry exploited there did not involve modular arithmetic [2]. In this way, it may be possible to partially account for why there are no more than three particle families.

V. A Mass Formula for All Quarks and Leptons

In order to accommodate both heavy and light quarks and leptons within a single formula, it is only necessary to alter the mass formula slightly

$$R(j,k,m) = \left(1 + \frac{1}{A} + B\right)^{\frac{j}{m}} \times A^k \times B^{\frac{1}{m}}$$

$$= 4.1^{\frac{j}{m}} \times 10^k \times 3^{\frac{1}{m}} \quad . \tag{6}$$

We let $k = 1$ for the t- and s-quarks, and 0 for all other particles. In addition, we carry out the following assignments for F_p

Heavy Particles $m = 1$	τ	e	μ	b	t	c
Light Particles $m = 2$	υ_3	υ_1	υ_2	d	s	u
F_p	−3	2	−1	1	0	1

and, we also assign values for F_p', with the particles swapping places as before.

Heavy Particles $m = 1$	μ	t	τ	c	e	b
Light Particles $m = 2$	υ_2	s	υ_3	u	υ_1	d
F_p'	3	2	5	1	0	1

Note that for the above parameter assignments the heavy particles are paired with light particles in a natural way, with all pairings governed by mass. So, the heaviest heavy quark (t) is paired with the heaviest light quark (s); the lightest heavy quark (c) is paired with the lightest light quark (u); and so on. Also note that the table assigns $m = 1$ for all heavy particles, and $m = 2$ for all light particles

On the one hand, the assignments in the above table are the same for the heavy particles as those used earlier, and, consequently, with m equal to 1, the mass ratios generated by Eq. (6) are the same as before.

On the other hand, for the light quarks and leptons, with m equal to 2, the mass ratios generated by Eq. (6) are either

$m = 2$	
$R\left(-\left(F_{\nu_3} - F_{\nu_1}\right), 0, m\right) = 58.95...$	$R\left(-\left(F_s - F_u\right), 1, m\right) = 35.07...$
$R\left(-\left(F_{\nu_2} - F_{\nu_1}\right), 0, m\right) = 14.37...$	$R\left(-\left(F_d - F_u\right), 0, m\right) = 1.73...$

or, identically,

$m = 2$	
$R\left(F'_{\nu_3} - F'_{\nu_1}, 0, m\right) = 58.95...$	$R\left(F'_s - F'_u, 1, m\right) = 35.07...$
$R\left(F'_{\nu_2} - F'_{\nu_1}, 0, m\right) = 14.37...$	$R\left(F'_d - F'_u, 0.m\right) = 1.73...$

Below these light quark mass ratios are compared against their experimental values, where it is seen that they are in agreement [8].

Mass Ratio	*Experimental Value*	*Calculated Value*
$\frac{M_u}{M_d}$	0.3 to 0.7	$\frac{1}{3^{\frac{1}{2}}} = 0.57735\ldots$
$\frac{M_s}{M_d}$	17 to 22	$\frac{4.1^{\frac{1}{2}}}{0.1} = 20.248\ldots$
$\frac{M_s}{(M_u + M_d)/2}$	25 to 30	25.674...
$\frac{M_s - \frac{M_d + M_u}{2}}{M_d - M_u}$	30 to 50	46.042...

VI. The Neutrino Squared-Mass Splittings

Equation (6) requires that the masses of the neutrino mass eigenstates occur in the following ratios

$$\sqrt{4.1^5 \times 3} : \sqrt{4.1^3 \times 3} : \sqrt{1} \ ,$$

and that the neutrino squared-mass splittings, in turn, fulfill the following ratios

$$\left(4.1^5 \times 3 - 1\right) : \left(4.1^5 \times 3 - 4.1^3 \times 3\right) : \left(4.1^3 \times 3 - 1\right) \ .$$

It follows that

$$\frac{\left|M(\nu_3)^2 - M(\nu_1)^2\right|}{\left|M(\nu_2)^2 - M(\nu_1)^2\right|} = \frac{4.1^5 \times 3 - 1}{4.1^3 \times 3 - 1} = 16.8868\ldots \;, \tag{7a}$$

and,

$$\frac{\left|M(\nu_3)^2 - M(\nu_2)^2\right|}{\left|M(\nu_2)^2 - M(\nu_1)^2\right|} = \frac{4.1^5 \times 3 - 4.1^3 \times 3}{4.1^3 \times 3 - 1} = 15.8868\ldots \;. \tag{7b}$$

Observational data exist for two neutrino squared-mass splittings, namely [8]

$$1.5 \times 10^{-3}\ \Delta\mathrm{eV}^2 < \left|M(\nu_\mu)^2 - M(\nu_x)^2\right| < 3.9 \times 10^{-3}\ \Delta\mathrm{eV}^2$$

and [9]

$$\left|M(\nu_e)^2 - M(\nu_x)^2\right| = 7.1 \times 10^{-5}\,{}^{+1.2}_{-0.6}\ \Delta\mathrm{eV}^2 \;.$$

As this second neutrino squared-mass splitting is the more precisely-known of the two, it may be used as a starting point to calculate $\left|M(\nu_\mu)^2 - M(\nu_x)^2\right|$, as well as the remaining unknown neutrino squared-mass splitting:

$$\frac{4.1^5 \times 3 - 1}{4.1^3 \times 3 - 1} \times 7.1 \times 10^{-5}\,{}^{+1.2}_{-0.6}\ \Delta \mathrm{eV}^2 \approx 1.19^{+0.2}_{-0.1} \times 10^{-3}\ \Delta \mathrm{eV}^2\ , \tag{8a}$$

and,

$$\frac{4.1^5 \times 3 - 4.1^3 \times 3}{4.1^3 \times 3 - 1} \times 7.1 \times 10^{-5}\,{}^{+1.2}_{-0.6}\ \Delta \mathrm{eV}^2 \approx 1.12^{+0.2}_{-0.1} \times 10^{-3}\ \Delta \mathrm{eV}^2\ . \tag{8b}$$

These predictions offer an opportunity to test the mass formulae's validity, especially as the Eq. (8a)'s value for $\left|M(\nu_\mu)^2 - M(\nu_x)^2\right|$ is predicted to be slightly below its experimental value. The predicted values for the neutrino mass eigenstates are discussed in greater detail elsewhere by the author [2].

VII. Summary and Conclusion

In summary, in this article four mass ratios between quarks, and another four mass ratios between leptons, are generated by a mass formula that is formed by the product of powers of constants that are derived from the fine structure constant (10 and 3), where this mass formula takes its parameters from a portion of the Fibonacci sequence. More exactly, the mass formula exploits two sets of Fibonacci-related parameters, which possess a symmetry that may be exploited to furnish at least a partial explanation of why there are no more than three particle families. The details of how one might use the above Fibonacci-related symmetry to automatically limit the number of particle families to three is only hinted at above. But elsewhere the author analyzes in detail a symmetrical pair of mass formulae that are demonstrated to give consistent values for mass only when there are three or fewer particle families [2].

Also, the appendix offers what is by far the most symmetrical way of linking the values F and F' to the quark and lepton charges Q. This method also appears to imply that no more than three particle families are possible, as it employs the twelve edges of a Rubik's ® cube in making its highly symmetrical parameter assignments.

Finally, the ease with which so many mass ratios can be approximated by the mass formula, and the symmetry, economy, and duality of its parameter specification, inevitably raises the question of why it should be so successful.

References

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Appendix

By far the most symmetrical way of linking the values F or F' to the quark and lepton charges Q exploits the twelve edges of a 3 x 3 x 3 "*Rubik's*" ® cube.

We begin by noting that a Rubik's cube possesses twelve *edge-subcubes*, as is seen in Figure 1.

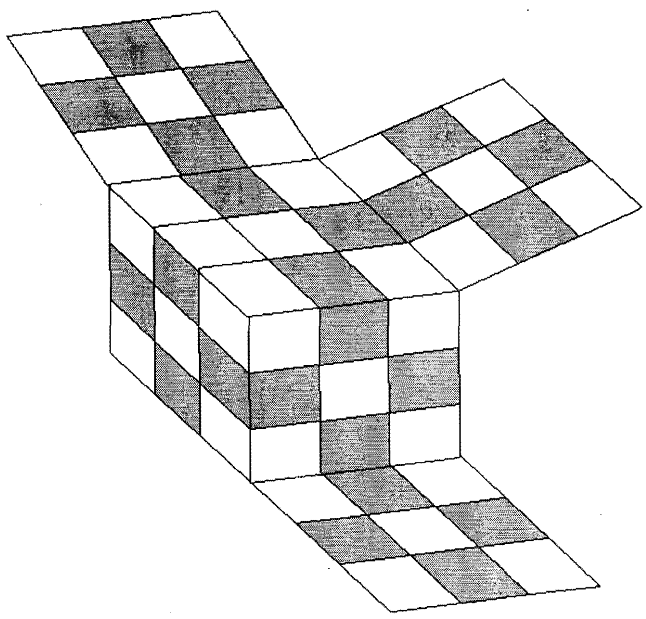


Figure 1. The 12 *edge-subcubes* of the 3 x 3 x 3 Rubik's Cube.

It also possesses 9 distinct *slices*, as is seen in Figure 2.

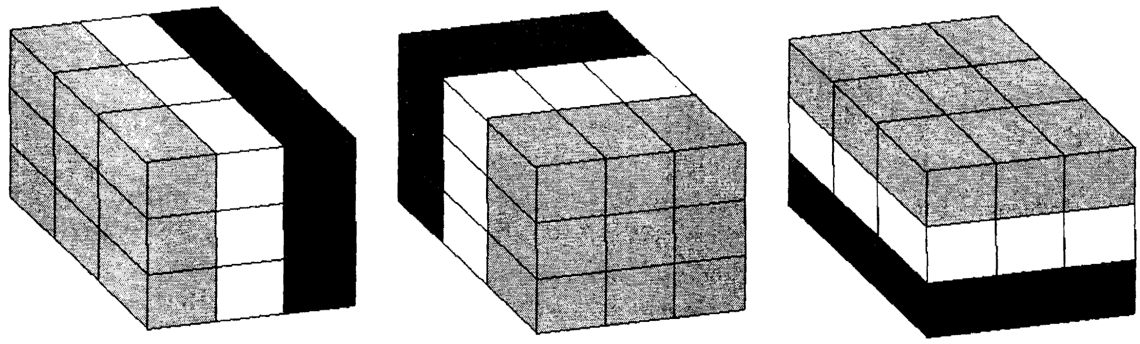


Figure 2. The 9 *slices* of the Rubik's Cube.

Lastly, it also possesses 6 pairs of *opposing edge-subcubes*, which are subcubes separated by the 180-degree rotation of a middle slice. Three of six opposing edge-subcubes appear in Figure 3.

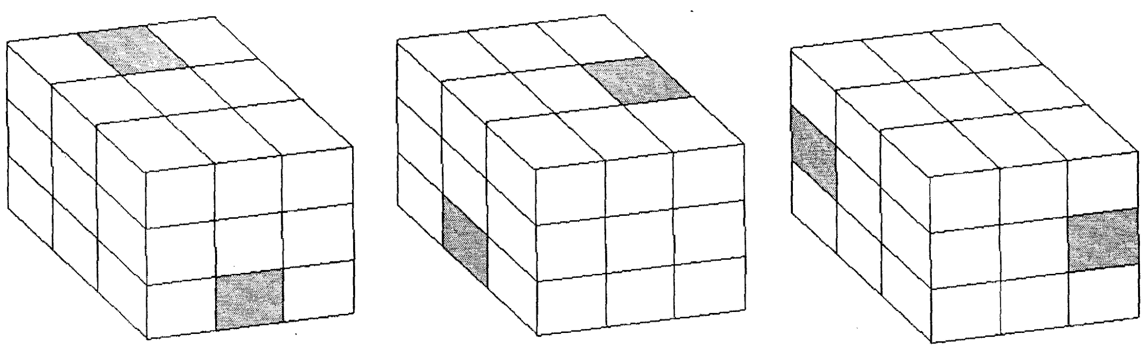


Figure 3. Examples of *opposing edge-subcubes*.

Now assume that the lowest unit of electric charge equals 1, the charge of the d-quark. Then the twelve quarks and leptons will possess the following charges: 0, 0, 0, 1, 1, 1, 2, 2, 2, 3, 3, 3. We assign these twelve values to the twelve edge-subcubes of the Rubik's cube in such a way that these values sum to 6 for all 9 of the cube's slices.

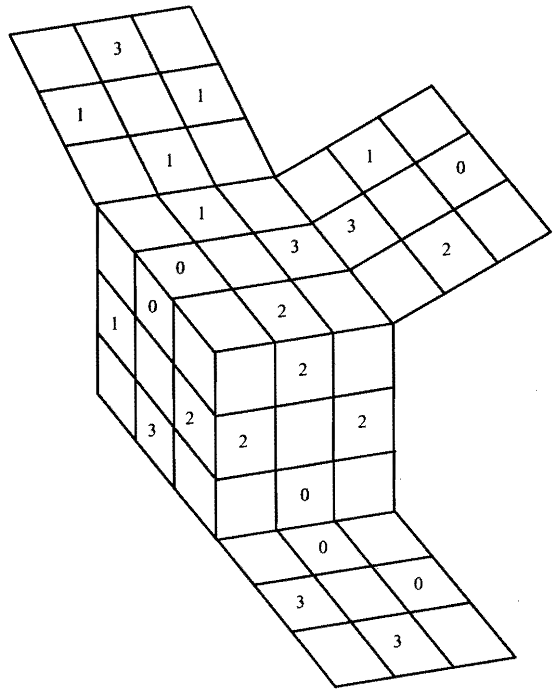
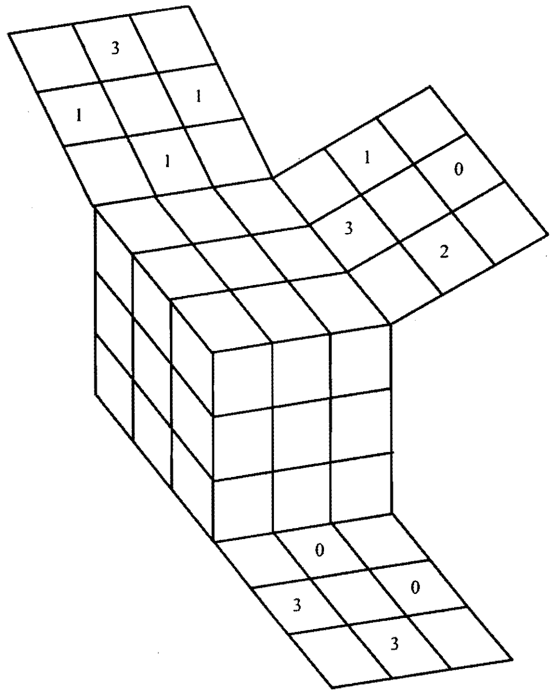


Figure 4. The arrangement of electric charge Q_p for the twelve quarks and leptons.

So, for the three slices made visible by "folding-out" the hidden faces of the cube



we have the sums:

$$1+3+1+1=6 \quad ,$$

and

$$1+0+2+3=6 \quad ,$$

and

$$0+0+3+3=6 \quad .$$

In the same way, the remaining 6 slices also contain values that sum to 6.

Now, note that the Rubik's cube also possesses 8 *corner-triads*, as shown in Figure 5.

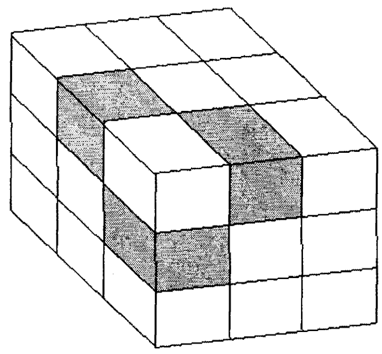


Figure 5. Example of one of the 8 *corner-triads* possessed by Rubik's cube.

We now assign the values

$$0, 1, 1, 2, 3, 5,$$

and

$$0, 1, 1, 2, 3, 5$$

to the twelve edge-subcubes of the cube in such a way that the cube's 8 corner-triads also sum to 6, as is seen in Figure 6. Note that the above values, which we will term the *mass charge* F'_p for each particle *p,* are merely the first six Fibonacci numbers repeated.

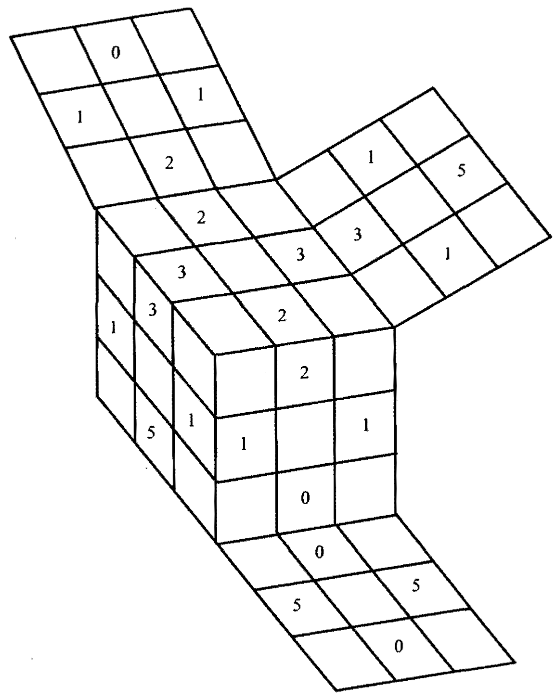


Figure 6. The mass charge F_p' for the twelve quarks and leptons.

So, beginning clockwise from the north-west corner, the four top corner-triads produce the sums

$$1+2+3=6 \quad ,$$

and

$$1+3+2=6 \quad ,$$

and

$$1+2+3=6 \quad ,$$

and

$$1+3+2=6 \quad .$$

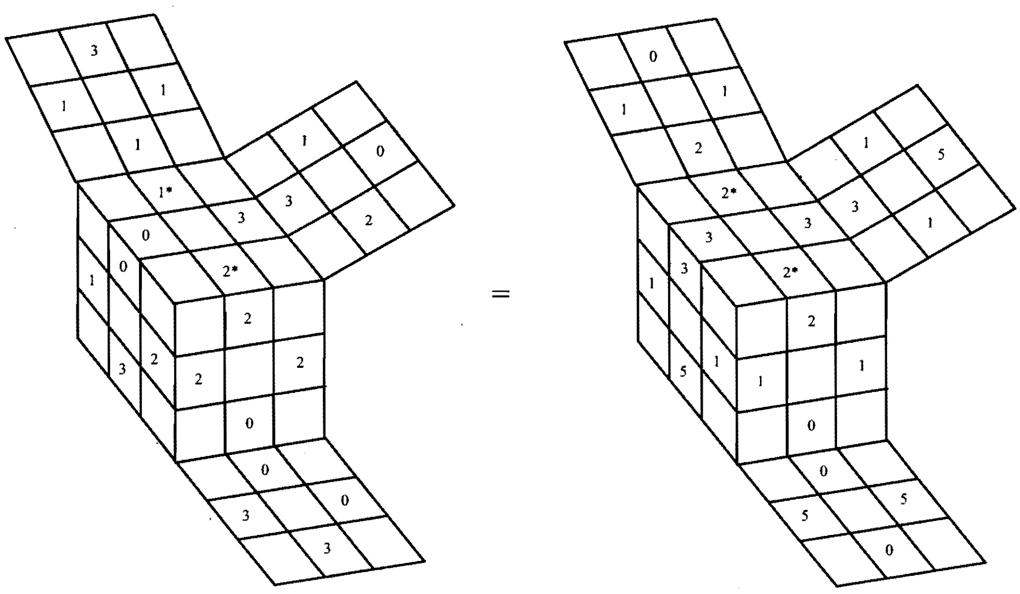
In the same way, the remaining four bottom corner-triads also possess values that sum to 6.

We now link the electric charge of Figure 4 to the mass charge of Figure 6 by direct assignment, as is seen in Figure 7. This assignment sees to it that the mass formula

$$\frac{M_p}{M_q} = 4.1^{\left(\frac{F'_p - F'_q}{m}\right)} \times 10^{k_p - k_q} \times 3^{\frac{1}{m}} \tag{A1}$$

will generate mass ratios consistent with those produced earlier, while each particle p is also assigned its proper electric charge Q_p.

Sequence: 0, 1, 2, 3 Sequence: 0, 1, 1, 2, 3, 5



Electric Charge Q_p Mass Charge F'_p

Figure 7. Electric charge linked to mass charge in such a way that accurate mass ratios for particles p and q are generated by $\dfrac{M_p}{M_q} = 4.1^{\left(\frac{F'_p - F'_q}{m}\right)} \times 10^{k_p - k_q} \times 3^{\frac{1}{m}}$. The value $k =$ 1 for the t- and s-quarks, which are marked by asterisks; k = 0, otherwise.

As particles that are heavy have a value for m of 1, while particles that are light have a value for m of 2, it is important that we be able to distinguish heavy particles from light in Figure 7. Of course, in Figure 7 we know that the heavy leptons have a charge of 3, and the light leptons have a charge of 0; but no such simple guide exists to tell heavy quarks from light. So, in order to make such identification possible, we adopt the following rule:

Rule: Particles occupying opposing edge-subcubes must be either both heavy, or both light.

This rule is sufficient to allow ready distinction of heavy particles from light in Figure 7. This also allows Figure 7 to generate mass ratios identical to those deduced earlier, namely, $\frac{M_{tau}}{M_{electron}}$, $\frac{M_{top}}{M_{charmed}}$, $\frac{M_{muon}}{M_{electron}}$, $\frac{M_{bottom}}{M_{charmed}}$, $\frac{M_{\nu_3}}{M_{\nu_1}}$, $\frac{M_{\nu_2}}{M_{\nu_1}}$, $\frac{M_{strange}}{M_{up}}$, and $\frac{M_{down}}{M_{up}}$, provided that one recognizes that $k = 1$ for the t- and s-quarks (marked by asterisks in Figure 7), and that $k = 0$ otherwise.

As before, the particle appearing in the denominator is either:

1) the lightest heavy quark (the charmed quark),
2) the lightest light quark (the up quark),
3) the lightest heavy lepton (the electron), or
4) the lightest neutrino mass eigenstate (υ_1).

Note that Figure 7 uses mass charge F'_p to generate the mass ratios. It is logical to ask whether identical results could be achieved using F_p, as was demonstrated earlier. We accomplish this first by assigning the mass charges

$$-3, 2, -1, 1, 0, 1,$$

and

$$-3, 2, -1, 1, 0, 1$$

to the twelve edge-subcubes of the cube in such a way that the cube's 8 corner-triads sum to 0, as is seen in Figure 8. The above values, of course, are the six Fibonacci numbers produced when the sequence is extended in the reverse of its normal direction.

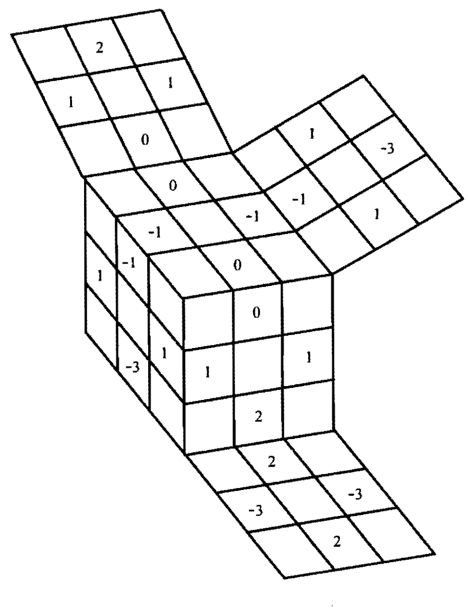
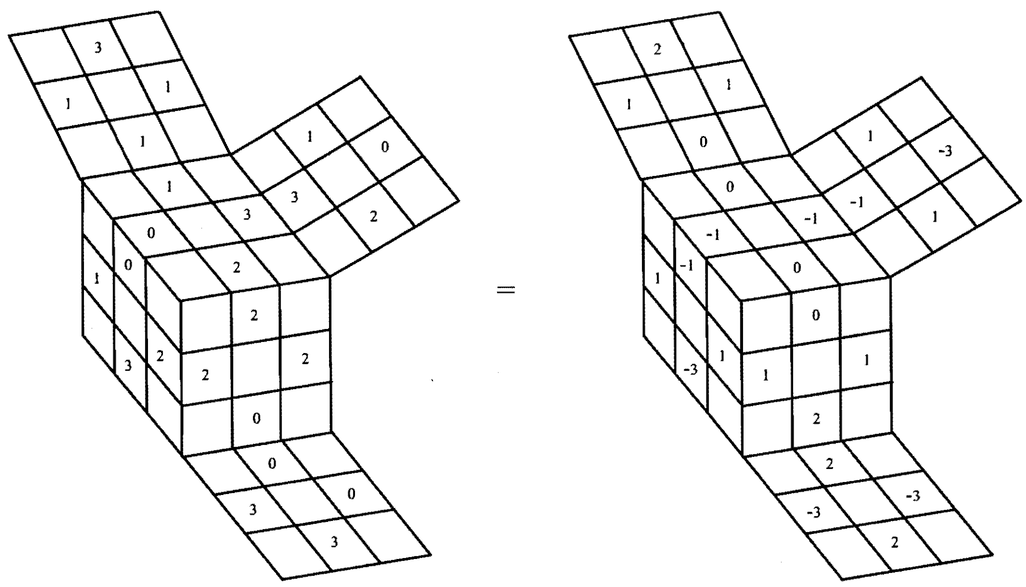


Figure 8. The mass charges F_p for the twelve quarks and leptons. Note that when the above values are divided modulo 6, and the six pairs of opposing edge-subcubes are swapped, the cube yielded is identical with that of Figure 6, which contains values for F'_p.

We now link the electric charge of Figure 4 to mass charge of Figure 8, again using direct assignment, as is seen in Figure 9.

Sequence: 0, 1, 2, 3   Sequence: −3, 2, −1, 1, 0, 1



Electric Charge Q_p Mass Charge F_p

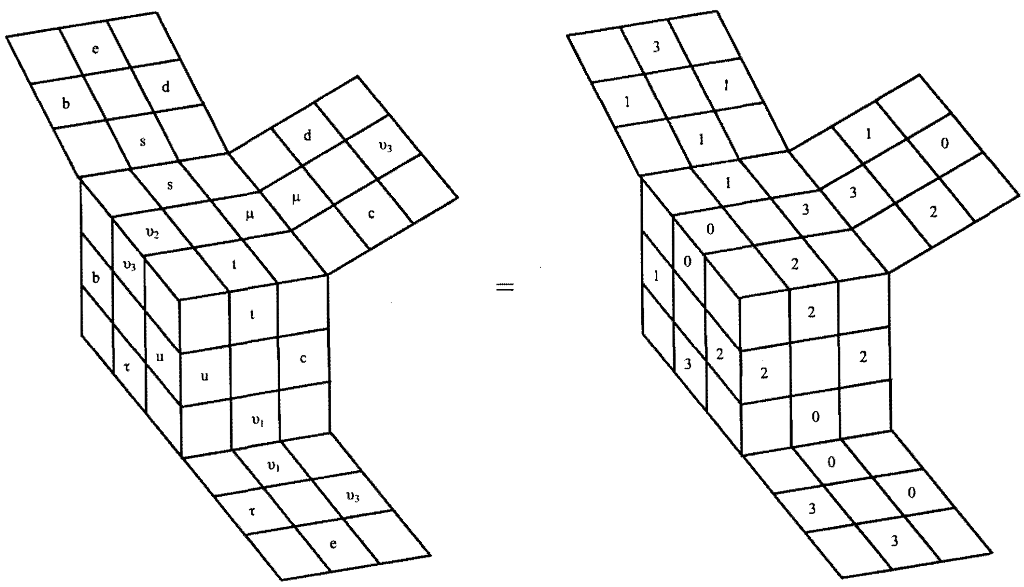
Figure 9. Electric charge linked to mass charge in such a way that accurate mass ratios for particles p and q are generated by $\frac{M_p}{M_q} = 4.1^{-\left(\frac{F_p - F_q}{m}\right)} \times 10^{k_p - k_q} \times 3^{\frac{1}{m}}$.

This assignment again sees to it that the appropriate particle mass will be generated for each quark and lepton, this time by

$$\frac{M_p}{M_q} = 4.1^{-\left(\frac{F_p - F_q}{m}\right)} \times 10^{k_p - k_q} \times 3^{\frac{1}{m}} \ . \qquad \text{(A2)}$$

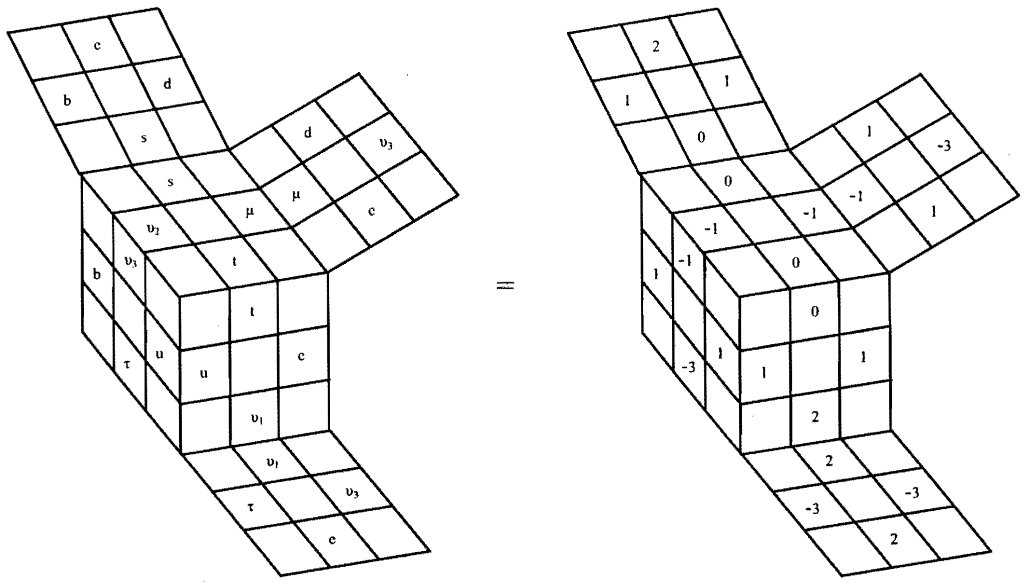
Notice that Eq. (A2) is identical with Eq. (A1), except that the sign for the exponent of 4.1 has been toggled. Furthermore, note that when Figure 9's values for F_p are divided modulo 6, and its six pairs of opposing edge-subcubes are swapped, the result is the mass charge assignments of Figure 7 for F'_p.

Finally, the actual charge assignments implicit in Figure 9 are summarized explicitly in Figures 10 and 11.



Particle p Electric Charge Q_p

Figure 10. Assignment of electric charge Q_p for quarks and leptons.



Particle p Mass Charge F_p

Figure 11. Assignment of mass charge F_p for quarks and leptons.