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An economical means of generating the mass ratios of the quarks and leptons

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Abstract

A mass formula that generates the mass ratios of the quarks and leptons is described. The formula succeeds by exploiting a symmetry among the Fibonacci numbers that helps explain why there are no more than three particle families.

I. A Formula for the Fine Structure Constant Inverse

The fine structure constant inverse $\frac{1}{\alpha}$ can be approximated closely with the aid of the constants 10 and 3

$$\frac{1}{\alpha} \approx \frac{10^3}{3^3} + 10^2 \quad ,$$

$$\approx 137.037037\ldots \quad , \tag{1}$$

where the 2002 CODATA value for $\frac{1}{\alpha}$ equals 137.03599911 (46) [1]. The effectiveness of this approximation lends support to the conjecture that the constants 10 and 3, which will be employed later in a mass formula, are fundamental constants of nature.

Of course, one could plausibly object that the above approximation achieves its close fit of $\frac{1}{\alpha}$ by coincidence, and that other approximations of the same form might achieve a better fit while employing even smaller integers.

To resolve this issue, a computer searched for a better approximation of $\frac{1}{\alpha}$ in the form

$$\frac{A^a}{B^b} + C^c \ ,$$

where the exponents a, b, and c were integers arbitrarily allowed to range from 0 to 5, inclusive, and A, B, and C were integers allowed to range from 1 to 10, inclusive. Across these ranges no better approximation was found.

As it is, to find a better approximation requires that A, B, and C be allowed to range up to **37**, as follows

`28^3 / `**`37`**`^2 + 11^2 = 137.0350620... ,` (2)

with, once again, a, b, and c limited to between 0 and 5, inclusive. Accordingly, *for values of A, B, and C less than 37*, the best fit is achieved by the unusually small integers

$$A = C = 10 \ , \tag{3a}$$

$$B = 3 \ , \tag{3b}$$

which will appear as the key values in the mass formula to be introduced.

Additional suggestive results can be obtained by carrying out a search for a refined version of the approximation $\frac{10^3}{3^3}+10^2$, specifically one in the form

$$\frac{10^3-D^d}{3^3}+10^2-E^e,$$

where the exponents d and e are integers arbitrarily allowed to range from 0 to −3, inclusive, and D and E are integers arbitrarily allowed to range from 1 to 30, inclusive (a total of 4 × 4 × 30 × 30 = 14,400 possibilities). Within these restrictions the best fit of the experimental value of the fine structure constant inverse is provided when

$$D=E=10$$

and

$$d=e=-3\quad,$$

so that

$$D^d=E^e=10^{-3}\quad,$$

and

$$\frac{1}{\alpha}\approx\frac{10^3-10^{-3}}{3^3}+10^2-10^{-3}\quad.$$

$$\approx \frac{999.999}{3^3} + 99.999 \ ,$$

$$\approx 137.036 \ . \tag{4}$$

Remarkably, the integer 10 now occurs no less than four times, while reproducing exactly the celebrated 137.036, which fits the fine structure constant inverse to within 6.5 parts per billion [1]. This four-fold repetition of 10 is further evidence that the constants 10 and 3 are fundamental constants of nature.

II. A Mass Formula for Heavy Quarks and Leptons

We begin by exploiting the terms A and B of Eqs. (3a) and (3b) to define the mass formula

$$R(j,k) = \left(1 + \frac{1}{A} + B\right)^j \times A^k \times B$$

$$= 4.1^j \times 10^k \times 3 \ . \tag{5}$$

Note that the absence of a physical understanding of the nature of mass precludes explaining why this equation can generate key mass ratios. For the same reason, it cannot now be put in its canonical form. As it is, it follows the general method the author has used elsewhere [2,3,4,5,6].

The above formula now allows the mass ratios

$\dfrac{M_{tau}}{M_{electron}}$	$\dfrac{M_{top}}{M_{charmed}}$
$\dfrac{M_{muon}}{M_{electron}}$	$\dfrac{M_{bottom}}{M_{charmed}}$

to be generated as follows

$R(F_5,0)=3475.68603$	$R(F_1,1)=123$
$R(F_4,0)=206.763$	$R(F_0,0)=3$

Note that the Fibonacci sequence extends in both directions and includes the terms

$$\ldots \quad -3 \quad 2 \quad -1 \quad 1 \quad \underline{\mathbf{0}} \quad \underline{\mathbf{1}} \quad \mathbf{1} \quad \mathbf{2} \quad \mathbf{3} \quad \mathbf{5} \quad \ldots$$

where each term equals the sum of the preceding two, and where 0 and 1 are the sequence initiators (underlined above). In the equations above, the values for F_n are taken from the Fibonacci sequence's first six terms (in boldface). Accordingly,

$$F_0 = 0 \ ,$$

$$F_1 = 1 \ ,$$

$$F_4 = 3 \ ,$$

and,

$$F_5 = 5 \ .$$

III. Analysis of Results

The calculated values for $\frac{M_{top}}{M_{charmed}}$ and $\frac{M_{bottom}}{M_{charmed}}$ fit the roughly-known experimental quark mass ratios within, or close to, their broad limits of error. These experimental mass ratios are calculated below by choosing from the experimental values' upper or lower bounds in an effort to fit the calculated values. Experimentally, the t-quark's mass equals 172,700 ± 2,900 MeV [7]; while the b-quark's mass ranges from 4,100 to 4,400 MeV [8]; and the c-quark's mass ranges from 1,150 to 1,350 MeV [8]. It follows that

$$\frac{172{,}700\text{ MeV} - 2{,}900\text{ MeV}}{1350\text{ MeV}} = 125.77...$$

and

$$\frac{4{,}100\text{ MeV}}{1{,}350\text{ MeV}} = 3.037... \ ,$$

which are in rough accord with their calculated values of 123.04... and 3.000... .

Furthermore, the calculated values for $\frac{M_{tau}}{M_{electron}}$ and $\frac{M_{muon}}{M_{electron}}$ fit their corresponding experimental values to roughly 1 part in 2,000, and 1 part in 40,000, respectively [8].

IV. A Dual Set of Mass Formula Parameters

An alternative way of specifying the above mass formula parameters involves the same Fibonacci sequence extended in the *reverse* of its usual direction

−3	2	−1	1	0	1

Again, each term equals the sum of the preceding two. Interestingly, the terms of the above sequence, when divided modulo 6, yield a second sequence consisting of the *first* six Fibonacci numbers (albeit out of order):

3	2	5	1	0	1

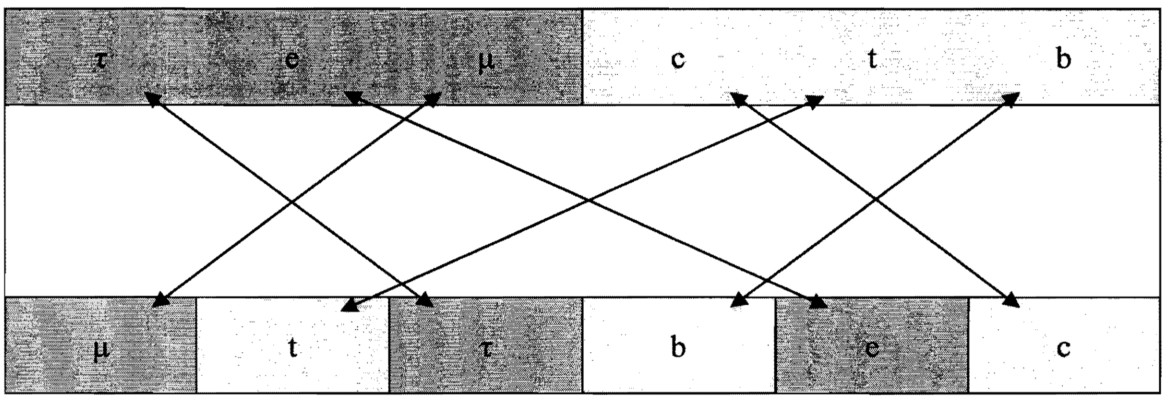
Note that, above, $-3 \equiv 3 \pmod 6$, and $-1 \equiv 5 \pmod 6$.

We exploit the above pair of six-term sequences, which we will label $\boldsymbol{F}$ and $\boldsymbol{F'}$ respectively, in making the assignments for the mass formula parameters F_p and F'_p; note that, in the upper and lower tables below, only leptons appear heavily shaded:

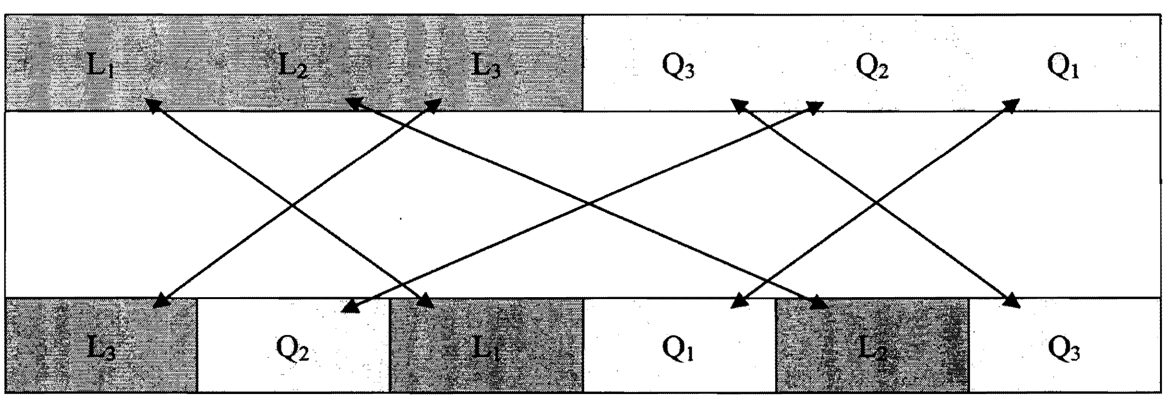
Particle p	τ	e	μ	c	t	b
F_p	−3	2	−1	1	0	1

Particle p	μ	t	τ	b	e	c
F'_p	3	2	5	1	0	1

Notice how the leptons and quarks are rearranged in going from the first table to the second:



Or, schematically:



The above assignments now make it possible to produce mass ratios $\frac{M_p}{M_q}$ for particles p and q while using as a parameter, either $-\left(F_p - F_q\right)$,

$R(-(F_\tau - F_e),0) = 3475.68603$	$R(-(F_t - F_c),1) = 123$
$R(-(F_\mu - F_e),0) = 206.763$	$R(-(F_b - F_c),0) = 3$

or, with identical results, $F'_p - F'_q$:

$R(F'_\tau - F'_e,0) = 3475.68603$	$R(F'_t - F'_c,1) = 123$
$R(F'_\mu - F'_e,0) = 206.763$	$R(F'_b - F'_c,0) = 3$

That this dual solution exists is remarkable. It is also noteworthy that the quarks and leptons, which appear segregated in the upper table of parameter assignments (LLLQQQ), symmetrically interleave in the lower table (LQLQLQ). But, once we decide to exploit the integer sequences $\boldsymbol{F}$ and $\boldsymbol{F'}$ when making our parameter assignments, if the mass formula is to fit the mass data, then the sequences LLLQQQ and LQLQLQ are forced—no other order will yield parameters that fit the mass data. Accordingly, the $\frac{\text{LLLQQQ}}{\text{LQLQLQ}}$ symmetry of the above parameter assignments is not the result of choice, but a consequence of the experimental data.

Another important aspect of the above assignments is that they see to it that there is no "first", or "ground state", particle. The upper table (using sequence ***F***), and the lower table (using sequence ***F′***), can equally lay claim to being the best way to assign the mass formula parameters, and so the particles occur in no particular order. It would be undesirable to single out one particle as a "ground state", because the quark and lepton mass spectrum does not appear to furnish the indefinite number of "excited states" that should go with it.

Furthermore, because the parameters exploited by the mass formula

n	-4	-3	-2	-1	0	1
F_n	-3	2	-1	1	0	1
$F'_n \equiv F_n \pmod 6$	3	2	5	1	0	1

are symmetrical in ways that cannot be extended to encompass more than the six Fibonacci terms of sequence ***F***, one can readily exploit the symmetries of ***F′*** and ***F′*** to formulate conditions that automatically limit the number of heavy quarks and leptons. In fact, this was done in another article by the author, although the key symmetry exploited there did not involve modular arithmetic [2]. In this way, it may be possible to partially account for why there are no more than three particle families.

V. A Mass Formula for All Quarks and Leptons

In order to accommodate both heavy and light quarks and leptons within a single formula, it is only necessary to alter the mass formula slightly

$$R(j,k,m)=\left(1+\frac{1}{A}+B\right)^{\frac{j}{m}}\times A^k\times B^{\frac{1}{m}}$$

$$=4.1^{\frac{j}{m}}\times 10^k\times 3^{\frac{1}{m}} \quad . \tag{6}$$

We let $k = 1$ for the t- and s-quarks, and 0 for all other particles. In addition, we carry out the following assignments for F_p

Heavy Particles $m = 1$	τ	e	μ	b	t	c
Light Particles $m = 2$	υ_3	υ_1	υ_2	d	s	u
F_p	−3	2	−1	1	0	1

and, we also assign values for F'_p, with the particles swapping places as before.

Heavy Particles $m = 1$	μ	t	τ	c	e	b
Light Particles $m = 2$	υ_2	s	υ_3	u	υ_1	d
F'_p	3	2	5	1	0	1

Note that for the above parameter assignments the heavy particles are paired with light particles in a natural way, with all pairings governed by mass. So, the heaviest heavy quark (t) is paired with the heaviest light quark (s); the lightest heavy quark (c) is paired with the lightest light quark (u); and so on. Also note that the table assigns $m = 1$ for all heavy particles, and $m = 2$ for all light particles

On the one hand, the assignments in the above table are the same for the heavy particles as those used earlier, and, consequently, with m equal to 1, the mass ratios generated by Eq. (6) are the same as before.

On the other hand, for the light quarks and leptons, with m equal to 2, the mass ratios generated by Eq. (6) are either

$m = 2$	
$R(-(F_{\nu_3} - F_{\nu_1}),0,m) = 58.95...$	$R(-(F_s - F_u),1,m) = 35.07...$
$R(-(F_{\nu_2} - F_{\nu_1}),0,m) = 14.37...$	$R(-(F_d - F_u),0,m) = 1.73...$

or, identically,

$m = 2$	
$R(F'_{\nu_3} - F'_{\nu_1},0,m) = 58.95...$	$R(F'_s - F'_u,1,m) = 35.07...$
$R(F'_{\nu_2} - F'_{\nu_1},0,m) = 14.37...$	$R(F'_d - F'_u,0.m) = 1.73...$

Below these light quark mass ratios are compared against their experimental values, where it is seen that they are in agreement [8].

Mass Ratio	*Experimental Value*	*Calculated Value*
$\dfrac{M_u}{M_d}$	0.3 to 0.7	$\dfrac{1}{3^{\frac{1}{2}}} = 0.57735...$
$\dfrac{M_s}{M_d}$	17 to 22	$\dfrac{4.1^{\frac{1}{2}}}{0.1} = 20.248...$
$\dfrac{M_s}{(M_u + M_d)/2}$	25 to 30	25.674...
$\dfrac{M_s - \dfrac{M_d + M_u}{2}}{M_d - M_u}$	30 to 50	46.042...

VI. The Neutrino Squared-Mass Splittings

Equation (6) requires that the masses of the neutrino mass eigenstates occur in the following ratios

$$\sqrt{4.1^5 \times 3} : \sqrt{4.1^3 \times 3} : \sqrt{1} \quad ,$$

and that the neutrino squared-mass splittings, in turn, fulfill the following ratios

$$\left(4.1^5 \times 3 - 1\right) : \left(4.1^5 \times 3 - 4.1^3 \times 3\right) : \left(4.1^3 \times 3 - 1\right) \quad .$$

It follows that

$$\frac{\left|M(\nu_3)^2 - M(\nu_1)^2\right|}{\left|M(\nu_2)^2 - M(\nu_1)^2\right|} = \frac{4.1^5 \times 3 - 1}{4.1^3 \times 3 - 1} = 16.8868\ldots \ , \tag{7a}$$

and,

$$\frac{\left|M(\nu_3)^2 - M(\nu_2)^2\right|}{\left|M(\nu_2)^2 - M(\nu_1)^2\right|} = \frac{4.1^5 \times 3 - 4.1^3 \times 3}{4.1^3 \times 3 - 1} = 15.8868\ldots \ . \tag{7b}$$

Observational data exist for two neutrino squared-mass splittings, namely [8]

$$1.5 \times 10^{-3}\ \Delta\mathrm{eV}^2 < \left|M(\nu_\mu)^2 - M(\nu_x)^2\right| < 3.9 \times 10^{-3}\ \Delta\mathrm{eV}^2$$

and [9]

$$\left|M(\nu_e)^2 - M(\nu_x)^2\right| = 7.1 \times 10^{-5}{}^{+1.2}_{-0.6}\ \Delta\mathrm{eV}^2 \ .$$

As this second neutrino squared-mass splitting is the more precisely-known of the two, it may be used as a starting point to calculate $\left|M(\nu_\mu)^2 - M(\nu_x)^2\right|$, as well as the remaining unknown neutrino squared-mass splitting:

$$\frac{4.1^5 \times 3-1}{4.1^3 \times 3-1} \times 7.1 \times 10^{-5}\,{}^{+1.2}_{-0.6}\ \Delta\mathrm{eV}^2 \approx 1.19^{+0.2}_{-0.1} \times 10^{-3}\ \Delta\mathrm{eV}^2\ , \tag{8a}$$

and,

$$\frac{4.1^5 \times 3-4.1^3 \times 3}{4.1^3 \times 3-1} \times 7.1 \times 10^{-5}\,{}^{+1.2}_{-0.6}\ \Delta\mathrm{eV}^2 \approx 1.12^{+0.2}_{-0.1}\ \times 10^{-3}\ \Delta\mathrm{eV}^2\ . \tag{8b}$$

These predictions offer an opportunity to test the mass formulae's validity, especially as the Eq. (8a)'s value for $\left|M(\nu_\mu)^2 - M(\nu_x)^2\right|$ is predicted to be slightly below its experimental value. The predicted values for the neutrino mass eigenstates are discussed in greater detail elsewhere by the author [2].

VII. Summary and Conclusion

In summary, in this article four mass ratios between quarks, and another four mass ratios between leptons, are generated by a mass formula that is formed by the product of powers of constants that are derived from the fine structure constant (10 and 3), where this mass formula takes its parameters from a portion of the Fibonacci sequence. More exactly, the mass formula exploits two sets of Fibonacci-related parameters, which possess a symmetry that may be exploited to furnish at least a partial explanation of why there are no more than three particle families. The details of how one might use the above Fibonacci-related symmetry to automatically limit the number of particle families to three is only hinted at above. But elsewhere the author analyzes in detail a symmetrical pair of mass formulae that are demonstrated to give consistent values for mass only when there are three or fewer particle families [2].

Finally, the ease with which so many mass ratios can be approximated by the mass formula, and the symmetry, economy, and duality of its parameter specification, inevitably raises the question of why it should be so successful.

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