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Dual mass formulae that generate the quark and lepton masses

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Abstract

Two symmetrical mass formulae are introduced that closely reproduce seven experimentally known mass ratios of the quarks and leptons. Consistency of results between these mass formulae is achieved by exploiting a symmetry present in the initial terms of the Fibonacci sequence. This symmetry determines the mass formulae parameters and requires that there exist either one, or three, particle families. It is this three-family solution that produces the quark and lepton mass ratios at or near their experimental values.

I. Introduction

Two symmetrical mass formulae are introduced that closely reproduce seven experimental mass ratios of the quarks and leptons, while exploiting the initial terms of the Fibonacci sequence. The formulae are not intended to explain the origin of mass, but rather to establish underlying phenomenological connections that may serve as a guide to a physical explanation. The formulae exploit constants equal to the beta coefficients $b_1 = 41/10$ and $\tilde{b}_1 = 1/10$ of the extra-dimensional, non-supersymmetric GUT described by Dienes, Dudas, and Gherghetta [1]. Earlier, the author used these same constants to reproduce the π^0 meson-, J/ψ meson-, muon-, and neutron-electron mass ratios to an accuracy at, or very near, their experimental limits [2], while also demonstrating, by using information theory, that this result is unlikely to be purely coincidental [3]. In addition, the author has demonstrated that toroids dimensioned by these values possess surface areas proportional to some particle masses [4].

IIa. The mass formulae parameters

The Fibonacci sequence extends in both directions and includes the following terms

$$\ldots \quad -3 \quad 2 \quad -1 \quad 1 \quad \mathbf{0} \quad \mathbf{1} \quad 1 \quad 2 \quad 3 \quad 5 \quad \ldots \; .$$

Above, each term of the Fibonacci sequence equals the sum of the two terms that precede it. The initiators of the sequence 0 and 1 appear in boldface.

To generate the mass formula parameters, we begin by selecting the six Fibonacci terms that are initiated by 0 and 1 and extended *right*wards (Sequence ***R***):

$$\mathbf{0} \quad \mathbf{1} \quad 1 \quad 2 \quad 3 \quad 5 \quad .$$

We then select the six Fibonacci terms that are initiated by 0 and 1 and extended *left*wards (Sequence ***L***):

$$-3 \quad 2 \quad -1 \quad 1 \quad \mathbf{0} \quad \mathbf{1} \quad .$$

We then pair the terms of ***L*** and ***R*** so that they sum uniformly to 2.

−3	2	−1	1	0	1
+ 5	+ 0	+ 3	+ 1	+ 2	+ 1
2	2	2	2	2	2

Note that the symmetry of this pair of 6-term sequences will be exploited below when assigning values to the mass formulae parameters n and $\hat{n}$ for the quarks and leptons. In addition, it will

be shown later how the non-existence of an equivalent pair of N-term sequences, with $N \geq 8$, automatically disallows the possibility of 4 or more particle families.

As the Fibonacci numbers may be written $F(-4) = -3$, $F(-3) = 2$, $F(-2) = -1$, $F(-1) = 1$, $F(0) = 0$, $F(1) = 1$, $F(2) = 1$, $F(3) = 2$, $F(4) = 3$, $F(5) = 5$, ... etc., the above sums may be restated as follows.

$$\begin{array}{cccccc} F(-4) & F(-3) & F(-2) & F(-1) & F(0) & F(1) \\ + F(5) & + F(0) & + F(4) & + F(2) & + F(3) & + F(1) \\ \hline 2 & 2 & 2 & 2 & 2 & 2 \end{array}$$

Conveniently, the above sums serve as a ready template for assigning values to $\hat{n}$ and n for the quarks and leptons

	τ	e	μ	b	t	c
	υ_3	υ_1	υ_2	d	s	u
$\hat{n}$	-4	-3	-2	-1	0	1
n	5	0	4	2	3	1

so that for any particle p

$$F(\hat{n}_p) + F(n_p) = 2 \quad . \tag{1a}$$

This implies that for any particles p and q

$$F(\hat{n}_p) + F(n_p) = 2 = F(\hat{n}_q) + F(n_q) \tag{1b}$$

or

$$F(n_p) - F(n_q) = -(F(\hat{n}_p) - F(\hat{n}_q)) \quad , \tag{1c}$$

a key relation that below will guarantee consistency of results between the two mass formulae.

Finally, note that the above parameter assignments are carried out with the heavy particles paired with the light particles in a natural way, with all pairings governed by mass. So, the heaviest heavy quark (t) is paired with the heaviest light quark (s); the lightest heavy quark (c) is paired with the lightest light quark (u); and so on.

IIb. The mass formulae

Now define $\lfloor x \rfloor$ as equal to the largest integer that is less than or equal to x; and define $\lceil x \rceil$ as equal to the smallest integer that is greater than or equal to x. Also define the symmetrical mass formulae

$$M_p = 4.1^{\frac{F(n_p)}{m_p}} \times 0.1^{F\left(\left\lceil \frac{|\hat{n}_p|}{2} \right\rceil\right)} \times 3^{\frac{F\left(\left\lfloor \frac{n_p}{2} \right\rfloor\right)}{m_p}} \tag{2a}$$

$$\hat{M}_p = 4.1^{\frac{-F(\hat{n}_p)}{m_p}} \times 0.1^{F\left(\left\lceil \frac{|\hat{n}_p|}{2} \right\rceil\right)} \times 3^{\frac{F\left(\left\lfloor \frac{n_p}{2} \right\rfloor\right)}{m_p}} , \tag{2b}$$

which differ only in their exponents for 4.1.

Above, M_p and $\hat{M}_p$ equal relative mass for a particle p, while the values for m_p are as follows.

	m
Light quarks and leptons	2
Heavy quarks and leptons	1

The values for $\hat{n}$, n, and m—the only parameters used by Eqs. (2a) and (2b)—are summarized in Table I, where the quarks and leptons are grouped by charge.

Note that the above mass formulae are only meant to accurately reproduce the quark and lepton mass ratios that hold *within* these particle sub-groups: the heavy quarks, the heavy leptons, the light quarks, and neutrinos. Accordingly, because the muon and electron are both heavy leptons, the muon-electron mass ratio $\frac{M(\mu)}{M(e)}$ may be calculated with either Eq. (2a)

$$\frac{M(\mu)}{M(e)} = \frac{4.1^{\frac{F(n_\mu)}{m_\mu}} \times 0.1^{F\left(\left\lceil \frac{|\hat{n}_\mu|}{2} \right\rceil\right)} \times 3^{\frac{F\left(\left\lfloor \frac{n_\mu}{2} \right\rfloor\right)}{m_\mu}}}{4.1^{\frac{F(n_e)}{m_e}} \times 0.1^{F\left(\left\lceil \frac{|\hat{n}_e|}{2} \right\rceil\right)} \times 3^{\frac{F\left(\left\lfloor \frac{n_e}{2} \right\rfloor\right)}{m_e}}} = \frac{4.1^{\frac{F(4)}{1}} \times 0.1^{F\left(\left\lceil \frac{|-2|}{2} \right\rceil\right)} \times 3^{\frac{F\left(\left\lfloor \frac{4}{2} \right\rfloor\right)}{1}}}{4.1^{\frac{F(0)}{1}} \times 0.1^{F\left(\left\lceil \frac{|-3|}{2} \right\rceil\right)} \times 3^{\frac{F\left(\left\lfloor \frac{0}{2} \right\rfloor\right)}{1}}} \tag{3a}$$

$$= \frac{4.1^{F(4)} \times 0.1^{F(1)} \times 3^{F(2)}}{4.1^{F(0)} \times 0.1^{F(2)} \times 3^{F(0)}} = \frac{4.1^3 \times 0.1^1 \times 3^1}{4.1^0 \times 0.1^1 \times 3^0} = 4.1^3 \times 3 = 206.763$$

or Eq. (2b)

$$\frac{M(\mu)}{M(e)}=\frac{4.1^{\frac{-F(\hat{n}_\mu)}{m_\mu}}\times 0.1^{F\left(\left\lceil\frac{|\hat{n}_\mu|}{2}\right\rceil\right)}\times 3^{\frac{F\left(\left\lfloor\frac{n_\mu}{2}\right\rfloor\right)}{m_\mu}}}{4.1^{\frac{-F(\hat{n}_e)}{m_e}}\times 0.1^{F\left(\left\lceil\frac{|\hat{n}_e|}{2}\right\rceil\right)}\times 3^{\frac{F\left(\left\lfloor\frac{n_e}{2}\right\rfloor\right)}{m_e}}}=\frac{4.1^{\frac{-F(-2)}{1}}\times 0.1^{F\left(\left\lceil\frac{|-2|}{2}\right\rceil\right)}\times 3^{\frac{F\left(\left\lfloor\frac{4}{2}\right\rfloor\right)}{1}}}{4.1^{\frac{-F(-3)}{1}}\times 0.1^{F\left(\left\lceil\frac{|-3|}{2}\right\rceil\right)}\times 3^{\frac{F\left(\left\lfloor\frac{0}{2}\right\rfloor\right)}{1}}} \tag{3b}$$

$$=\frac{4.1^{-F(-2)}\times 0.1^{F(1)}\times 3^{F(2)}}{4.1^{-F(-3)}\times 0.1^{F(2)}\times 3^{F(0)}}=\frac{4.1^{1}\times 0.1^{1}\times 3^{1}}{4.1^{-2}\times 0.1^{1}\times 3^{0}}=4.1^{3}\times 3=206.763 \quad ,$$

with identical results. Likewise, Eqs. (2a) and (2b) allow calculation of the mass ratios $\frac{M(\nu_2)}{M(\nu_1)}$, $\frac{M(\tau)}{M(e)}$, $\frac{M(\nu_3)}{M(\nu_1)}$, $\frac{M(t)}{M(c)}$, $\frac{M(s)}{M(u)}$, $\frac{M(b)}{M(c)}$, and $\frac{M(d)}{M(u)}$, as these are also ratios between the masses of particles within the same sub-group.

Note that Eqs. (3a) and (3b) produce consistent results because, for the muon and electron,

$$\frac{4.1^{\frac{F(n_\mu)}{m_\mu}}}{4.1^{\frac{F(n_e)}{m_e}}}=\frac{4.1^{\frac{-F(\hat{n}_\mu)}{m_\mu}}}{4.1^{\frac{-F(\hat{n}_e)}{m_e}}} \tag{3c}$$

or substituting

$$\frac{4.1^{3}}{4.1^{0}} = \frac{4.1^{1}}{4.1^{-2}} \quad .$$

As this equation makes clear, the only difference between Eqs. (3a) and (3b) is that their exponents for 4.1 have been uniformly shifted by 2. Crucially, the uniformity of this shift leaves the differences between exponents unchanged; that is,

$$3 - 0 = 1 - (-2) \quad .$$

It follows that Eqs. (3a) and (3b) produce identical values for the muon-electron mass ratio, albeit in a slightly different manner.

But this should not be surprising as Eq. (1c) implies that, for any particles p and q, if $m_p = m_q$ then

$$\frac{4.1^{\frac{F(n_p)}{m_p}}}{4.1^{\frac{F(n_q)}{m_q}}} = \frac{4.1^{\frac{-F(\hat{n}_p)}{m_p}}}{4.1^{\frac{-F(\hat{n}_q)}{m_q}}} \quad .$$

Accordingly, for any particles p and q that share the same value for m, Eqs. (2a) and (2b) will produce mass ratios that are equal.

It should now be clear why, earlier, such care was taken in assigning values for n and $\hat{n}$. It is the symmetry of the Fibonacci numbers $F(\hat{n})$ and $F(n)$ that ultimately guarantees mass

formulae consistency. More specifically, it is the symmetry possessed by the two six-term sequences generated by the Fibonacci initiators 0 and 1, extended in either direction, that sees to it the mass formulae produce the same ratios.

As will be explained later, the absence of an equivalent symmetry for Fibonacci sequences of 8 or more terms automatically imposes a limit of 3 on the number of particle families. So, if 4 or more particle families are to be modeled, it will be impossible for the mass formulae to fit such data—irrespective of what the mass data is. No correct model of such masses will be possible, because no consistent model will be possible. Conversely, and perhaps surprisingly, the mere requirement that the values chosen for n and $\hat{n}$ achieve consistent results is enough to ensure mass formulae accuracy; that is to say, the values for n and $\hat{n}$ that produce consistent results automatically produce mass ratios that fit the experimental mass data.

III. Comparison of the calculated mass ratios against their experimental values

As was explained earlier, either Eq. (2a) or (2b) allows one to closely reproduce the experimental mass ratios that hold within these particle sub-groups: the heavy quarks, the heavy leptons, the light quarks, and neutrinos, where these equations take their parameters from Table I. The following mass ratios are a consequence of Eqs. (2a) and (2b) and Table I:

$$\frac{M(\tau)}{M(e)} = \left(\frac{M(\nu_3)}{M(\nu_1)}\right)^2 = 4.1^5 \times 3 \tag{4a}$$

$$\frac{M(\mu)}{M(e)} = \left(\frac{M(\nu_2)}{M(\nu_1)}\right)^2 = 4.1^3 \times 3 \tag{4b}$$

$$\frac{0.1 \times M(t)}{M(c)} = \left(\frac{0.1 \times M(s)}{M(u)}\right)^2 = 4.1^1 \times 3 \tag{4c}$$

$$\frac{M(b)}{M(c)} = \left(\frac{M(d)}{M(u)}\right)^2 = 4.1^0 \times 3 \ . \tag{4d}$$

The ratios that Eqs. (4a)-(4d) imply are:

For the heavy leptons: $4.1^5 \times 3 \ : \ 4.1^3 \times 3 \ : 1$

For the neutrinos: $\sqrt{4.1^5 \times 3} : \sqrt{4.1^3 \times 3} : \sqrt{1}$

For the heavy quarks: $4.1 \times 10 \times 3 : 3 : 1$

For the light quarks: $\sqrt{4.1} \times 10 \times \sqrt{3} : \sqrt{3} : \sqrt{1}$.

When we compare the mass ratios of Eqs. (4a)-(4d) against their corresponding experimental values [5], we find a remarkable fit:

The tau's measured mass equals $1776.99^{+0.29}_{-0.26}$ MeV, while the electron's measured mass is 0.510998918 MeV. Dividing the tau's mass by the electron mass yields a mass ratio of $\frac{1776.99}{0.510998918} = 3477.48....$ This is not very different from its calculated value of $4.1^5 \times 3 = 3475.68603$, which differs by roughly 1 part in 1,900.

Similarly, the experimental value for the muon-electron mass ratio equals 206.76828..., versus a calculated value of 206.763. These differ by roughly 1 part in 40,000.

The t-quark's mass of 178,000 ± 4,300 MeV and the c-quark's mass of 1,150 to 1,350 MeV suggest a possible t-quark / c-quark mass ratio of 173,700 / 1,350 = 128.66..., which is near its calculated value of 123. (Note: in the general mass equation introduced later, this discrepancy is resolved by raising the c-quark mass slightly.)

The b-quark's mass of 4,100 to 4,400 MeV and the c-quark's mass of 1,150 to 1,350 MeV suggest a possible b-quark / c-quark mass ratio of 4,100 / 1,350 = 3.037..., which is near its calculated value of 3.

The s/d experimental mass ratio 17 to 22 encompasses its calculated value of 20.248 .

The u/d experimental mass ratio 0.3 to 0.7 encompasses its calculated value of 0.57735 .

The above comparisons are summarized in Table II.

Finally, observational data exist for two neutrino squared-mass splittings, namely [6]

$$1.5 \times 10^{-3}\ \Delta\text{eV}^2 < \left|M(\nu_\mu)^2 - M(\nu_X)^2\right| < 3.9 \times 10^{-3}\ \Delta\text{eV}^2$$

and [7]

$$\left|M(\nu_e)^2 - M(\nu_X)^2\right| = 7.1 \times 10^{-5\,{}^{+1.2}_{-0.6}}\ \Delta\text{eV}^2 \quad .$$

If the three neutrino mass eigenstates differ in mass by sufficiently large amounts, as they are predicted to here, then the square root of the ratio of the above squared-mass splittings offers a reasonable estimate of the mass ratio between two of the neutrino mass eigenstates. The above experimental values, if they are taken at their lower and upper bounds, respectively, produce a value that comes close to the calculated value for $\frac{M(\nu_3)}{M(\nu_2)}$, which equals 4.1:

$$\sqrt{\frac{1.5\times10^{-3}\ \Delta\mathrm{eV}^2}{8.3\times10^{-5}\,\Delta\mathrm{eV}^2}} = 4.25\ldots \quad . \tag{5}$$

This is a key additional way in which the mass formulae reproduce the experimental mass data.

IV. Prediction of the unknown mass splitting

Equations (4a)–(4d) fit seven experimental values. An eighth ratio, a squared-mass splitting, is uncertain at this time, but can be inferred from Eqs. (4a) and (4b). These equations predict that the neutrinos mass eigenstates must fulfill the following ratios

$$\sqrt{4.1^5\times3} : \sqrt{4.1^3\times3} : \sqrt{1} \ .$$

It follows that their squared-mass splittings must in turn fulfill the following ratios

$$\left(4.1^5 \times 3 - 1\right) : \left(4.1^5 \times 3 - 4.1^3 \times 3\right) : \left(4.1^3 \times 3 - 1\right) .$$

This places the unknown squared-mass splitting at approximately

$$\frac{4.1^5 \times 3 - 4.1^3 \times 3}{4.1^3 \times 3 - 1} \times 7.1 \times 10^{-5}\ \Delta\text{eV}^2 \approx 1.1\ldots \times 10^{-3}\ \Delta\text{eV}^2 .$$

This prediction offers an opportunity to test the mass formulae's validity.

V. An automatic limit of three on the number of particle families

It is helpful to examine in detail the two Fibonacci sequences responsible for reproducing the quark and lepton mass ratios; this is to say, the sequences that arise when the Fibonacci sequence initiators 0 and 1 are extended in both directions to a length of six terms.

Values for $F(n)$:					**0**	**1**	1	2	3	5
Values for $F(\hat{n})$:	−3	2	−1	1	**0**	**1**				

It is by appropriately coupling the quarks and leptons with the above terms that we see to it that the n and $\hat{n}$ sum to a common value, in this case 2:

$$-3 + 5 = 2 + 0 = -1 + 3 = 1 + 1 = 0 + 2 = 1 + 1 = 2 .$$

In this way Eq. (1a) is fulfilled and the mass formulae produce the quark and lepton masses.

It is interesting that the above Fibonacci sequences cannot be lengthened to accommodate 4 or more particle families. To see why, consider that if more than 4 particle families were modeled, the above Fibonacci sequences would have to be correspondingly extended to contain 8 or more terms. But an inspection of the first 8 terms of the Fibonacci sequence

0	1	1	2	3	5	8	13
+ ?	+ ?	+ ?	+ ?	+ ?	+ ?	+ ?	+ ?
k	k	k	k	k	k	k	k

shows that no other sequence of 8 consecutive Fibonacci numbers can be found to pair with them to sum to a common integer k. Furthermore, this problem remains even if the sequence is extended to more than 8 terms (see Appendix for proof).

This inevitable mismatch of terms sees to it that the mass formulae cannot produce consistent results for 4 or more particle families. Nor, for that matter, can it accommodate just 2 particle families, for the same reason, though it can accommodate just 1, as the sequence initiators may be paired with each other to produce a common sum. This "single-family solution" takes the following form.

$$\begin{array}{cc} 1 & 0 \\ +0 & +1 \\ \hline 1 & 1 \end{array}$$

Although the above reasoning should not be taken as absolute, particularly as it is inevitable that a sufficient "loosening of the framework" may make it possible to accommodate more than 3 families, nevertheless, it should not be overlooked that the above framework offers a natural way to limit the number of particle families to 3, and that any modifications to the above framework might very well rob it of its simplicity.

VI. Unambiguous steps that generate the quark and lepton mass ratios

It is instructive to identify an unambiguous set of steps—as well as the key assumptions that underlie these steps—that will generate the quark and lepton mass ratios of Eqs. (4a)-(4d), while automatically disallowing 4 or more particle families:

Step One: We begin by assuming that Eqs. (2a) and (2b) govern quark and lepton mass and that for those quarks and leptons the values for $\hat{n}$ and n are sequences of consecutive integers that are identical for heavy and light particles, and that the values for n are initiated by 0, 1.

Step Two: We further assume that for all particles the sum $F(n_p)+F(\hat{n}_p)$ equals the same constant. In addition, we require that the values for m equal 1 for those quarks and leptons that are heavy, and 2 for those that are light.

Step Three: Under the above restrictions, Eqs. (2a) and (2b) can achieve consistency in either of two ways: Via the single-family solution noted earlier where

$$n = \{ 0, 1 \} \text{ and } \hat{n} = \{ 0, 1 \} ,$$

and via the 3-family solution described at the outset of this article, where

$$n = \{ 0, 1, 2, 3, 4, 5 \} \text{ and } \hat{n} = \{ -4, -3, -2, -1, 0, 1 \} .$$

We discard the single-family solution and retain the 3-family solution. This 3-family solution is then used to compute each particle's relative mass, thereby producing the eight mass ratios $\frac{M_p}{M_q}$ or $\frac{\hat{M}_p}{\hat{M}_q}$ that hold within these particle sub-groups: the heavy quarks, the heavy leptons, the light quarks, and light leptons.

VII. Are the mass formulae successful for physical, or accidental, reasons?

The above framework generates the particle masses via formulae that take the general form of

$$M_p = J^{\frac{F(n_p)}{m_p}} \times K^{F\left(\left\lceil \frac{|\hat{n}_p|}{2} \right\rceil\right)} \times L^{\frac{F\left(\left\lfloor \frac{n_p}{2} \right\rfloor\right)}{m_p}} \tag{6a}$$

and

$$\hat{M}_p = J^{\frac{-F(\hat{n}_p)}{m_p}} \times K^{F\left(\left\lceil \frac{|\hat{n}_p|}{2} \right\rceil\right)} \times L^{\frac{F\left(\left\lfloor \frac{n_p}{2} \right\rfloor\right)}{m_p}} , \tag{6b}$$

where the values for the constants J, K, and L are as follows

$$J = \frac{41}{10} \ ,$$

$$K = \frac{1}{10} \ ,$$

$$L = 3 \ .$$

In Table I, the values for the parameters $\hat{n}$ and n are listed. Clearly these parameters are not easily fine-tuned in order to make Eqs. (6a) and (6b) fit the mass data. This is so partly because these parameters are sequences of consecutive integers, but more importantly because the values for n and $\hat{n}$ are rigidly constrained by the need for consistency between Eqs. (6a) and (6b). Consequently the parameters of Table I offer virtually no opportunity for fine-tuning the mass formulae parameters to fit the mass data by accident.

But it must be noted that J, K, and L also cannot be fine-tuned in order to make Eqs. (6a) and (6b) fit the mass data. This is because J, K, and L are not constants specifically chosen to fit the quark and lepton mass data, but instead are constants originally selected to generate the mass ratios of a quite different set of particles. More specifically, the constants 41/10, 1/10, 3 and were first introduced by the author to generate the π^0 meson-, J/ψ meson-, muon- and neutron-electron mass ratios [2,3,4]. Their reuse here, therefore, merely maintains consistency with earlier work.

Accordingly, the constants 41/10, 1/10, 3 cannot be regarded as values selected to accommodate the quark and lepton masses. That 41/10, 1/10, 3 can, despite this independent origin, still manage to generate the quark and lepton masses, must be taken as key evidence for their physical, rather than accidental, origin.

It is also suggestive that the fine structure constant reciprocal $\frac{1}{\alpha}$ may be approximated closely with the aid of the constants $K = \frac{1}{10}$ and $L = 3$ of Eqs. (6a) and (6b)

$$\frac{1}{\alpha} \approx \frac{1}{(KL)^3} + \frac{1}{K^2} = \frac{10^3}{3^3} + 10^2 = 137.037037\ldots \ , \tag{7a}$$

where the 2002 CODATA value for $\frac{1}{\alpha}$ equals 137.03599911 (46) [8]. The effectiveness of this approximation lends key additional support to the conjecture that the constants 1/10, and 3 are not arbitrary.

Of course, one could plausibly object that the above approximation achieves its close fit of $\frac{1}{\alpha}$ by coincidence and that other approximations of the same form might achieve a better fit while employing even smaller integers.

To resolve this issue, a computer searched for a better approximation of $\frac{1}{\alpha}$ in the form

$$\frac{A^a}{B^b} + C^c \ ,$$

where the exponents a, b, and c were integers arbitrarily allowed to range from 0 to 5, inclusive, and A, B, and C were integers allowed to range from 1 to 10, inclusive. Across these ranges no better approximation was found.

As it is, to find a better approximation requires that A, B, and C be allowed to range up to 37:

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28^3 / 37^2 + 11^2 = 137.0350620...
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(with, once again, a, b, and c limited to between 0 and 5, inclusive). Accordingly, *for values of A, B, and C less than 37*, the best fit is achieved by the unusually small integers

$$A = C = 10 \ ,$$

$$B = 3 \ ,$$

which, of course, are the same constants relied upon by the mass formulae.

Finally, it is interesting to carry out an additional search for a refined version of the approximation $\frac{10^3}{3^3} + 10^2$, specifically one in the form $\frac{10^3 - D^d}{3^3} + 10^2 - E^e$, where the exponents d and e are integers arbitrarily allowed to range from 0 to −3, inclusive, and D and E are integers arbitrarily allowed to range from 1 to 30, inclusive. Within these restrictions the best fit of the

experimental value of the fine structure constant inverse is provided when $D = E = 10$ and $d = e = -3$, so that

$$\frac{1}{\alpha} \approx \frac{10^3 - 10^{-3}}{3^3} + 10^2 - 10^{-3} = \frac{999.999}{3^3} + 99.999 = 137.036 \quad . \tag{7b}$$

Remarkably, the integer 10 now occurs no less than four times, while reproducing exactly the celebrated 137.036. This four-fold repetition of 10 is suggestive that Eq. (7b) is physically significant, and that the constants 10 and 3 may be fundamental constants of nature.

Because $A = 10 = \frac{1}{K}$ and $B = 3 = L$, one may readily restate Eqs. (6a) and (6b) in terms of A and B as follows

$$M_p = \left(\frac{1}{A} + B + 1\right)^{\frac{F(n_p)}{m_p}} \times \left(\frac{1}{A}\right)^{F\left(\left\lceil \frac{|\hat{n}_p|}{2} \right\rceil\right)} \times B^{\frac{F\left(\left\lfloor \frac{n_p}{2} \right\rfloor\right)}{m_p}} \tag{8a}$$

and

$$\hat{M}_p = \left(\frac{1}{A} + B + 1\right)^{\frac{-F(\hat{n}_p)}{m_p}} \times \left(\frac{1}{A}\right)^{F\left(\left\lceil \frac{|\hat{n}_p|}{2} \right\rceil\right)} \times B^{\frac{F\left(\left\lfloor \frac{n_p}{2} \right\rfloor\right)}{m_p}} \quad . \tag{8b}$$

Note that, above, 4.1 has been replaced by $\frac{1}{A}+B+1=\frac{1}{10}+3+1=4.1$.

It is especially significant that these equations generate the seven experimental quark and lepton mass ratios of Table II, because they make use of so few values chosen purely to fit the quark and lepton mass data. Their key values are either the interdependent and symmetric parameters n and $\hat{n}$, whose values are determined by the requirement of mass formulae consistency; or are small integers (the constants A and B) that were introduced earlier by the author to fit other mass data [2,3,4], and which, in any case, may be derived from the fine structure constant, as just demonstrated. The remaining values of the mass formulae are inherently trivial: the constant 1, which is used in the expression that substitutes for 4.1; the constant 2, which plays the same role in two exponents; and the parameter m_p, which equals either 1 or 2 for heavy and light particles, respectively.

In contrast, the mass ratios reproduced are non-trivial: They range across three orders of magnitude, and, where the tau- and muon-electron mass ratios are concerned, they are fit to roughly 1 part in 1,900, and 1 part in 40,000, respectively. All this supports the broad conclusion that the mass formulae work for physical, rather than accidental, reasons.

VIII. A general quark and lepton mass formula

Up until now all of the mass ratios produced have been valid only within these particle groups: the heavy quarks, the heavy leptons, the light quarks, and light leptons. These ratios therefore model only the mass ratios that hold within these four groups, and as a result constitute

four independent "islands of knowledge". It is logical to ask whether it is possible, via some general mass formula, to achieve an economical unification of these islands of knowledge. The answer is yes, by means of the following equation

$$M = \left(\frac{41}{10}\right)^{\frac{F(n)-F(\hat{n})}{2m}+\frac{s_1}{2}} \times \left(\frac{1}{10}\right)^{F\left(\left\lceil\frac{|\hat{n}|}{2}\right\rceil\right)+\frac{s_2}{(-1/3)^m}} \times 3^{\frac{F\left(\left\lfloor\frac{n}{2}\right\rfloor\right)}{m}-\frac{s_1+s_2+s_3}{2}} \quad . \tag{9}$$

Note that in Eq. (9) only the parameters s_1, s_2, and s_3 are new—the remaining parameters are the same as in Eqs. (2a) and (2b), and they retain the same values as were assigned earlier. The values for s_1, s_2, and s_3 will be as follows:

	s_1	s_2	s_3
Heavy Quarks	1	1	1
Neutrinos	0	1	1
Heavy Leptons	0	0	1
Light Quarks	0	0	0

Note that the values for s_n are assigned symmetrically, and that Eq. (9)'s exploitation of $\left(\frac{1}{10}\right)^{\left(\frac{s_2}{(-1/3)^m}\right)}$ mirrors Eq. (7b)'s three-fold use of $10^{\pm 3}$.

Although this general mass formula is more complicated than Eqs. (2a) and (2b), it is important in that it demonstrates at least one way to unify all four particle groups via a single mass equation. This formula now allows the calculation of all quark and lepton masses from the precisely known electron mass. These calculated values appear alongside their experimental values in Table III. In additional, five additional experimental values known for the light quarks appear alongside their calculated values in Table IV. Collectively, they fit the known mass data well, although the value for the c-quark mass is predicted to be somewhat higher than its experimental value, and the value of one neutrino mass splitting is predicted to be somewhat lower. Observe that the calculated mass of the heaviest neutrino mass eigenstate (Mass(ν_3) = 0.03521... eV) is near the lower end of the range determined by cosmological considerations such as [6]

$$0.03\,\text{eV} < \text{Mass}\,[\text{Heaviest}\ \nu_i] < 0.23\,\text{eV}\ .$$

IX. Summary

In summary, in this article seven experimentally known mass ratios of the quarks and leptons are reproduced by a symmetrical pair of mass formulae that generate the quark and lepton mass ratios that hold within these particle sub-groups: the heavy quarks, the heavy leptons, the light quarks, and neutrinos. It is then shown that the requirement that these mass

formulae be consistent automatically limits the number of particle families to three, and that the calculated masses they produce fit the experimental mass data closely. Finally, key mass formulae constants (specifically, 0.1, and 3) are shown to derive from the fine structure constant, and a single general mass formula is described that yields mass ratios between all quarks and leptons.

X. Conclusion

It is noteworthy that the Fibonacci numbers conveniently generate the proper values for the $2 \times 12 \times 3 = 72$ exponents of Eqs. (2a) and (2b), the mass formulae. If any of these 72 exponents were altered by just 1, its corresponding mass would have its value shifted by a factor of at least 3, which in almost all instances would shift the corresponding mass ratio to well outside its range of experimental error. This congruence of 72 exponents inevitably suggests that the mass formula works for some as yet unknown physical reason.

But why should Fibonacci numbers play such a role? Within the realm of physics Fibonacci numbers appear at least twice. They govern the self-organization into spirals of magnetized droplets in a magnetic field [9], and they play a role in helping understand non-periodic long-range order in quasicrystals [10].

Finally, it is interesting to conjecture what physical considerations might underpin the constants 4.1 and 0.1 of the mass formula. As the beta coefficients b_1 and $\tilde{b}_1$ of the extra-dimensional, non-supersymmetric GUT described by Dienes, Dudas, and Gherghetta [1] also equal 4.1 and 0.1, it is tempting to speculate whether a physical basis ties one, or both, of these beta coefficients to the mass formula.

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Appendix

*Assume a portion of the Fibonacci sequence **R** is initiated by 0 and 1 and extended rightwards to include at least eight terms*

$$\boldsymbol{R} = \{ 0, 1, 1, 2, 3, 5, 8, 13, \ldots \} .$$

*Then another portion of the Fibonacci sequence **L** cannot exist whose terms when paired one-to-one with those of **R** sum to a common value k.*

This follows because ***R*** contains two, and only two, repeated terms { 1, 1 }, and therefore ***L*** must likewise contain two, and only two, repeated terms, which when paired with { 1, 1 } sum to k. This requires that ***L*** take the form

$$\boldsymbol{L} = \{ \ldots , -8, 5, -3, 2, -1, 1, 0, 1 \} .$$

and that $k = 2$. (Note that ***L*** cannot be extended further rightwards as this would give ***L*** *three* 1s, and cannot be shortened on the right, as it would then have no repeated term.) Now if $k = 2$ there is no Fibonacci number that can be found to pair with the value 8 in ***R*** to sum to 2. Accordingly, a sequence ***L*** meeting the above requirement cannot exist.

Table I. Assignment of the values for the parameters $\hat{n}$, n, and m for all quarks and leptons. These parameters, along with the mass formulae Eqs. (2a) and (2b), are all that is needed to generate the quark and lepton mass ratios of Eqs. (4a)-(4d). Solid lines group those particles that possess the same electric charge Q. The Fibonacci numbers are $F(-4) = -3$, $F(-3) = 2$, $F(-2) = -1$, $F(-1) = 1$, $F(0) = 0$, $F(1) = 1$, $F(2) = 1$, $F(3) = 2$, $F(4) = 3$, $F(5) = 5$, while for all particles $F(\hat{n}) + F(n) = 2$.

	Light Particles	*Heavy Particles*			
	$m = 2$	$m = 1$	$\hat{n}$	n	$F(\hat{n}) + F(n)$
Quarks	$Q = +2/3$				
	u	c	1	1	2
	s	t	0	3	2
	d	b	−1	2	2
	$Q = -1/3$				
Leptons	$Q = 0$	$Q = -1$			
	ν_2	μ	−2	4	2
	ν_1	e	−3	0	2
	ν_3	τ	−4	5	2

Table II. Experimental versus calculated values for the quark and lepton mass ratios calculated using Eq. (1a) or (1b) and the parameters of Table I [4]. The experimental mass ratios for $\frac{M_t}{M_c}$, $\frac{M_b}{M_c}$, and $\frac{M_{\nu_3}}{M_{\nu_2}}$ below were formed by choosing from the experimental values' upper or lower bounds, in an effort to fit the calculated values [4,5,6]. (See text for discussion.) Experimentally, the t-quark's mass equals 178,000 ± 4,300 MeV; the b-quark's mass is from 4,100 to 4,400 MeV; while the c-quark's mass is from 1,150 to 1,350 MeV [4].

Mass Ratio	*Experimental Value*	*Calculated Value*
$\frac{M_\tau}{M_e}$	$\frac{1776.99}{0.510998918} = 3477.48...$	$4.1^5 \times 3 = 3475.68603$
$\frac{M_\mu}{M_e}$	206.76828	$4.1^3 \times 3 = 206.763$
$\frac{M_t}{M_c}$	$\frac{173{,}700}{1{,}350} = 128.66...$	$4.1 \times 3 \times 0.1^{-1} = 123$
$\frac{M_b}{M_c}$	$\frac{4{,}100}{1{,}350} = 3.037...$	3
$\frac{M_{\nu_3}}{M_{\nu_2}}$	$\sqrt{\frac{1.5\times10^{-3}\ \Delta\text{eV}^2}{8.3\times10^{-5}\ \Delta\text{eV}^2}} = 4.25...$	4.1
$\frac{M_u}{M_d}$	0.3 to 0.7	$\frac{1}{3^{\frac{1}{2}}} = 0.57735...$
$\frac{M_s}{M_d}$	17 to 22	$4.1^{\frac{1}{2}} \times 0.1^{-1} = 20.248...$

Table III. Values for the particle masses produced by Eq. (9), the general quark and lepton mass formula. All masses are calculated from the precisely known electron mass.

Particle	*Exp. Mass*[a] *(MeV)*	*Calc. Mass (MeV)*
t	178,000 ± 4,300 MeV	173,932.0...
b	4,100 – 4,400	4,242.2...
c	1,150 – 1,350	1,414.0...
s	80 – 130	127.26...
d	4 – 8	6.28...
u	1.5 – 4.0	3.62...
τ	$1776.99^{+0.29}_{-0.26}$	1776.07...
μ	105.658...	105.655...
e	0.510998918	(*used as base value*)
υ_3	$> 3 \times 10^{-8}$ and $< 23 \times 10^{-8}$ [b]	$3.521... \times 10^{-8}$
υ_2	-	$8.589... \times 10^{-9}$
υ_1	-	$5.973... \times 10^{-10}$

Mass Splitting	*Exp. Mass Splitting* (ΔeV^2)	*Calc. . Mass Splitting* (ΔeV^2)
$\lvert m(\nu_3)^2 - m(\nu_1)^2 \rvert$	$> 1.5 \times 10^{-3}$ and $< 3.9 \times 10^{-3}$ [a] (90% CL)	$1.239... \times 10^{-3}$
$\lvert m(\nu_3)^2 - m(\nu_2)^2 \rvert$	-	$1.166... \times 10^{-3}$
$\lvert m(\nu_2)^2 - m(\nu_1)^2 \rvert$	$7.1 \times 10^{-5\,+1.2}_{-0.6}$ [c] (99% CL)	$7.342... \times 10^{-5}$

[a]Reference 5.
[b]Based on cosmological considerations. Reference 6.
[c]Reference 7.

Table IV. The experimental versus calculated values for five additional values involving light quarks [5].

	Exp. Value	*Calc. Value*
$\frac{M_u}{M_d}$	0.3 to 0.7	0.57735...
$\frac{M_u + M_d}{2}$	3 to 5.5 MeV	4.95 MeV
$\frac{M_s}{M_d}$	17 to 22	20.248...
$\frac{M_s}{(M_u + M_d)/2}$	25 to 30	25.67...
$\frac{M_s - \frac{M_d + M_u}{2}}{M_d - M_u}$	30 to 50	46.0...