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Dual mass formulae that generate the quark and lepton masses

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Abstract

Two symmetrical mass formulae are introduced that closely reproduce nine experimentally known mass ratios of the quarks and leptons. Consistency of results between these mass formulae is achieved by exploiting a symmetry present in the initial terms of the Fibonacci sequence. This symmetry determines the mass formulae parameters and requires that there exist either one, or three, particle families. It is this three-family solution that produces the quark and lepton mass ratios that approximate their experimental values.

I. Introduction

Two symmetrical mass formulae are introduced that closely reproduce nine experimental mass ratios of the quarks and leptons. The formulae are not intended to explain the origin of mass, but rather to establish underlying phenomenological connections that may serve as a guide to a physical explanation. The formulae exploit constants equal to the beta coefficients $b_1 = 41/10$ and $\widetilde{b}_1 = 1/10$ of the extra-dimensional, non-supersymmetric GUT described by Dienes, Dudas, and Gherghetta [1]. Earlier, the author used these same constants to reproduce the experimental values of the fine structure constant, as well as the neutron-, and muon-electron mass ratios, at or very near their experimental limits [2,3], while also demonstrating, by using information theory, that this result is unlikely to be purely coincidental [4].

IIa. The mass formulae parameters

The Fibonacci sequence extends in both directions and includes the following terms

$$\ldots \quad -3 \quad 2 \quad -1 \quad 1 \quad \underline{0} \quad \underline{1} \quad 1 \quad 2 \quad 3 \quad 5 \quad \ldots \; .$$

Above, each term of the Fibonacci sequence equals the sum of the two terms that precede it. The initiators of the sequence 0 and 1 appear underlined.

To generate the mass formulae parameters, we begin by defining the sequence ***L*** as the six Fibonacci terms that are initiated by 0 and 1, and extended *left*wards:

$$-3 \quad 2 \quad -1 \quad 1 \quad \underline{0} \quad \underline{1} \ .$$

We then define the sequence ***R*** as equal to the six Fibonacci terms that are initiated by 0 and 1, and extended *right*wards:

$$\underline{0} \quad \underline{1} \quad 1 \quad 2 \quad 3 \quad 5 \ .$$

We then pair the terms of the sequences ***L*** and ***R*** so that they sum uniformly to 2.

$$\begin{array}{cccccc} -3 & 2 & -1 & 1 & 0 & 1 \\ +5 & +0 & +3 & +1 & +2 & +1 \\ \hline 2 & 2 & 2 & 2 & 2 & 2 \end{array}$$

As the Fibonacci numbers may be written $F(-4) = -3$, $F(-3) = 2$, $F(-2) = -1$, $F(-1) = 1$, $F(0) = 0$, $F(1) = 1$, $F(2) = 1$, $F(3) = 2$, $F(4) = 3$, $F(5) = 5$, ... etc., the above sums may be restated as follows.

$$\frac{F(-4) + F(5)}{2} \quad \frac{F(-3) + F(0)}{2} \quad \frac{F(-2) + F(4)}{2} \quad \frac{F(-1) + F(2)}{2} \quad \frac{F(0) + F(3)}{2} \quad \frac{F(1) + F(1)}{2}$$

Conveniently, the above sums serve as a ready template for assigning values to $\hat{n}$ and n for the quarks and leptons.

	τ	e	μ	b	t	c
	υ_3	υ_1	υ_2	d	s	u
$\hat{n}$	−4	−3	−2	−1	0	1
n	5	0	4	2	3	1

Note that the above parameter assignments are carried out with heavy particles paired with light particles in a natural way, with all pairings governed by mass. So, the heaviest heavy quark (t) is paired with the heaviest light quark (s); the lightest heavy quark (c) is paired with the lightest light quark (u); and so on.

In addition, the values for one additional variable m will be assigned as follows.

Particle Subgroups	*m*
Light Quarks & Light Leptons	2
Heavy Quarks & Heavy Leptons	1

The values for $\hat{n}$, *n*, and *m*—the only parameters used by the mass formulae—are summarized in Table I.

Finally, note that the above assignments guarantee that for any particle *p*

$$F(\hat{n}_p) + F(n_p) = 2 \quad . \tag{1a}$$

This implies that for any particles *p* and *q*

$$F(\hat{n}_p) + F(n_p) = 2 = F(\hat{n}_q) + F(n_q) \tag{1b}$$

or, equivalently,

$$F(n_p) - F(n_q) = -(F(\hat{n}_p) - F(\hat{n}_q)) \quad , \tag{1c}$$

which is the key relation that will guarantee consistency of results between the mass formulae defined below.

IIb. The mass formulae

Define $\lfloor x \rfloor$ as equal to the largest integer that is less than or equal to x; and define $\lceil x \rceil$ as equal to the smallest integer that is greater than or equal to x. Also define the symmetrical mass formulae

$$M(p) = 4.1^{\frac{F(n_p)}{m_p}} \times 0.1^{F\left(\left\lceil \frac{\lfloor \hat{n}_p \rfloor}{2} \right\rceil\right)} \times 3^{\frac{F\left(\left\lfloor \frac{n_p}{2} \right\rfloor\right)}{m_p}} \tag{2a}$$

$$\hat{M}(p) = 4.1^{\frac{-F(\hat{n}_p)}{m_p}} \times 0.1^{F\left(\left\lceil \frac{\lfloor \hat{n}_p \rfloor}{2} \right\rceil\right)} \times 3^{\frac{F\left(\left\lfloor \frac{n_p}{2} \right\rfloor\right)}{m_p}} , \tag{2b}$$

where $M(p)$ and $\hat{M}(p)$ equal relative mass for a particle p.

It is important to note that the only differences between the right sides of Eqs. (2a) and (2b) are in their exponents for 4.1; they are otherwise identical.

Also important is the fact that the above mass formulae are limited in scope, in that they are only meant to reproduce the quark and lepton mass ratios *within* these four particle subgroups: the heavy quarks, the heavy leptons, the light quarks, and neutrino mass eigenstates. That is to say, Eqs. (2a) and (2b) only are meant to reproduce the following eight mass ratios.

Particle Subgroup	*Mass Ratios*	
Neutrino mass eigenstates	$\frac{M(\nu_2)}{M(\nu_1)} = \frac{\hat{M}(\nu_2)}{\hat{M}(\nu_1)}$	$\frac{M(\nu_3)}{M(\nu_1)} = \frac{\hat{M}(\nu_3)}{\hat{M}(\nu_1)}$
Light quarks	$\frac{M(d)}{M(u)} = \frac{\hat{M}(d)}{\hat{M}(u)}$	$\frac{M(s)}{M(u)} = \frac{\hat{M}(s)}{\hat{M}(u)}$
Heavy leptons	$\frac{M(\mu)}{M(e)} = \frac{\hat{M}(\mu)}{\hat{M}(e)}$	$\frac{M(\tau)}{M(e)} = \frac{\hat{M}(\tau)}{\hat{M}(e)}$
Heavy quarks	$\frac{M(b)}{M(c)} = \frac{\hat{M}(b)}{\hat{M}(c)}$	$\frac{M(t)}{M(c)} = \frac{\hat{M}(t)}{\hat{M}(c)}$

The application of the mass formula is straightforward. By way of example, consider the muon-electron mass ratio, which appears in the above list, and which is therefore within the scope of the formulae. The muon-electron mass ratio may be calculated using either Eq. (2a)

$$\frac{M(\mu)}{M(e)} = \frac{4.1^{\frac{F(n_\mu)}{m_\mu}} \times 0.1^{F\left(\left\lceil \frac{|\hat{n}_\mu|}{2} \right\rceil\right)} \times 3^{\frac{F\left(\left\lfloor \frac{n_\mu}{2} \right\rfloor\right)}{m_\mu}}}{4.1^{\frac{F(n_e)}{m_e}} \times 0.1^{F\left(\left\lceil \frac{|\hat{n}_e|}{2} \right\rceil\right)} \times 3^{\frac{F\left(\left\lfloor \frac{n_e}{2} \right\rfloor\right)}{m_e}}} = \frac{4.1^{\frac{F(4)}{1}} \times 0.1^{F\left(\left\lceil \frac{|-2|}{2} \right\rceil\right)} \times 3^{\frac{F\left(\left\lfloor \frac{4}{2} \right\rfloor\right)}{1}}}{4.1^{\frac{F(0)}{1}} \times 0.1^{F\left(\left\lceil \frac{|-3|}{2} \right\rceil\right)} \times 3^{\frac{F\left(\left\lfloor \frac{0}{2} \right\rfloor\right)}{1}}} \tag{3a}$$

$$=\frac{4.1^{F(4)}\times 0.1^{F(1)}\times 3^{F(2)}}{4.1^{F(0)}\times 0.1^{F(2)}\times 3^{F(0)}}=\frac{4.1^{3}\times 0.1^{1}\times 3^{1}}{4.1^{0}\times 0.1^{1}\times 3^{0}}=4.1^{3}\times 3=206.763$$

or Eq. (2b)

$$\frac{\hat{M}(\mu)}{\hat{M}(e)}=\frac{4.1^{\frac{-F(\hat{n}_\mu)}{m_\mu}}\times 0.1^{F\left(\left\lceil\frac{|\hat{n}_\mu|}{2}\right\rceil\right)}\times 3^{\frac{F\left(\left\lfloor\frac{n_\mu}{2}\right\rfloor\right)}{m_\mu}}}{4.1^{\frac{-F(\hat{n}_e)}{m_e}}\times 0.1^{F\left(\left\lceil\frac{|\hat{n}_e|}{2}\right\rceil\right)}\times 3^{\frac{F\left(\left\lfloor\frac{n_e}{2}\right\rfloor\right)}{m_e}}}=\frac{4.1^{\frac{-F(-2)}{1}}\times 0.1^{F\left(\left\lceil\frac{|-2|}{2}\right\rceil\right)}\times 3^{\frac{F\left(\left\lfloor\frac{4}{2}\right\rfloor\right)}{1}}}{4.1^{\frac{-F(-3)}{1}}\times 0.1^{F\left(\left\lceil\frac{|-3|}{2}\right\rceil\right)}\times 3^{\frac{F\left(\left\lfloor\frac{0}{2}\right\rfloor\right)}{1}}} \tag{3b}$$

$$=\frac{4.1^{-F(-2)}\times 0.1^{F(1)}\times 3^{F(2)}}{4.1^{-F(-3)}\times 0.1^{F(2)}\times 3^{F(0)}}=\frac{4.1^{1}\times 0.1^{1}\times 3^{1}}{4.1^{-2}\times 0.1^{1}\times 3^{0}}=4.1^{3}\times 3=206.763 \quad ,$$

with identical results: $\frac{M(\mu)}{M(e)}=\frac{\hat{M}(\mu)}{\hat{M}(e)}=4.1^{3}\times 3=206.763$.

Note that Eqs. (3a) and (3b) produce consistent results because, for the muon and electron

$$\frac{4.1^{\frac{F(n_\mu)}{m_\mu}}}{4.1^{\frac{F(n_e)}{m_e}}}=\frac{4.1^{\frac{-F(\hat{n}_\mu)}{m_\mu}}}{4.1^{\frac{-F(\hat{n}_e)}{m_e}}}=4.1^{3} \quad ; \tag{3c}$$

or, substituting and simplifying,

$$\frac{4.1^3}{4.1^0} = \frac{4.1^1}{4.1^{-2}} = 4.1^3 \ .$$

As this equation makes clear, the only difference between Eqs. (3a) and (3b) is that their exponents for 4.1 have been uniformly shifted by 2. Crucially, the uniformity of this shift leaves the differences between exponents unchanged. It is this, along with the fact that the muon and electron share the same value for *m*, which enables Eqs. (3a) and (3b) to produce the same values for the muon-electron mass ratio, albeit in a slightly different manner.

As it is, equations equivalent to (3a) and (3b) for any other particles p and q will also undergo the same uniform shift, and therefore produce consistent mass ratios, provided of course that p and q share the same value for *m*, which of course they will if they are members of the same subgroup. The uniformity of the above shift is a direct consequence of Eq. (1c).

It should now be clear why, earlier, such care was taken in assigning values for $\hat{n}$ and n. It is the symmetry of the Fibonacci numbers $F(\hat{n}_p)$ and $F(n_p)$ that ultimately guarantees mass formulae consistency. More specifically, it is the symmetry possessed by the two 6-term sequences generated by the Fibonacci initiators 0 and 1 that allows the assignment of the values for $\hat{n}$ and n in such a way that $F(\hat{n}_p) + F(n_p) = 2$ for all particles p; and it is this which, in turn, guarantees that the mass formulae to produce consistent results.

As will be analyzed later, the absence of an equivalent symmetry for Fibonacci sequences of 8 or more terms automatically imposes a limit of 3 on the number of particle families. So, if the masses of 4 or more particle families were to be modeled, it would be impossible for Eqs.

(2a) and (2b) to fit such mass ratios—irrespective of what they were. No correct model of such mass ratios would be possible, because there would be no way to assign consecutive values to $\hat{n}$ and n so as to allow Eqs. (2a) and (2b) to yield consistent results.

Conversely, and perhaps surprisingly, the mere requirement that the values assigned for $\hat{n}$ and n achieve consistent results is enough to ensure mass formulae accuracy. That is to say, if $\hat{n}$ and n are chosen to produce consistent results, they will automatically produce accurate results.

III. Comparison of the calculated mass ratios against their experimental values

As was noted earlier, either Eq. (2a) or (2b) allows one to closely reproduce the experimental mass ratios that hold within these particle subgroups: the heavy quarks, the heavy leptons, the light quarks, and neutrino mass eigenstates, where these equations take their parameters from Table I. The following ratios are a consequence of Eqs. (2a) and (2b) and Table I:

$$\left(\frac{M(\nu_3)}{M(\nu_1)}\right)^2 = \frac{M(\tau)}{M(e)} = 4.1^5 \times 3 \tag{4a}$$

$$\left(\frac{M(\nu_2)}{M(\nu_1)}\right)^2 = \frac{M(\mu)}{M(e)} = 4.1^3 \times 3 \tag{4b}$$

$$\left(\frac{0.1 \times M(s)}{M(u)}\right)^2 = \frac{0.1 \times M(t)}{M(c)} = 4.1^1 \times 3 \tag{4c}$$

$$\left(\frac{M(d)}{M(u)}\right)^2 = \frac{M(b)}{M(c)} = 4.1^0 \times 3 \quad . \tag{4d}$$

And Eqs. (4a)-(4d) imply the mass ratios:

For the heavy leptons: $4.1^5 \times 3 \ : \ 4.1^3 \times 3 \ : 1$

For the neutrino mass eigenstates: $\sqrt{4.1^5 \times 3} : \sqrt{4.1^3 \times 3} : \sqrt{1}$

For the heavy quarks: $4.1 \times 10 \times 3 : 3 : 1$

For the light quarks: $\sqrt{4.1} \times 10 \times \sqrt{3} : \sqrt{3} : \sqrt{1}$.

When we compare these ratios against their corresponding experimental values, we find a remarkable fit:

The tau's measured mass equals $1776.99^{+0.29}_{-0.26}$ MeV [5], while the electron's measured mass is 0.510998918 MeV [5]. Dividing the lower end of the tau's mass by the electron mass yields a mass ratio of $\frac{1776.99 - 0.26}{0.510998918} = 3476.97...$. This is not very different from its calculated value of $4.1^5 \times 3 = 3475.686...$, from which it differs by roughly 1 part in 2,700.

Similarly, the experimental value for the muon-electron mass ratio equals 206.7682838 [5], versus a calculated value of 206.763. These differ by roughly 1 part in 40,000.

The t-quark's mass of 172,700 ± 2,900 MeV [6] and the c-quark's mass of 1,150 to 1,350 MeV [5] suggest a possible t-quark / c-quark mass ratio of 169,800 / 1,350 = 125.77..., which is near its calculated value of 123.

The b-quark's mass of 4,100 to 4,400 MeV [5] and the c-quark's mass of 1,150 to 1,350 MeV suggest a possible b-quark / c-quark mass ratio of 4,100 / 1,350 = 3.037..., which is near its calculated value of 3.

The above comparisons are summarized in Table II. In addition, Table III provides four additional calculated mass ratios involving light quarks, that are within their ranges of experimental error.

IVa. The neutrino squared-mass splittings

Equations (4a) and (4b) require that the masses of the neutrino mass eigenstates occur in the following ratios

$$\sqrt{4.1^5 \times 3} : \sqrt{4.1^3 \times 3} : \sqrt{1} \quad ,$$

and that the neutrino squared-mass splittings, in turn, fulfill the following ratios

$$\left(4.1^5 \times 3 - 1\right) : \left(4.1^5 \times 3 - 4.1^3 \times 3\right) : \left(4.1^3 \times 3 - 1\right) \quad .$$

It follows that

$$\frac{\left|M(\nu_3)^2 - M(\nu_1)^2\right|}{\left|M(\nu_2)^2 - M(\nu_1)^2\right|} = \frac{4.1^5 \times 3 - 1}{4.1^3 \times 3 - 1} = 16.8868\ldots \quad \text{, and,} \tag{5a}$$

$$\frac{\left|M(\nu_3)^2 - M(\nu_2)^2\right|}{\left|M(\nu_2)^2 - M(\nu_1)^2\right|} = \frac{4.1^5 \times 3 - 4.1^3 \times 3}{4.1^3 \times 3 - 1} = 15.8868\ldots \quad . \tag{5b}$$

Observational data exist for two neutrino squared-mass splittings, namely [5]

$$1.5 \times 10^{-3}\ \Delta\mathrm{eV}^2 < \left|M(\nu_\mu)^2 - M(\nu_x)^2\right| < 3.9 \times 10^{-3}\ \Delta\mathrm{eV}^2$$

and [9]

$$\left|M(\nu_e)^2 - M(\nu_x)^2\right| = 7.1 \times 10^{-5}\,{}^{+1.2}_{-0.6}\ \Delta\mathrm{eV}^2 \ .$$

As this second neutrino squared-mass splitting is the more precisely-known of the two, it may be used as a starting point to calculate $\left|M(\nu_\mu)^2 - M(\nu_x)^2\right|$, as well as the remaining unknown neutrino squared-mass splitting:

$$\frac{4.1^5 \times 3 - 1}{4.1^3 \times 3 - 1} \times 7.1 \times 10^{-5}\,{}^{+1.2}_{-0.6}\ \Delta\mathrm{eV}^2 \approx 1.19^{+0.2}_{-0.1} \times 10^{-3}\ \Delta\mathrm{eV}^2 \quad \text{, and,} \tag{5c}$$

$$\frac{4.1^5 \times 3 - 4.1^3 \times 3}{4.1^3 \times 3 - 1} \times 7.1 \times 10^{-5}\,{}^{+1.2}_{-0.6}\ \Delta\mathrm{eV}^2 \approx 1.12^{+0.2}_{-0.1} \times 10^{-3}\ \Delta\mathrm{eV}^2 \ . \tag{5d}$$

These predictions offer an opportunity to test the mass formulae's validity, especially as the Eq. (5c)'s value for $\left|M(\nu_\mu)^2 - M(\nu_x)^2\right|$ is predicted to be slightly below its experimental value.

IVb. The neutrino mass eigenstates

The above neutrino squared-mass splittings allow one to calculate $M(\nu_1)$, $M(\nu_2)$, and $M(\nu_3)$ from the observed value for $\left|M(\nu_e)^2 - M(\nu_x)^2\right|$. Thus,

$$M(\nu_2) = \sqrt{\frac{\left|M(\nu_1)^2 - M(\nu_2)^2\right|}{1 - \dfrac{1}{4.1^3 \times 3}}} = \sqrt{\frac{7.1 \times 10^{-5}\,{}^{+1.2}_{-0.6}\ \Delta\mathrm{eV}^2}{1 - \dfrac{1}{4.1^3 \times 3}}} = 8.5 \times 10^{-3}\,{}^{+0.7}_{-0.4}\ \Delta\mathrm{eV}^2 \ .$$

If follows that

$$M(\nu_1) = \frac{M(\nu_2)}{\sqrt{4.1^3 \times 3}} = 5.9 \times 10^{-3}\,{}^{+0.5}_{-0.3}\Delta\mathrm{eV}^2$$

and

$$M(\nu_3) = M(\nu_2) \times 4.1 = 3.5 \times 10^{-3}\,{}^{+0.4}_{-0.2}\Delta\mathrm{eV}^2 \ .$$

Note that the calculated mass of the heaviest neutrino mass eigenstate $M(\nu_3)$ is within range of cosmological considerations such as [10]

$$0.03\ \text{eV} < \text{Mass}\,[\text{Heaviest}\ \nu_{\text{i}}] < 0.23\ \text{eV}\ .$$

V. An automatic limit of three on the number of particle families

It is helpful to examine in detail the two Fibonacci sequences responsible for reproducing the quark and lepton mass ratios; this is to say, the sequences that arise when the Fibonacci sequence initiators 0 and 1 are extended in both directions to a length of 6 terms.

Values for $F(n)$:					0	1	1	2	3	5
Values for $F(\hat{n})$:	−3	2	−1	1	0	1				

Earlier, by appropriately pairing the quarks and leptons with the above terms, we saw to it that $F(\hat{n})$ and $F(n)$ summed to a common value, in this case 2:

$$-3+5=2+0=-1+3=1+1=0+2=1+1=2\,.$$

In this way, Eq. (1c) was fulfilled and the mass formulae produced consistent values for the quark and lepton masses.

It is interesting that the above Fibonacci sequences cannot be lengthened to accommodate 4 or more particle families. To see why, consider that if more than 4 particle families were modeled, the above Fibonacci sequences would have to be correspondingly extended to contain 8 or more terms. But an inspection of the first 8 terms of the Fibonacci sequence

$$\begin{array}{cccccccc} 0 & 1 & 1 & 2 & 3 & 5 & 8 & 13 \\ +\,? & +\,? & +\,? & +\,? & +\,? & +\,? & +\,? & +\,? \\ \hline k & k & k & k & k & k & k & k \end{array}$$

shows that no other sequence of 8 consecutive Fibonacci numbers can be found to pair with them to sum to a common integer *k*. Furthermore, this problem remains even if the sequence is extended to more than 8 terms (see Appendix for proof).

This inevitable mismatch of terms sees to it that the mass formulae cannot produce consistent results for 4 or more particle families. Nor, for that matter, can it accommodate just 2 particle families, for the same reason, though it can accommodate just 1, as the sequence initiators may be paired with each other to produce a common sum. This "single-family solution" takes the following form.

$$\begin{array}{cc} 1 & 0 \\ +0 & +1 \\ \hline 1 & 1 \end{array}$$

Although the above conclusions should not be taken as absolute, particularly as it is inevitable that a sufficient "loosening of the framework" may make it possible to accommodate more than 3 families, nevertheless, it should not be overlooked that the above framework offers a natural way to limit the number of particle families to 3, and that any modifications to the above framework might very well rob it of its simplicity.

VI. Unambiguous steps that generate the quark and lepton mass ratios

It is instructive to identify an unambiguous set of steps that will generate the quark and lepton mass ratios of Eqs. (4a)-(4d), while automatically disallowing 4 or more particle families:

Step One: We begin by assuming that Eqs. (2a) and (2b) govern particle mass, and that their values for m equal 1 for heavy particles H_n, and 2 for light particles L_n.

Step Two: We then assume there exist N pairs of heavy and light particles (H_0, L_0), (H_1, L_1), ... , (H_{N-1}, L_{N-1}), where each pair is assigned an integer n, as follows:

Particle Pair	n
(H_0, L_0)	0
(H_1, L_1)	1
...	...
(H_{N-1}, L_{N-1})	$N-1$

Step Three: We further assume that a second set of N consecutive integers $\hat{n}$ is also mapped one-to-one to each pair of particles. The values for $\hat{n}$ need not map over in any particular order, but in order to assure consistency between Eqs. (2a) and (2b), the sum $F(\hat{n}_p)+F(n_p)$ must produce the same value for all particles p.

Step Four: Under the above restrictions, Eqs. (2a) and (2b) can achieve consistency in only two ways: via the *single-family solution* noted earlier, where

$$\hat{n} = \{ 0, 1 \} \quad \text{and} \quad n = \{ 0, 1 \} \ ,$$

and via the *three-family solution* described at the outset of this article, where

$$\hat{n} = \{ -4, -3, -2, -1, 0, 1 \} \ \text{and} \ n = \{ 0, 1, 2, 3, 4, 5 \} \ \ .$$

We discard the single-family solution and retain the three-family solution, where the values for $\hat{n}$ and n are paired and mapped to particles as in Table I. This three-family solution is then used to compute, in the form of either $\frac{M(p)}{M(q)}$ or $\frac{\hat{M}(p)}{\hat{M}(q)}$, the eight independent mass ratios that hold within the following subgroups: the heavy quarks, the heavy leptons, the light quarks, and neutrino mass eigenstates.

VII. Are the mass formulae successful for physical, or accidental, reasons?

The above framework generates the particle masses via formulae that take the general form of

$$M(p) = J^{\frac{F(n_p)}{m_p}} \times K^{F\left(\left\lceil \frac{|\hat{n}_p|}{2} \right\rceil\right)} \times L^{\frac{F\left(\left\lfloor \frac{n_p}{2} \right\rfloor\right)}{m_p}} \tag{6a}$$

and

$$\hat{M}(p) = J^{\frac{-F(\hat{n}_p)}{m_p}} \times K^{F\left(\left\lceil \frac{|\hat{n}_p|}{2} \right\rceil\right)} \times L^{\frac{F\left(\left\lfloor \frac{n_p}{2} \right\rfloor\right)}{m_p}} , \tag{6b}$$

where the values for the constants J, K, and L are as follows

$$J = \frac{41}{10} ,$$

$$K = \frac{1}{10} ,$$

$$L = 3 .$$

In Table I, the values for the parameters $\hat{n}$ and n are listed. Clearly these parameters are not easily fine-tuned in order to make Eqs. (6a) and (6b) fit the mass data. This is so partly because these parameters are sequences of consecutive integers, but more importantly because the values for $\hat{n}$ and n are rigidly constrained by the need for consistency of results between Eqs. (6a) and (6b). Consequently the parameters of Table I offer virtually no opportunity for fine-tuning the mass formulae parameters to fit the mass data by accident.

But it must be added that J, K, and L also cannot be fine-tuned in order to make Eqs. (6a) and (6b) fit the mass data. This is because J, K, and L were not specifically chosen to fit the quark and lepton mass data, but instead are constants originally selected to generate the mass ratios of a quite different set of particles. More specifically, the constants 4.1, 0.1, and 3 were first introduced by the author to generate the π^0 meson-, J/ψ meson-, muon- and neutron-electron mass ratios [2,3,4]. Their reuse here, therefore, merely maintains consistency with earlier work.

Accordingly, the constants 4.1, 0.1, and 3 cannot be regarded as values selected to accommodate the quark and lepton masses. That they can still manage to generate the quark and lepton masses, despite this independent origin, must be taken as key evidence for their physical, rather than accidental, origin.

It is also suggestive that the fine structure constant reciprocal $\frac{1}{\alpha}$ may be approximated closely with the aid of the constants $K = \frac{1}{10}$ and $L = 3$ of Eqs. (6a) and (6b)

$$\frac{1}{\alpha} \approx \frac{1}{(KL)^3} + \frac{1}{K^2} = \frac{10^3}{3^3} + 10^2 = 137.037037\ldots \, , \tag{7a}$$

where the 2002 CODATA value for $\frac{1}{\alpha}$ equals 137.03599911 (46) [11]. The effectiveness of this approximation lends key additional support to the conjecture that the constants 1/10, and 3 are not arbitrary.

Of course, one could plausibly object that the above approximation achieves its close fit of $\frac{1}{\alpha}$ by coincidence, and that other approximations of the same form might achieve a better fit while employing even smaller integers.

To resolve this issue, a computer searched for a better approximation of $\frac{1}{\alpha}$ in the form

$$\frac{A^a}{B^b} + C^c \ ,$$

where the exponents a, b, and c were integers arbitrarily allowed to range from 0 to 5, inclusive, and A, B, and C were integers allowed to range from 1 to 10, inclusive. Across these ranges no better approximation was found.

As it is, to find a better approximation requires that A, B, and C be allowed to range up to 37, as follows

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28^3 / 37^2 + 11^2 = 137.0350620... ,
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with, once again, a, b, and c limited to between 0 and 5, inclusive. Accordingly, *for values of A, B, and C less than 37*, the best fit is achieved by the unusually small integers

$$A = C = 10 \ ,$$

$$B = 3 \ ,$$

which, of course, are the same constants relied upon by the mass formulae.

Finally, it is interesting to carry out an additional search for a refined version of the approximation $\frac{10^3}{3^3} + 10^2$, specifically one in the form $\frac{10^3 - D^d}{3^3} + 10^2 - E^e$, where the exponents d and e are integers arbitrarily allowed to range from 0 to −3, inclusive, and D and E are integers arbitrarily allowed to range from 1 to 30, inclusive. Within these restrictions the best fit of the experimental value of the fine structure constant inverse is provided when $D = E = 10$ and $d = e = -3$, so that $D^d = E^e = 10^{-3}$, and

$$\frac{1}{\alpha} \approx \frac{10^3 - 10^{-3}}{3^3} + 10^2 - 10^{-3} = \frac{999.999}{3^3} + 99.999 = 137.036 \ . \qquad (7b)$$

Remarkably, the integer 10 now occurs no less than four times, while reproducing exactly the celebrated 137.036. This four-fold repetition of 10 is suggestive that Eq. (7b) is physically significant, and that the constants 10 and 3 may be fundamental constants of nature.

Because $A = 10 = \frac{1}{K}$ and $B = 3 = L$, one may readily restate Eqs. (6a) and (6b) in terms of A and B as follows

$$M(p) = \left(1 + \frac{1}{A} + B\right)^{\frac{F(n_p)}{m_p}} \times \left(\frac{1}{A}\right)^{F\left(\left\lceil \frac{|\hat{n}_p|}{2} \right\rceil\right)} \times B^{\frac{F\left(\left\lfloor \frac{n_p}{2} \right\rfloor\right)}{m_p}} \tag{8a}$$

and

$$\hat{M}(p) = \left(1 + \frac{1}{A} + B\right)^{\frac{-F(\hat{n}_p)}{m_p}} \times \left(\frac{1}{A}\right)^{F\left(\left\lceil \frac{|\hat{n}_p|}{2} \right\rceil\right)} \times B^{\frac{F\left(\left\lfloor \frac{n_p}{2} \right\rfloor\right)}{m_p}} . \tag{8b}$$

Note that, above, 4.1 has been replaced by $1 + \frac{1}{A} + B$.

It is especially significant that these equations generate the nine experimental quark and lepton mass ratios of Tables II and III, because they make use of no important values chosen to fit the quark and lepton mass data. Their key values are either the interdependent and symmetric parameters n and $\hat{n}$, whose values are determined by the requirement of mass formulae consistency; or are small integers (the constants A and B) that were introduced earlier by the author to fit other mass data [2,3,4], and which, in any case, may be derived from the fine structure constant, as just demonstrated. The remaining values of the mass formulae are inherently trivial: the constant 1, which is used in the expression that substitutes for 4.1; the constant 2, which plays the same role in two exponents; and the parameter *m*, which equals either 1 or 2 for heavy and light particles, respectively.

In contrast, the mass ratios reproduced are non-trivial: they range across three orders of magnitude, and, where the tau- and muon-electron mass ratios are concerned, they are fit to roughly 1 part in 2,700, and 1 part in 40,000, respectively. All this supports the broad conclusion that the mass formulae work for physical, rather than accidental, reasons.

VIII. Summary and Conclusion

In summary, in this article nine experimentally known mass ratios of the quarks and leptons are reproduced by a symmetrical pair of mass formulae that generate the quark and lepton mass ratios that hold within these particle subgroups: the heavy quarks, the heavy leptons, the light quarks, and neutrino mass eigenstates. It is shown that the requirement that these mass formulae be consistent automatically limits the number of particle families to 3, and that the calculated masses they produce approximate the experimental mass data. Finally, a link is established between the mass formulae constants 0.1 and 3, and the fine structure constant.

It is noteworthy that the Fibonacci numbers conveniently generate the proper values for the $2 \times 12 \times 3 = 72$ exponents of Eqs. (2a) and (2b), the mass formulae. If any of these 72 exponents were altered by just 1, its corresponding mass would have its value shifted by a factor of at least 3. In almost all instances this would shift the corresponding mass ratio to well outside its range of experimental error. This congruence of 72 exponents inevitably suggests that the mass formula works for some, as yet unknown, physical reason.

But why should the Fibonacci numbers play such a role? Within the realm of physics Fibonacci numbers appear at least three times. They govern the self-organization into spirals of magnetized droplets in a magnetic field [12]. They play a role in helping understand quasiperiodicity in quasicrystals [13]. And they may be generated from the "magic numbers" that correspond to the total number of electrons in filled electron shells; to be specific, the expression $\lfloor Z/18+1/2 \rfloor$ generates the first six Fibonacci numbers, as Z assumes the atomic numbers of the noble gases [14].

Noble Gas	*Atomic Number Z*	$\lfloor Z/18+1/2 \rfloor$
Helium	2	0
Neon	10	1
Argon	18	1
Krypton	36	2
Xenon	54	3
Radon	86	5

Finally, it is interesting to conjecture what physical considerations might underpin the constants 4.1 and 0.1 of the mass formula. As the beta coefficients b_1 and $\tilde{b}_1$ of the extra-dimensional, non-supersymmetric GUT described by Dienes, Dudas, and Gherghetta [1] also equal 4.1 and 0.1, it is tempting to speculate whether a physical basis ties one, or both, of these beta coefficients to the mass formula.

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Appendix

Assume a portion of the Fibonacci sequence ***R*** *is initiated by 0 and 1 and extended rightwards to include at least eight terms*

$$\boldsymbol{R} = \{\, 0,\ 1,\ 1,\ 2,\ 3,\ 5,\ 8,\ 13, \ldots \,\}\,.$$

Then another portion of the Fibonacci sequence ***L*** *cannot exist whose terms when paired one-to-one with those of* ***R*** *sum to a common value k.*

This follows because ***R*** contains two, and only two, repeated terms { 1, 1 }, and therefore ***L*** must likewise contain two, and only two, repeated terms, which when paired with { 1, 1 } sum to *k*. This requires that ***L*** take the form

$$L = \{\ \dots, -8, 5, -3, 2, -1, 1, 0, 1\ \}\ .$$

and that $k = 2$. (Note that $\boldsymbol{L}$ cannot be extended further rightwards as this would give $\boldsymbol{L}$ *three* 1s, and it cannot be shortened on the right, as it would then have *no* repeated terms.) Now if $k = 2$ there is no Fibonacci number that can be found to pair with the value 8 in $\boldsymbol{R}$ to sum to 2. Accordingly, a sequence $\boldsymbol{L}$ meeting the above requirements cannot exist.

Table I. Assignment of the values for the parameters $\hat{n}$, n, and m for all quarks and leptons. These parameters, along with the mass formulae Eqs. (2a) and (2b), are all that is needed to generate the quark and lepton mass ratios of Eqs. (4a)-(4d). Solid lines group those particles that possess the same electric charge Q. The Fibonacci numbers are $F(-4) = -3$, $F(-3) = 2$, $F(-2) = -1$, $F(-1) = 1$, $F(0) = 0$, $F(1) = 1$, $F(2) = 1$, $F(3) = 2$, $F(4) = 3$, $F(5) = 5$, while for all particles $F(\hat{n}) + F(n) = 2$.

	Light Particles	*Heavy Particles*			
	$m = 2$	$m = 1$	$\hat{n}$	n	$F(\hat{n}) + F(n)$
Quarks	$Q = +2/3$				
	u	c	1	1	2
	s	t	0	3	2
	d	b	−1	2	2
	$Q = -1/3$				
Leptons	$Q = 0$	$Q = -1$			
	ν_2	μ	−2	4	2
	ν_1	e	−3	0	2
	ν_3	τ	−4	5	2

Table II. Experimental versus calculated values for the quark and lepton mass ratios, calculated using Eq. (2a) or (2b) and the parameters of Table I. The experimental mass ratios for $\frac{M_t}{M_c}$, $\frac{M_b}{M_c}$, and $\frac{M_{\nu_3}}{M_{\nu_2}}$ below were formed by choosing from the experimental values' upper or lower bounds, in an effort to fit the calculated values. Experimentally, the t-quark's mass equals 172,700 ± 2,900 MeV [6], the b-quark's mass ranges from 4,100 to 4,400 MeV [5], while the c-quark's mass ranges from 1,150 to 1,350 MeV [5]. See text for discussion of the neutrino squared-mass splittings.

Mass Ratio	*Experimental Value*	*Calculated Value*
$\frac{M_\tau}{M_e}$	$\frac{1776.99-0.26}{0.510998918}=3476.97...$ [a]	$4.1^5\times 3=3475.686...$
$\frac{M_\mu}{M_e}$	206.7682838 [a]	$4.1^3\times 3=206.763$
$\frac{M_t}{M_c}$	$\frac{172{,}700-2{,}900}{1350}=125.77...$ [a,b]	$4.1\times 3\times 0.1^{-1}=123$
$\frac{M_b}{M_c}$	$\frac{4{,}100}{1{,}350}=3.037...$ [a]	3
$\left\|M(\nu_\mu)^2-M(\nu_x)^2\right\|$	$> 1.5\times 10^{-3}\ \Delta eV^2$ $< 3.9\times 10^{-3}\ \Delta eV^2$ [a,c]	$1.19^{+0.2}_{-0.1}\times 10^{-3}\ \Delta eV^2$

[a]Reference 5.
[b]Reference 6.
[c]Reference 9.

Table III. The experimental versus calculated values for four additional ratios involving light quarks [5].

Mass Ratio	*Experimental Value*	*Calculated Value*
$\frac{M_u}{M_d}$	0.3 to 0.7	$\frac{1}{3^{\frac{1}{2}}} = 0.57735\ldots$ [a]
$\frac{M_s}{M_d}$	17 to 22	$4.1^{\frac{1}{2}} \times 0.1^{-1} = 20.248\ldots$ [b]
$\frac{M_s}{(M_u + M_d)/2}$	25 to 30	25.674...
$\frac{M_s - \frac{M_d + M_u}{2}}{M_d - M_u}$	30 to 50	46.042...

[a]This calculated value is within 4% of the first order approximation produced by chiral perturbation theory [5,7,8]: $\frac{M_u}{M_d} = \frac{2M_{\pi^0}^2 - M_{\pi^+}^2 + M_{K^+}^2 - M_{K^0}^2}{M_{K^0}^2 - M_{K^+}^2 + M_{\pi^+}^2} = 0.5560\ldots$.

[b]This calculated value is within 1/2% of the first order approximation produced by chiral perturbation theory [5,7,8]: $\frac{M_s}{M_d} = \frac{M_{K^0}^2 + M_{K^+}^2 - M_{\pi^+}^2}{M_{K^0}^2 - M_{K^+}^2 + M_{\pi^+}^2} = 20.152\ldots$.