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Some comments on the ambiguity of string time-evolution *

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Abstract:

The Nambu-Goto equations of motion for a free string in flat space-time are brought into a form which makes the ambiguity of the time-evolution explicit. It is achieved by finding the "general solution" for the second time-derivatives. A possible application to path-integral motivated descriptions of string dynamics is pointed out. Moreover, one may use the formalism to write down a version of the equations of motion which drags the string into the gauge $\vec{x}'\dot{\vec{x}} = 0$ asymptotically.

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1. Reformulation of the Nambu-Goto equations

The Nambu-Goto equations of motion for a free string in flat space are invariant under reparametrizations on the world-sheet (for reviews see Refs. [1-3]). Hence, they constitute a constrained system. If the time-coordinate $t \equiv x^0$ is chosen as the timelike parameter, part of this invariance remains. In this case, the Nambu-Goto Lagrangian reads

$$\mathcal{L} = -\sqrt{(\dot{\vec{x}}\vec{x}\,')^2 + \vec{x}\,'^2(1 - \dot{\vec{x}}^{\,2})} \tag{1.1}$$

where $\vec{x} \equiv \vec{x}\,(t, \sigma)$, the corresponding equations of motion are

$$\frac{\partial}{\partial t}\frac{\partial}{\partial \dot{x}_i}\mathcal{L} = -\frac{\partial}{\partial \sigma}\frac{\partial}{\partial x_i'}\mathcal{L}, \tag{1.2}$$

where we will place all three-indices downstairs (as long as we are not dealing with four-vectors, as we will do in sect.4). One usually adopts a condition fixing a particular "gauge" for the coordinate σ, as for example [4] $\dot{\vec{x}}\vec{x}\,' = \dot{\vec{x}}^{\,2} + \vec{x}\,'^2 - 1 = 0$. Here, σ measures the total energy along the string, and (1.2) reduces to three two-dimensional wave equations $\ddot{\vec{x}} = \vec{x}\,''$. In this note, we will not assume such a choice but deal with (1.2) in its unconstrained form. Hence, prescribing initial values $\vec{x}\,(0, \sigma)$ and $\dot{\vec{x}}\,(0, \sigma)$, the time-evolution is not uniquely fixed. The purpose of this work is to bring (1.2) into a form which illustrates the ambiguity of the time-evolution explicitly but does not seem to be known to a broader audience. The idea is to solve (1.2) with respect to $\ddot{\vec{x}}$ (in the sense of finding the most general solution).

Let us choose

$$\begin{aligned} \vec{q}\,(t, \sigma) &= \vec{x}\,'(t, \sigma), \\ \vec{p}\,(t, \sigma) &= \dot{\vec{x}}\,(t, \sigma) \end{aligned} \tag{1.3}$$

as basic variables and introduce the projection matrix (from now on suppressing the dependence on t and σ)

$$P_{ij} = \delta_{ij} - \frac{q_i\, q_j}{\vec{q}^{\,2}}. \tag{1.4}$$

The component of $\dot{\vec{x}}$ orthogonal to the string tangent is given by (see e.g. [3])

$$v_i = P_{ij}\, \dot{x}_j \equiv p_i - \frac{\vec{q}\,\vec{p}}{\vec{q}^{\,2}}\, q_i \tag{1.5}$$

and may be denoted as the physical velocity of the string. Then the Nambu-Goto Lagrangian (1.1) reads

$$\mathcal{L} = -|\vec{q}|\sqrt{1-\vec{v}^{\,2}}. \tag{1.6}$$

Carrying out the derivatives in (1.2), and making use of the quantities defined so far, one obtains (upon separating $\ddot{\vec{x}}$)

$$\ddot{x}_j \frac{\partial^2}{\partial \dot{x}_i \partial \dot{x}_j}\sqrt{1-\vec{v}^{\,2}} = -\frac{1}{|\vec{q}|}X_i \tag{1.7}$$

with

$$X_i = \Big(\frac{\partial^2 \mathcal{L}}{\partial \dot{x}_i \partial x'_j} + \frac{\partial^2 \mathcal{L}}{\partial x'_i \partial \dot{x}_j}\Big)\dot{x}'_j + \frac{\partial^2 \mathcal{L}}{\partial x'_i \partial x'_j}\, x''_j. \tag{1.8}$$

The left-hand-side of (1.7) turns out to be

$$-\frac{1}{\sqrt{1-\vec{v}^{\,2}}} P_{ik}\, Q_{kl}\, P_{lj}\, \ddot{x}_j \tag{1.9}$$

where

$$Q_{kl} = \delta_{kl} + \frac{v_k\, v_l}{1-\vec{v}^{\,2}}. \tag{1.10}$$

Hence, multiplying by $-(1-\vec{v}^{\,2})^{1/2}$, the Nambu-Goto equations of motion (1.2) take the form

$$P_{ik}\, Q_{kl}\, P_{lj}\, \ddot{x}_j = \frac{\mathcal{L}}{\vec{q}^{\,2}} X_i \equiv Y_i. \tag{1.11}$$

The evaluation of Y_i requires some tedious but straightforward manipulations. For technical ease, one should define $\mathcal{L} = -\sqrt{W}$ and compute (1.8) in terms of W. The result may be brought into the form

$$Y_i = \frac{1}{\vec{q}^{\,2}} P_{ik}\, Q_{kl}\, P_{lj} \Big((1-\vec{p}^{\,2})\, x''_j + 2\,(\vec{q}\vec{p})\, p'_j\Big) \tag{1.12}$$

Putting everything together, the equations of motion (1.2) turn out to be identical to

$$P_{ik}\, Q_{kl}\, P_{lj} \Big(\dot{p}_j - \frac{1-\vec{p}^{\,2}}{\vec{q}^{\,2}}\, q'_j - 2\,\frac{\vec{q}\vec{p}}{\vec{q}^{\,2}}\, p'_j\Big) = 0. \tag{1.13}$$

2. "Constraint-free" formulation of time-evolution

In order to proceed, we note that the determinant of the matrix Q_{ij} from (1.10) is given by $(1-\vec{v}^{\,2})^{1/2}$ which is generically non-zero. Hence,

the ambiguity in the time-evolution as described by (1.13) stems from the projection matrix P_{ij}. Explicit computation gives

$$P_{ik}\, Q_{kl}\, P_{lj} = P_{ij} + \frac{v_i\, v_j}{1-\vec{v}^{\,2}} \tag{2.1}$$

which is (using $\vec{q}\vec{v} = 0$) generically of rank 2. Hence, an equation of the type

$$P_{ik}\, Q_{kl}\, P_{lj}\, a_j = 0 \tag{2.2}$$

has the general solution $a_j = f q_j$, where f is arbitrary. Note that (1.13) is precisely of this type. The dynamics of the free string may thus be written in the form

$$\dot{\vec{q}} = \vec{p}\,', \tag{2.3a}$$

$$\dot{\vec{p}} = \frac{1-\vec{p}^{\,2}}{\vec{q}^{\,2}}\,\vec{q}\,' + 2\,\frac{\vec{q}\vec{p}}{\vec{q}^{\,2}}\,\vec{p}\,' + f\vec{q}. \tag{2.3b}$$

The connection to the standard formalism is as follows: $\vec{x}\,(t,\sigma)$ satisfies the Nambu-Goto equations (1.2) if and only if there exists a function $f(t,\sigma)$ such that (2.3b) is satisfied. One may likewise adopt $\vec{q}$ and $\vec{p}$ as the basic ("phase space") variables and consider both equations (2.3) as the complete system.

It is possible to invert this argument, prescribe the function f (e.g. as $f(t,\sigma)$ or $f(\vec{q},\vec{p},\vec{q}\,',\vec{p}\,')$ or even as a functional) and integrate (2.3) from initial conditions $(\vec{q}\,(0,\sigma),\vec{p}\,(0,\sigma))$. Clearly, the resulting string configuration will satisfy the correct equations of motion.

Two simple choices are $f = 0$ and

$$f = \partial_\sigma\Big(\frac{1-\vec{p}^{\,2}}{\vec{q}^{\,2}}\Big). \tag{2.4}$$

In the latter case, (2.3b) becomes

$$\dot{\vec{p}} = \partial_\sigma\Big(\frac{1-\vec{p}^{\,2}}{\vec{q}^{\,2}}\,\vec{q}\Big) + 2\,\frac{\vec{q}\vec{p}}{\vec{q}^{\,2}}\,\vec{p}\,'. \tag{2.5}$$

A third possibility is mentioned in sect.5 (see eq.(5.4)). Note that there are no constraints necessary for the variables $(\vec{q},\vec{p})$ in this approach.

3. Reparametrizations and gauge-fixing

The invariance of the system under reparametrizations of the parameter σ,

$$\sigma^{\text{new}} = \Sigma(t,\sigma), \tag{3.1}$$

is reflected here by the freedom in choosing f. Taking (3.1) in its infinitesimal form

$$\sigma^{\text{new}} = \sigma + \epsilon\,(t, \sigma) \tag{3.2}$$

(i.e. neglecting $O(\epsilon^2)$), two string configurations are equivalent if they differ by a variation

$$\begin{aligned} \delta\vec{q} &= \epsilon'\vec{q} + \epsilon\,\vec{q}\,' \equiv \partial_\sigma(\epsilon\vec{q}), \\ \delta\vec{p} &= \dot{\epsilon}\vec{q} + \epsilon\,\vec{p}\,' \equiv \partial_t(\epsilon\vec{q}), \end{aligned} \tag{3.3}$$

the very last identity sign being valid only "on shell", i.e. on account of (2.3a). Their respective equations of motion differ by

$$\delta f = \ddot{\epsilon} - \frac{1-\vec{p}^{\,2}}{\vec{q}^{\,2}}\,\epsilon'' - 2\,\frac{\vec{q}\vec{p}}{\vec{q}^{\,2}}\,\dot{\epsilon}' + \epsilon\,f' - \epsilon' f. \tag{3.4}$$

This illustrates that arbitrary reparametrizations (3.1) are allowed within this formalism.

One may likewise use a specific choice of f in order to propagate a gauge-condition. If, for example, the usual constraints $\vec{q}\vec{p} = \vec{q}^{\,2} + \vec{p}^{\,2} - 1 = 0$ are adopted for the initial conditions, f may be determined such that they are conserved by (2.3). The result is $f = 0$, and (2.3b) reduces to $\dot{\vec{p}} = \vec{q}\,'$.

4. Covariant formalism

It is possible to start with the Nambu-Goto action for a string $x^\mu(\tau, \sigma)$ which is completely unconstrained in its world-sheet parametrization,

$$\mathcal{L} = -\sqrt{(\dot{x}x')^2 - x'^2\dot{x}^2}, \tag{4.1}$$

where $\dot{x}^2 \equiv \eta_{\mu\nu}\dot{x}^\mu\dot{x}^\nu$ etc. with $\eta = \text{diag}(1, -1, -1, -1)$, and a dot denoting ∂_τ.

The analysis of sect.1 may be repeated almost identically. Instead of (2.1), one encounters the matrix

$$M^{\mu\nu} = \eta^{\mu\nu} - \frac{x'^\mu x'^\nu}{x'^2} - \frac{v^\mu v^\nu}{v^2}, \tag{4.2}$$

where

$$v^\mu = \dot{x}^\mu - \frac{x'\dot{x}}{x'^2}\,x'^\mu. \tag{4.3}$$

Since (4.2) is generically of rank 2, an equation $M^{\mu\nu}a_\nu = 0$ has the general solution $a^\mu = fx'^\mu + g\dot{x}^\mu$. Hence, the analogue of (2.3b) turns out to be

$$\ddot{x}^\mu = -\frac{\dot{x}^2}{x'^2}\, x''^\mu + 2\,\frac{x'\dot{x}}{x'^2}\,\dot{x}'^\mu + fx'^\mu + g\dot{x}^\mu, \tag{4.4}$$

with f and g arbitrary functions on the world-sheet. These two functions correspond to the invariance under full world-sheet reparametrizations as given by

$$\delta x^\mu = (\epsilon^\tau \partial_\tau + \epsilon^\sigma \partial_\sigma) x^\mu \tag{4.5}$$

in its infinitesimal form. There are of course two identities analogous to (3.4), and (4.4) is equivalent to the Nambu-Goto equations of motion following from the Lagrangian (4.1).

If the world-sheet coordinates are restricted to $\tau = t$ (hence $x^0(\tau,\sigma) = \tau$), the $\mu = 0$-component of (4.4) implies $g = 0$, and identifying $\dot{x}^2 \equiv 1 - \vec{p}^{\,2}$ etc., we recover (2.3b).

The theoretical status of f and g may be illustrated by the analogy to the free particle motion $x^\mu(\tau)$ which is described by the reparametrization invariant Lagrangian

$$\mathcal{L} = -\sqrt{\dot{x}^2}. \tag{4.6}$$

Here, the equations of motion are equivalent to

$$\ddot{x}^\mu = g\dot{x}^\mu \tag{4.7}$$

with g an arbitrary function.

5. Applications

In statistical (path-integral motivated) approaches to cosmic string dynamics [5], one would like to deal with a probability density functional $\rho[\vec{q},\vec{p}]$. One draw-back in the conventional view is that the constraints induce delta-functionals that are difficult to handle. In the formalism presented above, one could use an unconstrained phase space and evolve any string in the ensemble according to (2.3). Thereby, one could either choose f once and for all, or use some (path-integral induced) average over all f's to define statistical expectation values.

Hence, the problems are shifted into a form that might be easier to handle: the density functional ρ should be invariant (or transform in a simple

way) under reparametrizations. The free time-evolution of ρ (or likewise of arbitrary statistical mean values) is then induced by (2.3), whereas the interaction (intercommutation) terms may (in a first approximation) be taken over from Ref.[5] without essential change.

Another application is the (numerical) evolution of particular initial conditions $(\vec{q}, \vec{p})$ that do not obey the standard constraints. Inspecting (2.3), one finds an even more drastic simplification of the time-evolution. Defining

$$A = \frac{1-\vec{p}^{\,2}}{\vec{q}^{\,2}}, \qquad B = \frac{\vec{q}\vec{p}}{\vec{q}^{\,2}}, \tag{5.1}$$

(2.3) results into

$$\dot{A} = -2AB' + 2B(A' - f), \qquad \dot{B} = -\frac{1}{2}A' + f, \tag{5.2}$$

which may be solved without any reference to vector quantities (as long as f depends only on A and B). The solutions are inserted into (2.3b) which becomes

$$\dot{\vec{p}} = A\vec{q}\,' + 2B\vec{p}\,' + f\vec{q}. \tag{5.3}$$

Again, f may again be chosen arbitrarily. An interesting version is to set

$$f = \frac{1}{2}A' - \lambda B \tag{5.4}$$

with λ a positive constant. (5.2) implies

$$\dot{B} = -\lambda B. \tag{5.5}$$

Hence, B vanishes asymptotically, and A becomes a constant. In other words, the solution is forced to approach the gauge-condition $\vec{q}\vec{p} = 0$ for large times (i.e. for times larger than λ^{-1}). There are of course a lot of related possibilities to choose f.

Let us finally note that the energy $\mathcal{E}$ along the string satisfies

$$\frac{\partial \mathcal{E}}{\partial \sigma} = \left(\frac{\vec{q}^{\,2}}{1-\vec{v}^{\,2}}\right)^{1/2} \equiv (A + B^2)^{1/2}, \tag{5.6}$$

the time-derivative of this quantity being independent of f as a consequence of (5.2).

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