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BETHE–ANSATZ SOLUTION OF A MODIFIED SU(3)-XXZ-MODEL

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Abstract

We investigate a solvable SU(3)-XXZ-model modified by elements of the Cartan-subalgebra. The nested Bethe-Ansatz is discussed in detail and the corresponding quantum spin model is stated. The coupled Bethe-equations are solved in the thermodynamic limit. Effects of the modification on the finite size scaling are derived.

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1 Introduction

Lowdimensional integrable models, both in the continous formulation and on the lattice play a dominant role in field theory and solid state physics. They provide rigorous results for physical quantities. On the one hand they serve as toy-models for testing conjectures and approximation techniques of more realistic 3-dimensional models. On the other hand technological progress allows to handle systems that behave essentially two or one dimensional as there are quantum Hall devices, quantum wires and polymers, realizing some of these possibilities.

The Yang-Baxter equation (YBE) for the corresponding monodromy matrix is of central importance in the realm of integrable two-dimensional statistical models. It guarantees the existence of an infinite number of conserved currents for the transfer matrix. Commutation relations for the elements of the so called monodromy matrix implied by the YBE allow the construction of eigenstates of the transfer matrix and the calculation of eigenvalues (of which the largest one gives the free energy in the thermodynamic limit) provided that a number of consistency equations are solved. This procedure is known as algebraic Bethe-Ansatz [1].

Expanding the logarithm of the transfer-matrix gives one dimensional quantum mechanical models with local interactions. In the continuum limit the connection to 1+1 dimensional quantum field theory is apparent. The models provide examples for conformal field theory, if there is a massless phase in the thermodynamic limit. Conformal field theory predicts the behaviour of integrable models for large but finite volume. In particular, the central charge of the model determines how the groundstate energy approaches the thermodynamic limit.

One way of generalization consists of turning on 'electrical fields' to the vertex model (for the quantum model this gives diffusion terms). This route will be presented in this paper which is organized as follows: in section 2 we describe the model and employ the YBE for its generalization. The quantum spin model will be derived. Section 3 is devoted to the nested Bethe-Ansatz solution. In section 4 we review the thermodynamic limit properties and in section 5 finite size effects are discussed and the central charge is computed.

2 The Model

We start with the statistical model presented in [2]. The R-matrix for this model reads

$$\begin{aligned} R_{ab,ij} &= \sinh\gamma\ \delta_{ia}\delta_{jb}\ e^{\lambda\,\mathrm{sign}(a-b)} + \sinh\lambda\ \delta_{ib}\delta_{ja} \qquad i \neq j \\ R_{aa,ii} &= \sinh(\gamma-\lambda)\ \delta_{ia}\ . \end{aligned} \tag{1}$$

R acts in $\mathbf{C}^Q \otimes \mathbf{C}^Q$. The first factor will be called 'horizontal space' $\mathbf{C}^Q_H$ and the second factor is called 'vertical space' $\mathbf{C}^Q_V$. The indices a, i act in the horizontal space while b, j act in $\mathbf{C}^Q_V$. λ is called the spectral parameter and γ determines the anisotropy. We shall

call this model the SU(Q)-XXZ-model, because for $\gamma = 0$ it enjoys SU(Q) invariance [3]. There are several choices for the range of the parameters γ, λ. In the antiferroelectric region $\gamma > \lambda > 0$, the $-$ sign is taken in eq.(1), while in the ferroelectric region we take the $+$ sign and $\lambda > 0$, $\gamma > 0$. The second regime is obtained by letting $\gamma \to -\gamma + i\pi$. A third regime is obtained by allowing imaginary parameters $\gamma \to i\gamma$, $\lambda \to i\lambda$. The weights then are trigonometric functions of the parameters and are complex due to the exponential in eq.(1). Contrary to the SU(2)-case, this complex weights can not be absorbed by a gauge transformation. The model in the trigonometric region is of interest because it is critical.

The vertex model has $Q(2Q-1)$ nonvanishing statistical weights determined by the local operator $l_{0n} := P_{0n}R_{0n}$. The subscripts 0 and n indicate, in which factors of the $N+1$-fold tensor product of $\mathbf{C}^Q$ the matrices act nontrivially and P_{0n} denotes the permutation operator. In the following, we use the abbreviation

$$\begin{array}{ll} a := \sinh(\gamma - \lambda) & b := \sinh\lambda \\ c := \sinh\gamma\, e^{\lambda} & d := \sinh\gamma\, e^{-\lambda} \end{array} \tag{2}$$

for the weights of the allowed vertex configurations (the definition $l_{0n} = R_{0n}P_{0n}$ would simply interchange the weigths c and d). With this conventions, the Yang-Baxter equation

$$R_{12}(\lambda - \lambda', \gamma)\, l_{13}(\lambda, \gamma)\, l_{23}(\lambda', \gamma) = l_{13}(\lambda', \gamma)\, l_{23}(\lambda, \gamma)\, R_{12}(\lambda - \lambda', \gamma) \tag{3}$$

is fulfilled. We are interested in solutions of the YBE for models that are generated from the above ones by multiplication with elements of the Cartan subalgebra of SU(Q). For ease of computation, we restrict ourself to the case of SU(3), but every step could in principle be made for SU(Q) as well. For our discussion let us define:

$$\begin{gathered} K(\vec{v}) := e^{(v_3\lambda^3 + v_8\lambda^8)/N} \qquad I(\vec{h}) := e^{(h_3\lambda^3 + h_8\lambda^8)/N} \\ L_{0n} := I_0\, K_n\, l_{0n}. \end{gathered} \tag{4}$$

In this context we use the notation $\vec{v} := (v_3, v_8)$, $\vec{h} := (h_3, h_8)$. λ^3 and λ^8 are the usual diagonal Gell-Mann generators of SU(3) and I_0 means $I \otimes \mathbf{1}$ in $\mathbf{C}_0^3 \otimes \mathbf{C}_n^3$ and similar for K_n. A trivial overall factor e^{v_0} or e^{h_0} corresponding to the unit matrix is omitted in eq.(2). $\vec{v}$ and $\vec{h}$ will be called 'horizontal' and 'vertical' fields respectivly as it is suggested by the six-vertex model analogy. In prinicple we would like to implement different magnitudes of the fields to every lattice site, but this goal cannot be fulfilled for the vertical fields as we shall see below.

The crucial point is, that with the given conventions the local operator $L_{0n}(\lambda, \vec{v}, \vec{h}_n)$ solves the YBE and thereby generates an integrable model:

$$\tilde{R}_{12}(\lambda'', \vec{v}, \vec{v}')\, L_{13}(\lambda, \vec{v})\, L_{23}(\lambda, \vec{v}') = L_{13}(\lambda', \vec{v}')\, L_{23}(\lambda, \vec{v}) \tilde{R}_{12}(\lambda'', \vec{v}, \vec{v}')\ , \tag{5}$$

where the new R-matrix $\tilde{R}$ reads

$$\tilde{R}_{12}(\lambda - \lambda', \gamma, \vec{v}, \vec{v}') = K_1(\vec{v})\, R_{12}(\lambda - \lambda', \gamma)\, K_1^{-1}(\vec{v}')\ . \tag{6}$$

In eq.(5) we have suppressed the dependence of the parameter γ and the horizontal fields. Notice that the vertical fields have the properties of spectral parameters, whereas the horizontal fields have to be kept fixed in eq.(5).[2]

In order to prove our claim we recall from [2] the symmetries of the original R-matrix R_{12}. Although there is no SU(3) invariance for $\gamma \neq 0$, R_{12} is still invariant under the abelian subgroup generated by λ^3, λ^8 in the sense that

$$[R_{12}, \lambda^i \otimes \lambda^i] = 0 \,, \tag{7}$$

for $i = 3, 8$. Whenever eq.(7) holds, there is a general strategy for adding horizontal fields [2]. Moreover the structure of the equations allow space dependent horizontal fields. Remarkably this step does not affect the R-matrix. Inclusion of the vertical fields is achieved in the following manner: write the original YBE in terms of the modified model as

$$R_{12}(\lambda'')\, K_3^{-1}(\vec{v})\, L_{13}\, K_3^{-1}(\vec{v}')\, L'_{23} = K_3^{-1}(\vec{v}')\, L'_{13}\, K_3^{-1}(\vec{v})\, L_{23}(\lambda)\, R_{12}(\lambda'') \,,$$

(with $\lambda'' = \lambda - \lambda'$) and multiply both sides with $K_3(\vec{v})\, K_3(\vec{v}')\, K_1(\vec{v})$ from the left and with $K_1^{-1}(\vec{v})$ from the right. Using the commutation relation eq.(7), the desired result is immediatly obtained. Since the constructions for horizontal and vertical fields are independent of each other this proves the YBE for our model.

By using the symbol [ab] for the 9×9-matrix with elements $([ab])_{ij} = \delta_{ia}\, \delta_{jb}$ and defining $b_1 := b\alpha/\beta$, $b_2 := b\alpha/\gamma$, $b_3 := b\beta/\gamma$, $b_4 := b\beta/\alpha$, $b_5 := b\gamma/\alpha$, $b_6 := b\gamma/\beta$, where

$$\begin{aligned} \alpha &= e^{(v_3+v_8)/N} & \qquad x &= e^{(h_3+h_8)/N} \\ \beta &= e^{(-v_3+v_8)/N} & \qquad y &= e^{(-h_3+h_8)/N} \\ \gamma &= e^{(-2v_8)/N} & \qquad z &= e^{(-2h_8)/N} \,, \end{aligned}$$

the $\tilde{R}_{12}$- matrix takes the form:

$$\begin{aligned} R_{12} = \quad & a\,([11] + [55] + [99]) + d\,([22] + [33] + [66]) + \\ + \quad & c\,([44] + [77] + [88]) + b_1\,[24] + b_2\,[37] + b_3\,[68] + \\ + \quad & b_4\,[42] + b_5\,[73] + b_6\,[86] \,. \end{aligned} \tag{8}$$

Next we investigate the spin model associated to eq.(2). The monodromy matrix M and the transfer matrix T for a lattice of N horizontal sites are defined as follows:

$$\begin{aligned} M(\lambda, \gamma, \{\vec{h}_n\}, \vec{v}) &= L_{01} \ldots L_{0N} \\ T(\lambda, \gamma, \{\vec{h}_n\}, \vec{v}) &= \mathrm{tr}_0\, M \,. \end{aligned} \tag{9}$$

Since $L_{0n}(\lambda = 0) = \sinh\gamma\ (I_0\, K_n)\ P_{0n}$, the derivative of the transfer matrix is readily

[2] In the free fermi case of the six vertex model, it is in addition possible to take $\vec{h}$ as spectral parameters since a decorated YBE is fulfilled [4]. This can't be done for $Q \geq 3$.

computed as:

$$\left.\frac{dT}{d\lambda}\right|_{\lambda=0} = \sinh^{N-1}\gamma \ \mathrm{tr}_0 \sum_{i=1}^{N} I_0 K_1 P_{01} \cdots I_0 K_i (P_{0i} P_{0i}) \frac{dl_{0i}(0)}{d\lambda} \cdots I_0 K_N P_{0N} \ ,$$

were we have inserted $\mathbf{1} = P_{0i} P_{0i}$. At this point we must be careful, since $\frac{dl_{0i}}{d\lambda}$ is nondiagonal. When commuting $\frac{dl_{0i}}{d\lambda}$ with $I_0 K_{i+1}$ for $i \neq N$ in order to evaluate the trace, we have to insert $I_i^{-1} I_i$; using again the symmetry property eq.(7) gives:

$$\cdots I_0 K_i P_{0i} \frac{dl_{0i}}{d\lambda} (I_i^{-1} I_i) I_0 K_{i+1} P_{0i+1} \cdots = I_0 K_i P_{0i} I_0 K_{i+1} I_i \frac{dl_{0i}}{d\lambda} I_i^{-1} P_{0i+1} =$$
$$= I_0 K_i P_{0i} I_0 K_{i+1} P_{0i+1} I_i \frac{dl_{ii+1}}{d\lambda} I_i^{-1} \ .$$

There is no such problem for $i = N$ and this is the reason why our model does not satisfy periodic boundary conditions. Combining this with the definition $H := \sinh\gamma \, T^{-1}(0) \left.\frac{dT}{d\lambda}\right|_0$ we eventually obtain for the Hamiltonian of the spin chain:

$$H = \sum_{i=1}^{N-1} I_i \frac{dl_{i,i+1}}{d\lambda} P_{i,i+1} I_i^{-1} + \frac{l_{1N}}{d\lambda} P_{1N} \ . \tag{10}$$

Expressing the involved operators in terms of SU(3) generators the 'bulk' term can be rewritten as

$$\begin{aligned} H_{ii+1} = \quad & \cosh\gamma \left[\frac{1}{3}\mathbf{1} + \frac{1}{2}\lambda_i^3 \lambda_{i+1}^3 + \frac{3}{2}\lambda_i^8 \lambda_{i+1}^8\right] + \frac{1}{2\sqrt{3}} \sinh\gamma \left[\lambda_i^3 \lambda_{i+1}^8 - \lambda_i^8 \lambda_{i+1}^3\right] \\ & + \frac{x_i}{y_i} \frac{(\lambda_i^1 + i\lambda_i^2)(\lambda_{i+1}^1 - i\lambda_{i+1}^2)}{4} + \frac{y_i}{x_i} \frac{(\lambda_i^1 - i\lambda_i^2)(\lambda_{i+1}^1 + i\lambda_{i+1}^2)}{4} \\ & + \frac{x_i}{z_i} \frac{(\lambda_i^4 + i\lambda_i^5)(\lambda_{i+1}^4 - i\lambda_{i+1}^5)}{4} + \frac{z_i}{x_i} \frac{(\lambda_i^4 - i\lambda_i^5)(\lambda_{i+1}^4 + i\lambda_{i+1}^5)}{4} \\ & + \frac{y_i}{z_i} \frac{(\lambda_i^6 + i\lambda_i^7)(\lambda_{i+1}^6 - i\lambda_{i+1}^7)}{4} + \frac{z_i}{y_i} \frac{(\lambda_i^6 - i\lambda_i^7)(\lambda_{i+1}^6 + i\lambda_{i+1}^7)}{4} \ . \end{aligned} \tag{11}$$

We first notice that H is independent of the vertical fields. Selfadjointness of the Hamiltonian only holds for pure imaginary fields. The offdiagonal elements cannot be chosen independently since for SU(Q) there are $Q(Q-1)$ of them but merely $Q-1$ horizontal fields.

3 Algebraic Bethe-Ansatz

As it is well known, the YBE for the elements of the monodromy matrix give commutation relations for the elements of the transfer matrix T. Together with the existence of a reference eigenvector ('local vacuum' $|\omega\rangle$), these relations enable the construction of

eigenvectors for T in many cases. For our models with $Q \geq 3$, one has to proceed in steps, called nested algebraic Bethe-Ansatz. Explicitly, the equations

$$M = \begin{pmatrix} A & B_1 & B_2 \\ C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{pmatrix} \qquad T = A_1 + D_{11} + D_{22}$$

$$R\,(M \otimes M') = (M' \otimes M)\,R$$

give 81 relations for the elements of M (the prime stands for a different set of variables, see eq.(7)). 15 of these relations are of special interest for the Bethe-Ansatz and can be summarized as:

$$\begin{aligned}
A\,A' &= A'\,A \\
A\,B_1' &= \frac{\beta}{\alpha}\frac{a(\lambda'-\lambda)}{b(\lambda'-\lambda)}\,B_1'\,A - \frac{\beta}{\alpha}\frac{d(\lambda'-\lambda)}{b(\lambda'-\lambda)}\,B_1\,A' \\
A\,B_2' &= \frac{\gamma}{\alpha}\frac{a(\lambda'-\lambda)}{b(\lambda'-\lambda)}\,B_2'\,A - \frac{\gamma}{\alpha}\frac{d(\lambda'-\lambda)}{b(\lambda'-\lambda)}\,B_2\,A' \\
B \otimes B' &= (B' \otimes B)\,R^{(2)}(\lambda-\lambda') = R^{(2)T}(\lambda-\lambda')\,(B' \otimes B) \\
D \otimes B' &= \frac{1}{b(\lambda-\lambda')}\,S^{(2)}\,(B' \otimes D)\,R^{(2)}(\lambda-\lambda') - \frac{c(\lambda-\lambda')}{b(\lambda-\lambda')}\,S^{(2)}\,(B \otimes D'),
\end{aligned} \tag{12}$$

where we have used the tensor notation $(B \otimes B')_{ij} = B_i\,B_j'$, $(D \otimes B')_{i,jk} = D_{ij}\,B_k'$, $(B \otimes D')_{i,jk} = B_j\,D_{ik}'$. Due to our modifications, the operator B_1 and B_2 give different expressions when commuted with A. The matrices $R^{(2)}$ and $S^{(2)}$ in eqs.(12) are given by:

$$R^{(2)}(\lambda) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & c/a & b_3/a & 0 \\ 0 & b_6/a & d/a & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad S^{(2)} = \begin{pmatrix} \beta/\alpha & 0 \\ 0 & \gamma/\alpha \end{pmatrix}. \tag{13}$$

Now we turn to the action of L and M on the reference vectors $|\omega_i\rangle := (1,0,0) \in \mathbf{C}^3$ and $|\Omega\rangle := \otimes^N |\omega_i\rangle$. From

$$L_{0i}(\lambda)\,|\omega_i\rangle = \alpha \begin{pmatrix} ax\,|\omega_i\rangle & \star & \star \\ 0 & by\,|\omega_i\rangle & 0 \\ 0 & 0 & bz\,|\omega_i\rangle \end{pmatrix}, \tag{14}$$

where $\star$ means some nonzero vector (nonproportional to $|\omega_i\rangle$). We get for the action of the monodromy matrix elements on $|\Omega\rangle$:

$$\begin{aligned}
A\,|\Omega\rangle &= \alpha^N \left(\textstyle\prod_{i=1}^N x_i\right) a^N(\lambda)\,|\Omega\rangle \\
D_{11}\,|\Omega\rangle &= \alpha^N \left(\textstyle\prod_{i=1}^N y_i\right) b^N(\lambda)\,|\Omega\rangle \\
D_{22}\,|\Omega\rangle &= \alpha^N \left(\textstyle\prod_{i=1}^N z_i\right) b^N(\lambda)\,|\Omega\rangle \\
D_{12}\,|\Omega\rangle &= D_{21}\,|\Omega\rangle = 0\,.
\end{aligned} \tag{15}$$

Notice that even the nondiagonal elements D_{12} and D_{21} annihilate $|\Omega\rangle$, a fact that will be important for the Bethe-Ansatz.

We seek for eigenvectors $|\psi, \vec{\mu}; r, s\rangle$ of T which are of the form

$$|\psi, \vec{\mu}; r, s\rangle = \sum_{\{i_n=2,3\}} X_{i_1,\cdots,i_s} B_{i_1}(\mu_1) \cdots B_{i_s}(\mu_s) |\Omega\rangle . \tag{16}$$

In this linear combination, B_1 is supposed to appear r-times and therefore, B_2 appears $(r-s)$-times. The parameters $\vec{\mu} = \{\mu_1, \ldots, \mu_s\}$ are determined by coupled Bethe equations as we shall see later on. Note that $X_{i_1,\cdots,i_s}$ can be regarded as a vector in $\otimes \mathbf{C}^2$!

Applying the operator $A(\lambda)$ onto $|\psi\rangle$ and commuting it with the B's gives 2^s terms according to eq.(12). The 2^s terms can be collected into $2s$ terms, one of them is of special interest. Using the first summand for every commutation reproduces $|\psi\rangle$ and therefore contributes to the eigenvalue of T. Taking into account the action of A on $|\Omega\rangle$ gives

$$\begin{aligned} A(\lambda) |\psi, \vec{\mu}; r, s\rangle = \quad & \beta^r \gamma^{s-r} \alpha^{N-s} (\prod x_i) a^N(\lambda) \prod_{i=1}^{s} \frac{a(\mu_i - \lambda)}{b(\mu_i - \lambda)} |\psi\rangle \\ & + \text{nondiagonal terms} . \end{aligned} \tag{17}$$

Computing the contribution of $D_{\alpha\alpha} = D_{11} + D_{22}$ to the eigenvalue is less easy, because the commutation relations are more involved. Again the diagonal term is obtained by using the first term in the relation:

$$\begin{aligned} D_{\alpha\alpha}|\psi\rangle = \quad & \prod_{i=1}^{s} \frac{a(\lambda - \mu_i)}{b(\lambda - \mu_i)} X_{i_1 \cdots i_s} \tilde{M}^{a_s c_s}_{b_s \cdots b_1, i_s \cdots i_1} B_{b_1} \cdots B_{b_s} D_{a_s c_s} |\Omega\rangle \\ & + \text{nondiagonal terms} . \end{aligned} \tag{18}$$

In this formula, the new monodromy matrix $\tilde{M}$ is given by

$$\begin{aligned} \tilde{M}(\lambda, \vec{\mu})^{a_s c_s}_{b_s \cdots b_1, i_s \cdots i_1} &= R^{(2)}_{b_s c_s, c_{s-1} i_s}(\lambda - \mu_s) \cdots R^{(2)}_{b_1 c_1, i i_1}(\lambda - \mu_1) \, S_{i a_1} \cdots S_{a_{s-1} a_s} \\ &= l^{(2)}_{c_s b_s, c_{s-1} i_s}(\lambda - \mu_s) \cdots l^{(2)}_{c_1 b_1, i i_1}(\lambda - \mu_1) \, (S^{(2)s})_{i a_s} , \end{aligned} \tag{19}$$

where we have defined $l^{(2)}_{ij,kl} = (P\, R^{(2)})_{ij,kl} = R^{(2)}_{ji,kl}$. Remembering that $D_{a_s c_s}|\Omega\rangle = 0$ for $a_s \neq c_s$ we are lead to define:

$$\begin{aligned} \tilde{T} \quad &:= \quad (\prod y_i) \, \tilde{M}^{11} + (\prod z_i) \, \tilde{M}^{22} \\ &= \quad \tfrac{\beta^s}{\alpha^s} (\prod y_i) \, \tilde{A}(\lambda, \vec{\mu}) + \tfrac{\gamma^s}{\alpha^s} (\prod z_i) \, \tilde{D}(\lambda, \vec{\mu}) , \end{aligned} \tag{20}$$

where we have extracted convenient factors for later purpose. $\tilde{T}$ is the transfer matrix of a modified six vertex model. The diagonal contribution of $D_{\alpha\alpha}$ can then be written as

$$\begin{aligned} D_{\alpha\alpha} |\psi\rangle = \quad & \alpha^{N-s} b^N(\lambda) \prod_{i=1}^{s} \frac{a(\lambda - \mu_i)}{b(\lambda - \mu_i)} (\tilde{T} X)_{i_1 \cdots i_s} B_{i_1} \cdots B_{i_s} |\Omega\rangle . \\ & + \text{nondiagonal terms} . \end{aligned} \tag{21}$$

This ends the discussion of the summands that suit the eigenvalue equation.

Non diagonal terms are generated if the second terms of the eqs.(12) are used at least once. Then one of the μ_i's is interchanged with the spectral parameter λ. To obtain a typical term, we first make a cyclic permutation of the B-operators, using the expression in the third line of eqs.(12). The calculation gives:

$$X_{i_1\cdots i_s}B_{i_1}(\mu_1)\cdots B_{i_s}(\mu_s) = X_{i_1\cdots i_s}H^{k-1}_{i_1\cdots i_s,j_k\cdots j_{k-1}}B_{j_k}(\mu_k)\cdots B_s(\mu_s)B_1(\mu_1)\cdots B_{k-1}(\mu_{k-1}),$$

with

$$H = l^{(2)}_{i_1,i_2,a_1j_2}(\mu_1-\mu_2)\cdots l^{(2)}_{a_{s-2}i_s,j_1j_s}(\mu_1-\mu_s)\ .$$

With these considerations, the general nondiagonal term induced by the action of $A(\lambda)$ is obtained by taking the second term in eqs.(12) for the first commutation and the first term for all the others. It reads:

$$-\beta^r\gamma^{s-r}\alpha^{N-s}\,a^N(\mu_k)\frac{d(\mu_k-\lambda)}{b(\mu_k-\lambda)}\,(M^{k-1}\,X)_{i_k\cdots i_{k-1}}B_{i_k}(\lambda)B_{i_{k+1}}\cdots B_{i_{k-1}}\,|\Omega\rangle\ . \tag{22}$$

On the other hand, the unwanted terms due to $D_{\alpha\alpha}$ are given by:

$$-\alpha^{N-s}\frac{c(\lambda-\mu_k)}{b(\lambda-\mu_k)}\,b^N(\mu_k)\prod_{i=1}^{s}\frac{a(\mu_k-\mu_i)}{b(\mu_k-\mu_i)}\,(M^{k-1}\tilde{T}X)_{i_k\cdots i_{k-1}}B_{i_k}(\lambda)B_{i_{k+1}}\cdots B_{i_{k-1}}\,|\Omega\rangle\ . \tag{23}$$

Forcing the terms given in eq.(22) and eq.(23) to cancel each other implies that X has to be an eigenvector of the transfer matrix $\tilde{T}$ with eigenvalue

$$\tilde{T}(\mu_k,\vec{\mu})|X\rangle = \beta^r\gamma^{s-r}(\prod x_i)\frac{a^N(\mu_k)}{b^N(\mu_k)}\prod_{1=i\neq k}^{s}(-1)\,\frac{a(\mu_i-\mu_k)}{a(\mu_k-\mu_i)}\,|X\rangle\ . \tag{24}$$

On the other hand, our prime object of interest, the eigenvalue of the transfer matrix T of our model is the sum of eq.(17) and eq.(18) and reads:

$$\beta^r\gamma^{r-s}\alpha^{N-s}(\prod x_i)a^N(\lambda)\prod_{i=1}^{s}\frac{a(\mu_i-\lambda)}{b(\mu_i-\lambda)} + \alpha^{N-s}b^N(\lambda)\prod_{i=1}^{s}\frac{a(\lambda-\mu_i)}{b(\lambda-\mu_i)}\,(\tilde{T}X)\ . \tag{25}$$

This finishes the first step of the nested Bethe-Ansatz. We are left with the determination of the eigenvectors X of $\tilde{T}$, which is much easier to perform, because the dimension of the horizotal space is reduced by 1. An ordinary Bethe-Ansatz of a modified six vertex model remains to be done.

Before doing so, we have to ascertain, that $\tilde{M}$ defined in eq.(19) itself satisfies a YBE and deduce the commutation relation from it. This is indeed the case, since $R^{(2)}(\lambda)$ is the 'gauge transformed' version of the usual six vertex model ([5],[6]). In addition, the fact that $R^{(2)}$ depends only on the difference of the spectral parameters $\lambda-\lambda'$ allows the 'inhomogeneous' parametrization $(\lambda-\mu_s),\ldots,(\lambda-\mu_1)$ in eq.(19). This property is essential in the nested Bethe-Ansatz scheme. For instance, it does not hold in the Hubbard model ([7][4]), and no algebraic Bethe-Ansatz is known to the authors in this

case up to now. Choosing the reference vectors $|\omega^{(2)}\rangle = (1,0) \in \mathbf{C^2}$ and $|\Omega^{(2)}\rangle = \otimes^s |\omega^{(2)}\rangle$, and using the definition of $l^{(2)}$, we have:

$$\tilde{A}(\lambda, \vec{\mu})\,|\Omega^{(2)}\rangle = |\Omega^{(2)}\rangle \qquad \tilde{D}(\lambda, \vec{\mu})\,|\Omega^{(2)}\rangle = \prod_{i=1}^{s} \frac{\beta}{\gamma}\frac{b(\lambda-\mu_i)}{a(\lambda-\mu_i)}\,|\Omega^{(2)}\rangle\,. \tag{26}$$

The commutation relations take the form:

$$\begin{aligned} \tilde{A}\tilde{B}' &= \frac{\gamma}{\beta}\frac{a(\lambda'-\lambda)}{b(\lambda'-\lambda)}\tilde{B}'\tilde{A} - \frac{\gamma}{\beta}\frac{d(\lambda'-\lambda)}{b(\lambda'-\lambda)}\tilde{B}\tilde{A}' \\ \tilde{D}\tilde{B}' &= \frac{\gamma}{\beta}\frac{a(\lambda-\lambda')}{b(\lambda-\lambda')}\tilde{B}'\tilde{D} - \frac{\gamma}{\beta}\frac{d(\lambda-\lambda')}{b(\lambda-\lambda')}\tilde{B}\tilde{D}'\,. \end{aligned} \tag{27}$$

Therefore, one finds that a vector $|X\rangle := \tilde{B}(\nu_1)\ldots\tilde{B}(\nu_p)|\Omega^{(2)}\rangle$ is an eigenvector of $\tilde{T}$ with eigenvalue

$$\beta^{s-p}\gamma^p\left[(\prod y_i)\prod_{i=1}^{p}\frac{a(\nu_i-\lambda)}{b(\nu_i-\lambda)} + (\prod z_i)\prod_{i=1}^{s}\frac{a(\lambda-\mu_i)}{b(\lambda-\mu_i)}\prod_{j=1}^{p}\frac{a(\lambda-\nu_j)}{b(\lambda-\nu_j)}\right]\,. \tag{28}$$

At the point $\lambda = \mu_k$, this expression has to coincide with eq.(24). Together with the consistency equations for the parameters ν_i (obtained from the second Bethe-Ansatz) this gives two coupled Bethe equations for the set of parameters μ_i, ν_i :

$$\prod_{i=1}^{s}\frac{b(\nu_k-\mu_i)}{a(\nu_k-\mu_i)} = (\prod \tfrac{y_i}{z_i})\prod_{1=i\neq k}^{p}(-1)\frac{a(\nu_i-\nu_k)}{a(\nu_k-\nu_i)} \tag{29}$$

$$\prod_{i=1}^{p}\frac{a(\nu_i-\mu_k)}{b(\nu_i-\mu_k)} = (\prod \tfrac{x_i}{y_i})\frac{a^N(\mu_k)}{b^N(\mu_k)}\prod_{1=i\neq k}^{s}(-1)\frac{a(\mu_i-\mu_k)}{a(\mu_k-\mu_i)}\,. \tag{30}$$

Combining eq.(25) and eq.(28), we thus conclude that the eigenvalue of the transfer matrix T acquires the form:

$$\begin{aligned} &\beta^r\gamma^{s-r}\alpha^{N-s}a^N(\lambda)\left[(\prod x_i)\prod_{i=1}^{s}\frac{a(\mu_i-\lambda)}{b(\mu_i-\lambda)} + \frac{b^N(\lambda)}{a^N(\lambda)}\prod_{i=1}^{s}\frac{a(\lambda-\mu_i)}{b(\lambda-\mu_i)}\tilde{\tau}(\lambda)\right] \\ &\tilde{\tau}(\lambda,\vec{\mu},\vec{\nu}) = (\prod y_i)\prod_{i=1}^{p}\frac{a(\nu_i-\lambda)}{b(\nu_i-\lambda)} + (\prod z_i)\prod_{i=1}^{s}\frac{b(\lambda-\mu_i)}{a(\lambda-\mu_i)}\prod_{i=1}^{p}\frac{a(\lambda-\nu_i)}{b(\lambda-\nu_i)}\,. \end{aligned} \tag{31}$$

We see, that the vertical fields do not show up in the Bethe equations eqs.(29,30) but amount to overall factors in the eigenvalue eq.(31). The detailed structure of the horizontal fields is not important, only their sum contributes. Taking the logarithmic derivative of eq.(31) at $\lambda = 0$, we obtain the energy levels of the spin chain eq.(11). Since $b(0) = 0$, only the first summand in eq.(31) contributes and we conclude that the energy is independent of all the fields $(\vec{h}, \vec{v})$.

As far as the Bethe equations are concerned, the horizontal fields amount to a constant shift of the quantum numbers attached to the solutions.

4 Thermodynamic Limit

We will be interested in the way the horizontal fields influence the behaviour at large but finite sample sizes. Since there will be an effect in the gapless regime, we use complex parametrization $\gamma \to i\gamma$, $\lambda \to i\lambda$, $\vec{h} \to i\vec{h}$ from now on. The solution of the Bethe equations (29,30) in the infinite chain limit is obtained by the standard method of deriving an integral equation for the density of roots. Since we need some of the formulae in our investigation of the finite size effects, we shall present the key steps of the derivation, following essentially [2]. We shall concentrate on the ground state behaviour.

In the critical (trigonometric) parameter region of the model it is convenient to define an auxiliary function $\Phi(z,\gamma)$ and its Fourier-transform $\tilde{\Phi}(\omega,\gamma)$ by

$$\begin{aligned} \Phi(z,\gamma) &= i\log\frac{\sinh(z+i\gamma)}{\sinh(z-i\gamma)}\,, \\ \tilde{\Phi}(\omega,\gamma) &= \pi\delta(w) + \frac{1}{i\omega}\frac{\sinh\omega(\pi/2-\gamma)}{\sinh(\omega\pi/2)}\,. \end{aligned} \tag{32}$$

The branch cut of the logarithm is chosen such that $\lim_{z\to-\infty}\Phi(z,\gamma) = 2\pi$ and $\lim_{z\to\infty}\Phi(z,\gamma) = 0$. Taking the logarithm of the Bethe equations and shifting μ_k,ν_k to a new set of variables σ_k,ρ_k defined by

$$\mu_k = i\sigma_k - \gamma \qquad \nu_k = i\rho_k - \frac{\gamma}{2}\,, \tag{33}$$

we arrive at the coupled equations:

$$\begin{aligned} \sum_{i=1}^{p}\Phi(\sigma_k-\rho_i,\gamma/2) - \sum_{i=1}^{s}\Phi(\sigma_k-\sigma_i,\gamma) + N\Phi(\sigma_k,\gamma/2) &= 2\pi I_k^\sigma + 2h_3 \\ -\sum_{i=1}^{p}\Phi(\rho_k-\rho_i,\gamma) + \sum_{i=1}^{s}\Phi(\rho_k-\sigma_i,\gamma/2) &= 2\pi I_k^\rho - (h_3+3h_8)\,. \end{aligned} \tag{34}$$

I_k^σ and I_k^ρ are half odd integers and can be interpreted as 'quantum numbers' characterizing the solutions of the Bethe equations.

In order to obtain an expression for the density of roots we consider the l. h. s of eqs.(34) as functions of variables σ,ρ. To be precise, we set

$$\begin{aligned} Z_N^\sigma(\sigma) &= \frac{1}{2\pi}\left[\Phi(\sigma,\gamma/2) - \frac{1}{N}\sum_{i=1}^{s}\Phi(\sigma-\sigma_i,\gamma) + \frac{1}{N}\sum_{i=1}^{p}\Phi(\sigma-\rho_i,\gamma/2) - \frac{2h_3}{N}\right] \\ Z_N^\rho(\rho) &= \frac{1}{2\pi}\left[\quad -\frac{1}{N}\sum_{i=1}^{p}\Phi(\rho-\rho_i,\gamma) + \frac{1}{N}\sum_{i=1}^{s}\Phi(\rho-\sigma_i,\gamma/2) + \frac{(h_3-3h_8)}{N}\right]. \end{aligned} \tag{35}$$

Therefore, at the Bethe roots σ_k,ρ_k this functions give the values $Z_N^\sigma(\sigma_k) = I_k^\sigma/N$, $Z_N^\rho(\rho_k) = I_k^\rho/N$. Since the ground state is achieved if the quantum numbers take consecutive values starting at $\frac{1}{2}$ and fulfilling $I_{k+1} - I_k = 1$, the ground state densities η_∞^σ, η_∞^ρ

are given by

$$\begin{aligned}
\eta^{\sigma}_{\infty} &= \lim_{N\to\infty} \frac{1}{N(\sigma_{k+1}-\sigma_k)} = \lim_{N\to\infty} \frac{dZ^{\sigma}_N}{d\sigma} \\
\eta^{\rho}_{\infty} &= \lim_{N\to\infty} \frac{1}{N(\rho_{k+1}-\rho_k)} = \lim_{N\to\infty} \frac{dZ^{\rho}_N}{d\rho} .
\end{aligned} \tag{36}$$

In the limit of large N we can replace the sum by an integral weighted with the corresponding density function, i. e. $\lim_{N\to\infty}\sum_i f(\lambda_i) = \int d\lambda\ \eta(\lambda) f(\lambda)$. Performing this for the functions $Z^{\sigma}_N(\sigma)$ and $Z^{\rho}_N(\rho)$ and differentiating with respect to the variable σ (ρ), we obtain a matrix integral equation for the vacuum density distributions:

$$\vec{\eta}_{\infty}(\lambda) + \frac{1}{2\pi}\mathbf{M}(\lambda-\lambda')\ \vec{\eta}_{\infty}(\lambda') = \vec{\epsilon}(\lambda) . \tag{37}$$

In this formula we used a matrix notation with $\vec{\eta}(\lambda) = (\eta^{\sigma}, \eta^{\rho})$, $\vec{\epsilon}(\lambda) = \frac{1}{2\pi}(\Phi_{,\lambda}(\lambda, \gamma/2), 0)$ where $\Phi_{,\lambda}$ is the derivative of $\Phi(\lambda)$ with respect to λ. $\mathbf{M}(\lambda-\lambda')$ is a matrix integral operator which is given by

$$\mathbf{M}(\lambda-\lambda') = \begin{pmatrix} \int d\lambda'\ \Phi_{,\lambda}(\lambda-\lambda',\gamma) & -\int d\lambda'\ \Phi_{,\lambda}(\lambda-\lambda',\gamma/2) \\ -\int d\lambda'\ \Phi_{,\lambda}(\lambda-\lambda',\gamma/2) & \int d\lambda'\ \Phi_{,\lambda}(\lambda-\lambda',\gamma) \end{pmatrix} . \tag{38}$$

The integral equation is easily solved by Fourier-transformation, giving

$$\vec{\tilde{\eta}}_{\infty}(\omega) = \mathbf{R}(\omega)\ \vec{\tilde{\epsilon}}(\omega) . \tag{39}$$

In this equation the resolvent $\mathbf{R}(\omega)$ is given by

$$\mathbf{R}(\omega) = \frac{\sinh(\omega\pi/2)}{\sinh(\omega/2(\pi-\gamma))\ \sinh(3\omega\gamma/2)} \begin{pmatrix} \sinh\omega\gamma & \sinh\omega\gamma/2 \\ \sinh\omega\gamma/2 & \sinh\omega\gamma \end{pmatrix} = \begin{pmatrix} R^1 & R^2 \\ R^2 & R^1 \end{pmatrix} . \tag{40}$$

Summing up, we find that in the thermodynamic limit the root densities and and the function $Z^{\sigma,\rho}_{\infty}(x) = \int_{-\infty}^{x} dy\ \eta^{\sigma,\rho}_{\infty}(y)$ take the form

$$\begin{aligned}
\tilde{\eta}^{\sigma}_{\infty}(\omega) &= \frac{1}{2\pi}\frac{\sinh\omega\gamma}{\sinh 3\omega\gamma/2} & \tilde{\eta}^{\rho}_{\infty}(\omega) &= \frac{1}{2\pi}\frac{\sinh\omega\gamma/2}{\sinh 3\omega\gamma/2} \\
\eta^{\sigma}_{\infty}(\sigma) &= \frac{1}{\sqrt{3}\gamma}\frac{\cosh\frac{\pi}{3\gamma}\sigma}{\cosh\frac{\pi}{\gamma}\sigma} & \eta^{\rho}_{\infty}(\rho) &= \frac{1}{\sqrt{3}\gamma}\frac{\sinh\frac{\pi}{3\gamma}\rho}{\sinh\frac{\pi}{\gamma}\rho} \\
Z^{\sigma}_{\infty}(\sigma) &= -\frac{1}{\pi}\arctan\frac{\coth\frac{\pi}{3\gamma}\sigma}{\sqrt{3}} + \frac{5[-1]}{6} & Z^{\rho}_{\infty}(\rho) &= -\frac{1}{\pi}\arctan(\sqrt{3}\coth\frac{\pi}{3\gamma}\rho) + \frac{2[-1]}{3}.
\end{aligned} \tag{41}$$

For negative values of the arguments one has to take the numbers in brackets in the formulae for $Z^{\sigma,\rho}_{\infty}$. With this convention, $Z^{\sigma,\rho}_{\infty}$ is continous at zero and the distinction arises from a branch cut that has to be introduced when computing the integral for $Z^{\sigma,\rho}_{\infty}$.

5 Finite Size Effects

5.1 The Density

In the last section we analysed some properties of the system in the thermodynamic limit and found that they are independent on the fields $\vec{h}$, $\vec{v}$. Studying the model for large but finite N it is expected, that there will be some dependence (see [2] for the XXZ–case).

To study the deviation of the densities and the eigenvalue from their limit values, we shall consider the differences (cf. eq.35):

$$Z_N^\sigma(x) - Z_\infty^\sigma(x), \qquad Z_N^\rho(x) - Z_\infty^\rho(x). \tag{42}$$

Differentiating gives an integral equation which can be recasted into the form:

$$(\vec{\eta}_N - \vec{\eta}_\infty)(x) = \int dx' \, \mathbf{F}(x - x') \, \vec{S}_N(x') \, , \tag{43}$$

where $\vec{\eta} = (\eta^\sigma, \eta^\rho)$ as in eq.(37). The matrix resolvent $\mathbf{F}(x - x')$ is connected to the resolvent in eq.(40) by

$$\mathbf{F}(x - x') = \mathbf{R}(x - x') - \frac{1}{2\pi}\delta(x - x')\mathbf{1} \, . \tag{44}$$

The 'inhomogenous' term $\vec{S}_N(x') = (S_N^\sigma(x'), S_N^\rho(x'))$ results to

$$\begin{aligned} S_N^\sigma(x') &= \frac{1}{N}\sum_{i=1}^{s} \delta(x' - \sigma_i) - \eta_N^\sigma(x') \\ S_N^\rho(x') &= \frac{1}{N}\sum_{i=1}^{p} \delta(x' - \rho_i) - \eta_N^\rho(x') \end{aligned} \tag{45}$$

Since the unknown functions $\eta_N^{(\sigma,\rho)}$ appear at the right hand side of eq.(43), this equation is no solution to the problem but eq.(43) is a suitable expression for a systematic approximation with the help of the Euler–Maclaurin formula. The calculation is skeched in the appendix.

5.2 The Eigenvalue

The expression for the eigenvalue, eq.(31), is a sum of two termes. For $\lambda < \gamma/2$, we have $b^N(\lambda)/a^N(\lambda) = [\sin(\lambda)/\sin(\gamma - \lambda)]^N < 1$ and one concludes, that the second summand is suppressed exponentially for large N. Since in our analysis of the finite size corrections we shall only keep the leading terms, we consider only the first term of the transfer matrix eigenvalue:

$$\Lambda_N(\lambda) = \beta^r \gamma^{s-r} \alpha^{N-s} a^N(\lambda) e^{i(h_3+h_8)} \prod_{i=1}^{s} \frac{a(\mu_i - \lambda)}{b(\mu_i - \lambda)} \, . \tag{46}$$

As in the discussion of the density, one considers the correction L_N of the 'free energy' to the thermodynamic limit value:

$$L_N^h(\lambda) = -\frac{1}{N}\log\Lambda_N(\lambda) + \lim_{N\to\infty}\frac{1}{N}\log\Lambda_N(\lambda)\,. \tag{47}$$

The superscript h indicates the dependence of the horizontal fields h_3, h_8. Inserting eq.(32) and performing elementary manipulations leads to

$$\begin{aligned} L_N^h(\lambda) &= -i\frac{h_3+h_8}{N} - i\int dx\,\eta_\infty^\sigma(x)\Phi(x+i\lambda,\gamma/2) + \frac{i}{N}\sum_{j=1}^{s}\Phi(\sigma_j+i\lambda,\gamma/2) = \\ &= -i\frac{h_3+h_8}{N} - i\int dx\,\Phi(x+i\lambda,\gamma/2)\left[(\eta_N^\sigma-\eta_\infty^\sigma)(x) - S_N^\sigma(x)\right]. \end{aligned}$$

If we plug in the equation for $(\eta_\infty^\sigma - \eta_N^\sigma)(x)$ (eq.(43)), we immedeatly arrive at

$$L_N^h(\lambda) = -i\frac{h_3+h_8}{N} + \int dxdy\,\Phi(x+i\lambda,\gamma/2)\left[\mathbf{R}^{11}(x-y)S_N^\sigma(y) + \mathbf{R}^{12}(x-y)S_N^\rho(y)\right].$$

Taking the Fourier transform of this equation and using eq.(32) together with

$$\begin{aligned} \tilde{\eta}^\sigma(\omega) &= \frac{d\tilde{Z}_N^\sigma(x)}{dx} = i\omega\tilde{Z}_N^\sigma(\omega) \\ \tilde{\eta}^\sigma(\omega) &= \mathbf{R}^{11}(\omega)\tilde{\Phi}'(\omega,\gamma/2)\,, \end{aligned}$$

where the second equation follows from eq.(37), we get

$$\begin{aligned} L_N^h(\lambda) = -i\frac{h_3+h_8}{N} &+ \int dy\,\left[K^\sigma - 2\pi i Z_N^\sigma(y+i\lambda)\right]S_N^\sigma(y) \\ &+ \int dy\,\left[K^\rho - 2\pi i Z_N^\rho(y+i\lambda)\right]S_N^\rho(y)\,. \end{aligned} \tag{48}$$

The constants K^σ, K^ρ arise from the δ–function in eq.(32) and are determined to

$$K^\sigma = \frac{4\pi}{3(1-\gamma/\pi)} \qquad K^\rho = \frac{2\pi}{3(1-\gamma/\pi)}\,.$$

Equations (43) and (48) cannot be solved in closed form, therefore an approximation with the help of the Euler–Maclaurin formula is done like before.

Since we calculate the function $L_N(\lambda)$ in the ground state, the constants K^σ, K^ρ drop out of eq.(48):

$$\begin{aligned} \int dy\,S_N^\sigma(y) &= \frac{s}{N} - \left[Z_N^\sigma(\infty) - Z_N^\sigma(-\infty)\right] = 0 \\ \int dy\,S_N^\rho(y) &= \frac{p}{N} - \left[Z_N^\rho(\infty) - Z_N^\rho(-\infty)\right] = 0\,. \end{aligned} \tag{49}$$

Here we used the fact, that the groundstate belongs to $s = \frac{2N}{3}$, $p = \frac{N}{3}$ and eqs.(35) were used to obtain the asymptotic values of $Z_N^{\sigma,\rho}$.

Applying the Euler–Maclaurin approximation to the free energy correction eq.(48) gives:

$$\begin{aligned}
L_N^h(\lambda) = \quad & +2\pi i \left(\int_{-\infty}^{-\Lambda^\sigma} dy + \int_{\Lambda^\sigma}^{\infty} dy \right) \eta_N^\sigma(y) Z_\infty^\sigma(y+i\lambda) + \\
& +2\pi i \left(\int_{-\infty}^{-\Lambda^\rho} dy + \int_{\Lambda^\rho}^{\infty} dy \right) \eta_N^\rho(y) Z_\infty^\rho(y+i\lambda) - \\
& -\frac{i\pi}{N} \left[Z_N^\sigma(\Lambda^\sigma + i\lambda) + Z_N^\sigma(-\Lambda^\sigma + i\lambda) + Z_N^\rho(\Lambda^\rho + i\lambda) + Z_N^\rho(-\Lambda^\rho + i\lambda) \right] - \\
& -\frac{i\pi}{6N^2} \left[\frac{\eta_N^\sigma(\Lambda^\sigma + i\lambda)}{\eta_N^\sigma(\Lambda^\sigma)} - \frac{\eta_N^\sigma(-\Lambda^\sigma + i\lambda)}{\eta_N^\sigma(-\Lambda^\sigma)} + \frac{\eta_N^\rho(\Lambda^\rho + i\lambda)}{\eta_N^\rho(\Lambda^\rho)} - \frac{\eta_N^\rho(-\Lambda^\rho + i\lambda)}{\eta_N^\rho(-\Lambda^\rho)} \right] \\
& -i\frac{h_3 + h_8}{N} \\
& = L_I + L_B - i\frac{h_3 + h_8}{N}.
\end{aligned}$$

The terms in the first two lines will be called integral contributions L_I, while the terms of the third and forth line are called boundary terms L_B. Substituting $Z_N^{\sigma,\rho}(\Lambda^{\sigma,\rho} + i\lambda)$, $\eta_N^{\sigma,\rho}(\Lambda^{\sigma,\rho} + i\lambda)$ by their limits $Z_\infty^{\sigma,\rho}$, $\eta_\infty^{\sigma,\rho}$ and using eqs.(41) and the fact, that Λ^σ and Λ^ρ are large, the boundary contributions L_B sum up to:

$$\begin{aligned}
L_B(\lambda) = \quad & -\frac{i\pi}{N} + \frac{\sqrt{3}}{N} \left[e^{-\alpha\Lambda^\sigma} + e^{-\alpha\Lambda^\rho} \right] \sin(\alpha\lambda) - \\
& -\frac{\pi}{3\sqrt{3}\gamma N} \left[\frac{e^{-\alpha\Lambda^\sigma}}{N\eta_N^\sigma(\Lambda^\sigma)} + \frac{e^{-\alpha\Lambda^\rho}}{N\eta_N^\sigma(\Lambda^\rho)} \right] \sin(\alpha\lambda).
\end{aligned} \tag{50}$$

The parameter α appearing in the above expression is defined as $\alpha = \frac{2\pi}{3\gamma}$. The integral contributions L_I are evaluated with the help of complex integration. As a prototype of the involved integrals lets us consider

$$\begin{aligned}
I_\sigma \quad := \quad & 2\pi i \int_{\Lambda^\sigma}^{\infty} dy\, \eta_N^\sigma(y) Z_\infty^\sigma(y+i\lambda) \\
= \quad & 2\pi i \int_{-\infty}^{\infty} d\omega\, X_+^\sigma(\omega) \tilde{Z}_\infty^\sigma(\omega) e^{i\omega(\Lambda^\sigma + i\lambda)},
\end{aligned}$$

where $X_+^\sigma(\omega)$ is defined in the appendix (eq. 56). Since X_+^σ is analytic in the upper half plane, only the singularities of $Z_\infty^\sigma(\omega) = \frac{-i}{2\pi\omega} \frac{\sinh \omega\gamma}{\sinh 3\omega/2\gamma}$ at $\omega = 0$ and on the positive imaginary axis contribute to the integral. Neglecting exponentially damped contributions again gives:

$$I^\sigma = \frac{2\pi i}{3} X_+^\sigma(0) - \sqrt{3} i\, X_+^\sigma(i\alpha) e^{-\alpha(\Lambda^\sigma + i\lambda)},$$

and analogous expressions for the other integrals, which involve X_+^ρ, Y_-^σ, Y_-^ρ. From the definition of $X_\pm, Y_\pm$ (eq.56 in the appendix) we see that the terms containing $X_+^{\sigma,\rho}(0)$, $Y_-^{\sigma,\rho}(0)$ cancel the $1/N$–terms in the expression for L_N^h and we get for $L_N^h(\lambda)$:

$$\begin{aligned}\frac{1}{\sin(\alpha\lambda)}L_N^h(\lambda) = \quad & \frac{\sqrt{3}}{N}\left[e^{-\alpha\Lambda^\sigma}+e^{-\alpha\Lambda^\rho}\right]-\frac{\pi}{3\sqrt{3}\gamma N}\left[\frac{e^{-\alpha\Lambda^\sigma}}{N\eta_N^\sigma(\Lambda^\sigma)}+\frac{e^{-\alpha\Lambda^\rho}}{N\eta_N^\sigma(\Lambda^\rho)}\right]- \\ & -2\sqrt{3}\left[X_+^\sigma(i\alpha)e^{-\alpha\Lambda^\sigma}+X_+^\rho(i\alpha)e^{-\alpha\Lambda^\rho}\right].\end{aligned} \tag{51}$$

In vector notation, the last term reads simply $\langle X_+(i\alpha)|V\rangle$ and we make use of the results for $|X(\omega)\rangle$ obtained in the appendix. Evaluating eq.(62) at $\omega = i\alpha$ and inserting into eq.(51), we observe that the contributions from $|A(\omega)\rangle$ compensate exactly the first line in eq.(51). Therefore, the correction simplifies to

$$\frac{L_N^h}{\sin\alpha\lambda} = -\frac{3}{2\pi}\langle \mathbf{G}_+(i\alpha)V|\mathbf{G}_+(i\alpha)V\rangle - 2\sqrt{3}\langle \mathbf{G}_+(i\alpha)V|P(i\alpha)\rangle.$$

We use eq.(63,66) of the appendix and find that we can express $N^2L_N^h/2\pi\sin\alpha\lambda$ by formula (70) and all this rather lenghty expressions boil down to a quite simple final result:

$$L_N^h(\lambda) = \frac{2\pi}{N^2}\left[C_\phi + C_\psi\right]\sin\alpha\lambda = \frac{\pi}{N^2}\left[\frac{2}{6}-\frac{(h_3^2+3h_8^2)}{\pi(\pi-\gamma)}\right]\sin\frac{2\pi}{3\gamma}\lambda. \tag{52}$$

For conformal invariance one must properly normalize the corresponding Hamiltonian eq.(10) in order to get a relativistic energy momentum relation. That means we have to divide eq.(52) by $-\sin\alpha\lambda$ (see Refs.[9],[2]). Comparison to the prediction of conformal field theory for the finite size corrections to the eigenvalue, $L_N^h = \frac{-\pi}{6N^2}c$, we find, that the vertical fields modify the central charge c to

$$c = 2 - \frac{6}{\pi(\pi-\gamma)}(h_3^2+3h_8^2) \tag{53}$$

6 Conclusion

In this paper we found, that certain diffusion terms implemented into the system do not spoil integrabilty of multistate vertex models. The horizontal fields effect the one dimensional quantum mechanical problem while the vertical fields just give overall factors. The nested Bethe-Ansatz procedure is done and we showed that the thermodynamic limit is independent of the fields. In order to discuss finite size properties, we computed the central charge of the model in the critical region and found a result which generalizes that of Ref.[2].

Appendix: The Riemann–Hilbert Problem

The finite size equations (43,48) for the densities and the eigenvalue contain integrals of the form:

$$\int dy\, f(y) S_N(y) \qquad \text{where} \qquad S_N(y) = \frac{1}{N}\sum_{i=1}^{k} \delta(y-\sigma_i) - \eta_N(y) \tag{54}$$

and σ_i are the Bethe roots. Integrals of this type can be conveniently approximated by the Euler–Maclaurin formula. Following ([8],[9], [2]) we keep only the first two terms and get:

$$\begin{aligned} \int_{-\infty}^{\infty} f(y) S_N(y) \sim -\left(\int_{-\infty}^{-\Lambda} + \int_{\Lambda}^{\infty}\right) \eta(y) f(y) \; &+ \; \frac{1}{2N}\left[f(\Lambda) + f(-\Lambda)\right] \\ &+ \; \frac{1}{12N^2\eta(\Lambda)}\left[f'(\Lambda) - f'(\Lambda)\right]. \end{aligned} \tag{55}$$

As we shall see later on, the term $N\eta(\Lambda)$ is of order one. Therefore, eq.(55) is basically an expansion up to order $\frac{1}{N}$. Define the 'half Fourier transforms' $X_{\pm}^{\sigma,\rho}(\omega)$, $Y_{\pm}^{\sigma,\rho}(\omega)$ as:

$$\begin{aligned} X_{\pm}^{\sigma,\rho}(\omega) &= \int_{-\infty}^{\infty} dt\; e^{i\omega t}\Theta(\pm t)\eta_N^{\sigma,\rho}(t+\Lambda^{\sigma,\rho}) \\ Y_{\pm}^{\sigma,\rho}(\omega) &= \int_{-\infty}^{\infty} dt\; e^{i\omega t}\Theta(\pm t)\eta_N^{\sigma,\rho}(t-\Lambda^{\sigma,\rho}). \end{aligned} \tag{56}$$

This are analytic functions on the upper resp. lower half of the complex plane and give continous functions when ω approaches the real axis. From eq.(56) it follows, that the Fourier transform of $\eta_N^{\sigma,\rho}$ can be written as:

$$\tilde{\eta}_N^{\sigma,\rho} = e^{-i\omega\Lambda^{\sigma,\rho}}\left[X_+^{\sigma,\rho}(\omega) + X_-^{\sigma,\rho}(\omega)\right] = e^{i\omega\Lambda^{\sigma,\rho}}\left[Y_+^{\sigma,\rho}(\omega) + Y_-^{\sigma,\rho}(\omega)\right]. \tag{57}$$

Applying eq.(55) to the density equation eq.(43) and using eqs.(56,57) after Fourier transformation, the Wiener-Hopf type equations are transformed into a inhomogenous matrix Riemann-Hilbert problem:

$$|X_-(\omega)\rangle + \hat{\mathbf{R}}(\omega)|X_+(\omega)\rangle = |\eta(\omega)\rangle + (\hat{\mathbf{R}} - \mathbf{1})(\omega)|A(\omega)\rangle\,. \tag{58}$$

Expressed in words, we are looking for piecewise analytic functions $|X_{\pm}(\omega)\rangle$ such that their continous limit functions on the real axis satisfy the above relation. A vector notation was introduced to achieve a concise expression:

$$|X_{\pm}(\omega)\rangle = \begin{pmatrix} X_{\pm}^{\sigma}(\omega) \\ X_{\pm}^{\rho}(\omega) \end{pmatrix} \quad |\eta(\omega)\rangle = \begin{pmatrix} e^{i\omega\Lambda^{\sigma}}\tilde{\eta}_{\infty}^{\sigma}(\omega) \\ e^{i\omega\Lambda^{\rho}}\tilde{\eta}_{\infty}^{\rho}(\omega) \end{pmatrix} \quad |A(\omega)\rangle = \begin{pmatrix} \frac{1}{2N} + \frac{i\omega}{12N^2\eta_N^{\sigma}(\Lambda^{\sigma})} \\ \frac{1}{2N} + \frac{i\omega}{12N^2\eta_N^{\rho}(\Lambda^{\rho})} \end{pmatrix}$$

The matrix function $\hat{\mathbf{R}}(\omega)$ is a modification of the resolvent eq.(40) and reads

$$\hat{\mathbf{R}}(\omega) = \begin{pmatrix} R^1(\omega) & e^{i\omega(\Lambda^{\sigma}-\Lambda^{\rho})}R^2(\omega) \\ e^{-i\omega(\Lambda^{\sigma}-\Lambda^{\rho})}R^2(\omega) & R^1(\omega) \end{pmatrix} \tag{59}$$

Note, that in deriving eq.(58) we neglected terms of order $e^{2i\omega\Lambda^{\sigma,\rho}}$, $e^{i\omega(\Lambda^{\sigma}+\Lambda^{\rho})}$. This terms sum up to give logarithmic corrections to the $1/N^2$ behaviour as discussed in [9].

Solving eq.(58) (see [2] or [10] for the details), we observe that the homogenous problem provides a factorization of the matrix $\hat{\mathbf{R}}(\omega)$ into a product of matrices

$$\hat{\mathbf{R}}(\omega) = \mathbf{G}_+^{-1}(\omega)\mathbf{G}_-^{-1}(\omega) \tag{60}$$

Again, $\mathbf{G}_\pm(\omega)$ are analytic in the upper/lower plane and satisfy $\mathbf{G}_+(\omega) = \mathbf{G}_-(-\omega)$. $\mathbf{G}_\pm$ could be expressed by a set of fundamental solutions [10], but since we do not need them explicitly, we just remark that $\mathbf{G}_\pm(\pm\infty) = \mathbf{1}$ and has a Taylor expansion at $\omega = \infty$.

Multiplying eq.(58) with $\mathbf{G}_-(\omega)$ from the left and using the analyticity properties, one is led to the auxiliary problem

$$|Q_+(\omega)\rangle + |Q_-(\omega)\rangle = \mathbf{G}_-(\omega)|\eta(\omega)\rangle,$$

which is easily solved by Cauchy integrals:

$$\begin{aligned} |Q_+(\omega)\rangle &= \int_{-\infty}^{\infty} \frac{dt}{2\pi i} \frac{\mathbf{G}_-(t)|\eta(t)\rangle}{t-\omega} \\ &\sim \frac{i}{\sqrt{3}\gamma} \frac{1}{i\alpha+\omega} \mathbf{G}_+(i\alpha)|V\rangle. \end{aligned} \tag{61}$$

$|V\rangle$ is defined as $|V\rangle = (e^{-\alpha\Lambda^\sigma}, e^{-\alpha\Lambda^\rho})$ and the second line is obtained by evaluating the integral up to order $e^{-\alpha\Lambda}$.

Now the solution of the inhomogenous equation is uniquely determined by the requirement $\lim_{\omega\to\infty} X_+^{\sigma,\rho} = 0$ and reads:

$$|X_+(\omega)\rangle = \mathbf{G}_+(\omega)\ [\ |P(\omega)\rangle + |Q_+(\omega)\rangle\ ] + |A(\omega)\rangle. \tag{62}$$

The polynomial vector is given by:

$$|P(\omega)\rangle = -\frac{1}{2N} \begin{pmatrix} 1 - \frac{ig_1 - i\omega}{6N\eta_N^\sigma} \\ 1 - \frac{ig_2 - i\omega}{6N\eta_N^\rho} \end{pmatrix}. \tag{63}$$

The constants g_1, g_2 appearing in the expression for $|P(\omega)\rangle$ are the first order coefficients of the diagonal elements of the analytic matrix function in the expansion at $\omega = \infty$. Since they will not show up in the final result, we do not present them explicitly. All necessary information to determine the finite size correction of the eigenvalue will be encoded in the two conditions that follow from eq.(56).

$$|X_+(0)\rangle = \begin{pmatrix} \frac{1}{2N} - \frac{h_3}{N\pi} \\ \frac{1}{2N} - \frac{h_3 - 3h_8}{N\pi} \end{pmatrix}, \tag{64}$$

$$-i \lim_{\omega\to\infty} \omega |X_+(\omega)\rangle = N \begin{pmatrix} \eta_N^\sigma(\Lambda^\sigma) \\ \eta_N^\rho(\Lambda^\rho). \end{pmatrix} \tag{65}$$

Evaluating the solution eq.(62) at $\omega = 0$ we conclude from eq.(64) that

$$N\mathbf{G}_+(i\alpha)|V\rangle = \begin{pmatrix} \frac{\phi}{2\sqrt{3}G} + \frac{\pi}{\sqrt{3}}[1 - \frac{ig_1}{6N\eta_N^\sigma}] \\ \frac{\psi}{2\sqrt{3}G} + \frac{\pi}{\sqrt{3}}[1 - \frac{ig_2}{6N\eta_N^\rho}] \end{pmatrix}, \tag{66}$$

where $G := \sqrt{3}(1-\gamma/\pi)^{\frac{1}{2}}$ and ϕ, ψ are determined by the fields h_3, h_8 through

$$\phi = -(\sqrt{3}+3)h_3 - 3(\sqrt{3}-1)h_8 \qquad \psi = -(\sqrt{3}-3)h_3 - 3(\sqrt{3}+1)h_8. \tag{67}$$

On the other hand, from eq.(65) we get an equation for $N\eta_N^{\sigma,\rho}$:

$$N\eta_N^{\sigma} = \frac{4\alpha + ig_1}{2}\left[\frac{1}{2} - \frac{ig_1}{12N\eta_N^{\sigma}} + \frac{\phi}{4\pi G}\right] + \frac{ig_1}{2}\left[\frac{1}{2} - \frac{\phi}{4\pi G}\right], \tag{68}$$

and a similar expression for $N\eta_N^{\rho}$ with g_1 and ϕ replaced by g_2, ψ. This is a quadratic equation for $N\eta_N^{\sigma,\rho}$ and we see that $N\eta_N^{\sigma,\rho}$ is of order one as asserted earlier. We define variables a, b which will appear in the expression for $L_N^h(\lambda)$ as

$$a = N\eta_N^{\sigma} - \frac{ig_1}{2}\left[\frac{1}{2} - \frac{\phi}{4\pi G}\right], \qquad b = N\eta_N^{\rho} - \frac{ig_2}{2}\left[\frac{1}{2} - \frac{\psi}{4\pi G}\right]. \tag{69}$$

Eq.(68) can then be recasted into the form:

$$\frac{4a^2}{ig_1(4\alpha + ig_1)} - \frac{4a}{(4\alpha + ig_1)}\left[\frac{\phi}{4\pi G} + \frac{2\alpha}{ig_1}\left(\frac{1}{2} + \frac{\phi}{4\pi G}\right)\right] = C_\phi \tag{70}$$

plus an analogous equation for b. The $C_{\phi,\psi}$ appearing in this equations are defined as

$$C_\psi = \left(\frac{1}{4} - \frac{\phi^2}{16\pi^2 G^2}\right) \qquad \text{and} \qquad C_\psi = \left(\frac{1}{4} - \frac{\psi^2}{16\pi^2 G^2}\right). \tag{71}$$

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