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# Supergravity in the Weitzenböck Space-time

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Abstract:

It is shown that it is possible to formulate a consistent local supergravity theory in the Weitzenböck space-time. It is likely that our theory has a close relationship with the theory of FSUPWT by J.G. Taylor et al.

Theory of gravitation in the Weitzenböck space-time is called new general relativity.[1)†)] This space-time is characterized by the absence of the curvature tensor and the presence of the torsion tensor.[†)] In accordance with this, the gauge symmetry of the Lorentz transformation is reduced to a global symmetry. It has been shown that new general relativity is equivalent to ordinary general relativity as far as the measurements of macroscopic phenomena are concerned; differences appear only in the microscopic region.[1b)]

The recent work of J.G. Taylor et al.[2)] can be viewed as an extention of the concept of new general relativity to the superspace. In this theory, it has been shown that the only divergence occurs in the overall coupling constant renormalization to all orders.

It is interesting to investigate whether new general relativity, the geometry of the Weitzenböck space-time, is compatible with supergravity. In this paper, we show that it is indeed possible to construct a consistent local supergravity theory in the Weitzenböck space-time; the Lagrangian of the conventional supergravity in the Riemann-Cartan space-time[3),4)] can be translated into that in the Weitzenböck space-time.

Our conclusions are summarized as follows:

(1) In ref.1b, there remains an undetermined constant $\lambda$, connected with a massless scalar particle mode of the antisymmetric part of the linearized vierbein (tetrad or the parallel vector field $b_\mu^m$). On the other hand, in our supersymmetric extension of new general relativity, this constant $\lambda$ is shown to vanish because of a strong symmetry of supergravity.

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†) This space-time was once examined by Einstein in 1928.

(2) The coupling between the torsion and the gravitino $\psi_\mu$ cannot be a minimal one in a strict sense as is shown later.

(3) The conservation law of the energy-momentum tensor and the equivalence principle hold exactly, since there is no additional term due to the antisymmetric part of the energy-momentum tensor.

The questions of closure of the gauge algebra and the renormalizability are discussed later.

Now the Euclid condition is of the form, $D_\mu g_{\rho\sigma} = 0$. This can be written in the Weitzenböck space-time as[1b)†)]

$$D_\mu^{(W)} e_\rho{}^m \equiv \partial_\mu e_\rho{}^m - \Gamma_{\mu\rho}{}^\sigma e_\sigma{}^m = 0 \quad . \qquad (1)$$

$D_\mu^{(W)}$ is the full covariant derivative corresponding to the derivative $D_\mu^{(R)} e_\rho^m = \partial_\mu e_\rho^m - \Gamma_{\mu\rho}^\sigma e_\sigma^m + \omega_\mu^{mn} e_{\rho n} - \mathcal{D}_\mu^{(R)} e_\rho^m - \Gamma_{\mu\rho}^\sigma e_\sigma^m$ in the Riemann-Cartan space-time. We can choose the Lorentz connexion in the Weitzenböck space-time to be identically zero from the definition of this space-time. We solve $\Gamma_{\mu\rho}^\sigma$ in (1) to be of the form,

$$\Gamma_{\mu\rho}{}^\sigma = e_m{}^\sigma \partial_\mu e_\rho{}^m \quad . \qquad (2)$$

The contorsion tensor and the torsion tensor are given by

$$K_{\rho\sigma\tau} \equiv \frac{1}{2}(T_{\rho\sigma\tau} - T_{\rho\tau\sigma} + T_{\tau\sigma\rho}) = -K_{\rho\tau\sigma} \quad , \qquad (3)$$

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†) Strictly speaking, $e_\mu^m$ in the Weitzenböck space-time is a parallel vector field symbolized as $b_\mu^m$ in ref.1b and has the relationship with the metric tensor: $b_\mu^m(x)\eta_{mn}b_\nu^n(x) = g_{\mu\nu}(x)$

$$T_{\rho\sigma}{}^{\mu} \equiv \Gamma_{\rho\sigma}{}^{\mu} - \Gamma_{\sigma\rho}{}^{\mu} = e_m{}^{\mu}(\partial_\rho e_\sigma{}^m - \partial_\sigma e_\rho{}^m) = -T_{\sigma\rho}{}^{\mu}. \quad (4)$$

respectively.

It is noticed that we can represent the Lorentz connexion $\omega_\mu{}^{mn}(e,\psi)$ of the conventional supergravity theory (in the second order formalism) in the Riemann-Cartan space-time[3),4)] in terms of <u>tensorial</u> quantities in the Weitzenböck space-time.

$$\omega_\mu{}^{mn}(e,\psi) = \frac{1}{2}(T_\mu{}^{mn} - T_\mu{}^{nm} + T^{nm}{}_\mu) + \frac{1}{2}(C_\mu{}^{mn} - C_\mu{}^{nm} + C^{nm}{}_\mu), \quad (5)$$

$$C_{\rho\sigma}{}^{\mu} \equiv -\frac{1}{2}\bar{\psi}_\rho \gamma^\mu \psi_\sigma = -C_{\sigma\rho}{}^{\mu}. \quad (6)$$

The equations (4), (5) and (6) enable us to translate the Lagrangian $\mathcal{L}^{(R)}$ of the conventional supergravity theory (in the second order formalism without auxiliary fields)[3),4)] into $\mathcal{L}^{(W)}$ in the Weitzenböck space-time. After some calculations we obtain

$$\mathcal{L}^{(R)} = \frac{1}{2} e R[e,\omega(e,\psi)] - \frac{i}{2}\varepsilon^{\mu\nu\rho\sigma}\bar{\psi}_\mu \gamma_5 \gamma_\rho [\partial_\sigma + \frac{i}{4}\omega_\sigma{}^{mn}(e,\psi)\sigma_{mn}]\psi_\nu$$

$$= \frac{1}{2} e R(\{\}) - \frac{i}{2}\varepsilon^{\mu\nu\rho\sigma}\bar{\psi}_\mu \gamma_5 \gamma_\rho \partial_\sigma \psi_\nu$$

$$- \frac{1}{8} e C_{\rho\sigma\mu} C^{\rho\sigma\mu} + \frac{1}{4} e C_{\rho\sigma\mu} C^{\mu\rho\sigma} + \frac{1}{2} e C_{\rho\sigma}{}^{\sigma} C^{\rho}{}_{\tau}{}^{\tau}$$

$$- \frac{1}{4} e C_{\rho\sigma\mu} T^{\rho\sigma\mu} + \frac{1}{2} e C_{\rho\sigma\mu} T^{\mu\rho\sigma} + e C_{\rho\sigma}{}^{\sigma} T^{\rho}{}_{\tau}{}^{\tau} \equiv \mathcal{L}^{(W)}. \quad (7)$$

up to a four-divergence. We use the notations $\eta_{mn} = \mathrm{diag.}(-+++)$, $\varepsilon^{0123} = +1$, $\sigma^{mn} \equiv \frac{1}{2i}[\gamma^m, \gamma^n]$, $\{\gamma_m, \gamma_n\} = +2\eta_{mn}$, $\gamma_5 \equiv i\gamma^1\gamma^2\gamma^3\gamma^0$ and

$$[\partial_\mu + \tfrac{i}{4}\omega_\mu{}^{mn}\sigma_{mn}, \partial_\nu + \tfrac{i}{4}\omega_\nu{}^{rs}\sigma_{rs}] = \tfrac{i}{4} R_{\mu\nu}{}^{mn}(\omega)\sigma_{mn},$$

$$R_\mu{}^m \equiv e_n{}^\nu R_{\mu\nu}{}^{mn}, \quad R \equiv e_m{}^\mu R_\mu{}^m. \tag{8}$$

The scalar curvature $R(\{\})$ is formed of the Levi-Civita connexion $\{{}^{\mu}_{\rho\sigma}\}$. It should be noticed that $R(\{\})$ can be expressible as quadratic forms of $T_{\rho\sigma}{}^\mu$ 1b) †)

$$R(\{\}) \equiv -\tfrac{1}{4} T_{\rho\sigma\mu}T^{\rho\sigma\mu} + \tfrac{1}{2} T_{\rho\sigma\mu}T^{\mu\rho\sigma} + T_{\rho\sigma}{}^\sigma T^\rho{}_\tau{}^\tau - 2e^{-1}\partial_\rho(eT^\rho{}_\sigma{}^\sigma). \tag{9}$$

Then $\mathcal{L}^{(W)}$ in (7) is put into the simpler form

$$\mathcal{L}^{(W)} = -\tfrac{1}{8} e R_{\rho\sigma\mu}R^{\rho\sigma\mu} + \tfrac{1}{4} e R_{\rho\sigma\mu}R^{\mu\rho\sigma} + \tfrac{1}{2} e R_{\rho\sigma}{}^\sigma R^\rho{}_\tau{}^\tau$$

$$- \tfrac{i}{2}\varepsilon^{\mu\nu\rho\sigma}\bar{\psi}_\mu \gamma_5 \gamma_\rho \partial_\sigma \psi_\nu . \tag{10}$$

where

$$R_{\rho\sigma}{}^\mu \equiv T_{\rho\sigma}{}^\mu + C_{\rho\sigma}{}^\mu = -R_{\sigma\rho}{}^\mu . \tag{11}$$

is a supercovariant torsion in the sense of ref.5.

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†) In ref.1b, the expression is given in terms of three irreducible components of $T_{\rho\sigma}{}^\mu$: $t_{\rho\sigma\mu}$, $v_\mu$ and $a_\mu$

The first line of (10) may be viewed as a natural extension of the scalar curvature (9) in ordinary general relativity. A simple replacement of $T_{\rho\sigma}{}^{\mu}$ in $R(\{\})$ by $R_{\rho\sigma}{}^{\mu}$ with the additional kinetic term of $\psi_\mu$ yields the Lagrangian of supergravity in the Weitzenböck space-time. Supersymmetry leaves no additonal parameter $\lambda = 9/(4c_3)$ of the antisymmetric part $a^\mu = \frac{1}{6}\epsilon^{\mu\rho\sigma\tau}T_{\rho\sigma\tau}$ of the torsion $T_{\rho\sigma\tau}$.

As in conventional supergravity, one must check the consistency condition for the field equation of $\psi_\mu$. However, the problem is already solved since our Lagrangian $\mathcal{L}^{(W)}$ is equivalent to $\mathcal{L}^{(R)}$ with the same variational principle. Then the field equations are the same as those in ref.3 (in the second order formalim). We can rewrite the field equation of $\psi_\mu$ in terms of $T_{\rho\sigma}{}^{\mu}$ and $R_{\rho\sigma}{}^{\mu}$ explicitly:

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\begin{aligned}
R^\mu \equiv \frac{\delta \mathcal{L}^{(R)}}{\delta \psi_\mu} &= \frac{\delta \mathcal{L}^{(W)}}{\delta \psi_\mu} \\
&= i\varepsilon^{\mu\nu\rho\sigma}\gamma_5\gamma_\rho\partial_\sigma\psi_\nu - \frac{i}{4}\varepsilon^{\mu\nu\rho\sigma}\gamma_5\gamma_\varphi\psi_\nu T_{\rho\sigma}{}^{\varphi} \\
&\quad - \frac{1}{2}e\gamma^\mu\psi_\rho R^\rho{}_\sigma{}^\sigma + \frac{1}{2}e\gamma^\rho\psi_\rho R^\mu{}_\sigma{}^\sigma \\
&\quad + \frac{1}{4}e\gamma^\rho\psi^\sigma(R_\rho{}^\mu{}_\sigma - R_{\rho\sigma}{}^\mu + R_\sigma{}^\mu{}_\rho) = 0 . \qquad (12)
\end{aligned}
$$

We can realize the equivalence of the "1.5-order formalism"[4b)] to the second order formalism. Each term of the equation (12) is covariant under the <u>global</u> Lorentz transformation but not under

the local one, since we regard $T_{\rho\sigma}{}^{I}$ and $R_{\rho\sigma}{}^{I}$ as tensors in the Weitzenböck space-time. The consistency check can be done in the same way as in ref.3, but the interpretation is slightly different. The equation $\mathcal{D}_{\mu}^{(R)} R^{\mu} = 0$ is now from (5), (6) and (11)

$$[D_{\mu}^{(L)} + \frac{i}{8}(R_{\mu}{}^{\rho\sigma} - R_{\mu}{}^{\sigma\rho} + R^{\sigma\rho}{}_{\mu})\sigma_{\rho\sigma}]\, e^{-1} R^{\mu} = 0 \,. \qquad (13)$$

where $D_{\mu}^{(L)}$ is a covariant derivative with the Levi-Civita connexion. Since the last three terms of (13) form a super-covariant contorsion tensor, they should not be confused with the connexion. These terms pose no problem for several reasons. First, it is natural that the contorsion tensor appears in the conservation law in the Weitzenböck space-time.[1b)] Secondly, the field equation $R^{\mu} = 0$ can be applied to (13) after the operation of the derivative in $D_{\mu}^{(L)}$, so that the last three terms have no contributions.[†)] Thirdly, the invariance of the action under the supersymmetry transformation of the fields, (20) and (21), leads to (13) by the use of equation (15) after the integration by parts. We also emphasize that our field equation $R^{\mu} = 0$ is the same as that in ref.3. From these considerations we can safely regard $\mathcal{L}^{(W)}$ as a consistent Lagrangian.

The absence of the minimal coupling between $T_{\rho\sigma}{}^{\mu}$ and $\psi_{\mu}$ in $\mathcal{L}^{(W)}$ can be understood as follows. If we add a minimal coupling between them in (10), there appears a term of the form $\sim \varepsilon^{\mu\nu\rho\sigma}\gamma_5 \bar{\psi}_{\nu} D_{\mu}^{(W)} T_{\rho\sigma}{}^{\psi}$ in (13). This term never vanishes by using

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†) There is a similar situation in the conservation law of the energy-momentum tensor. (see (19) and (17))

the field equation (15) and $R^{\mu} = 0$ itself, so that the consistency check fails.

Attention should be paid to the Einstein gravitational field equation. Suppose we regard the part $\frac{1}{2}eR(\{\})$ as the gravitational Lagrangian $\mathcal{L}_G^{(W)}$, and the rest in $\mathcal{L}^{(W)}$ as the <u>matter</u> Lagrangian $\mathcal{L}_M^{(W)}$;

$$\mathcal{L}_G^{(W)} \equiv \frac{1}{2} e R(\{\}) \ , \quad \mathcal{L}_M^{(W)} \equiv \mathcal{L}^{(W)} - \mathcal{L}_G^{(W)} . \tag{14}$$

Then the Einstein equation takes the form:

$$G_{\mu\nu}^{(W)}(\{\}) = T_{\mu\nu}^{(W)} . \tag{15}$$

Since the left hand side is symmetric in $\mu$ and $\nu$, (15) is equivalent to

$$G_{\mu\nu}^{(W)}(\{\}) = T_{(\mu\nu)}^{(W)} , \tag{16}$$

$$T_{[\mu\nu]}^{(W)} = 0 . \tag{17}$$

The Bianchi identity for the Einstein tensor is

$$D_{\rho}^{(L)} G_{(W)}^{\mu\rho}(\{\}) = 0 . \tag{18}$$

The response equation of $T_{(W)}^{\mu\nu}$ is [1b),6)]

$$D_{\rho}^{(L)} T_{(W)}^{\mu\rho} + K^{\mu}{}_{\rho\sigma} T_{(W)}^{\rho\sigma} = 0 \tag{19}$$

Since $K^{\mu}{}_{\rho\sigma}$ is antisymmetric in $\rho$ and $\sigma$, the second term of (19)

vanishes when the equation (17) is applied. In a general Weitzenböck space-time,[1b] the second term of (19) does not vanish even on the geodesic frame as far as the antisymmetric part of $T^{[\mu\nu]}$ is present. So the conservation law $D^{(L)}_{\rho} T^{\mu\rho} = 0$ and the equivalence principle are broken in the microscopic region. In our theory, on the other hand, the supersymmetry excludes such a term, and recovers the conservation law and the equivalence principle. The difference of $G^{(W)}_{\mu\nu}$ from that of ref.3 is due to the different way $\mathcal{L}^{(W)}_{G}$ and $\mathcal{L}^{(W)}_{M}$ are separated from each other, but the Einstein equation itself is the same, since $\mathcal{L}^{(W)} = \mathcal{L}^{(R)}$.

The transformation rule of the fields is[4]

$$\delta e_{\mu}{}^{m} = \bar{\varepsilon} \gamma^{m} \psi_{\mu} , \tag{20}$$

$$\delta \psi_{\mu} = 2 \partial_{\mu} \varepsilon + \frac{i}{4} (R_{\mu}{}^{\rho\sigma} - R_{\mu}{}^{\sigma\rho} + R^{\sigma\rho}{}_{\mu}) \sigma_{\rho\sigma} \varepsilon \tag{21}$$

As mentioned above, the last three terms of (21) are tensors.

We next turn to the question of closure of the gauge algebra. It is instructive to notice the peculiar property of the generator $J_{rs}$ in the graded Poincaré algebra (GP).

$$[J_{mn}, J_{rs}] = (i \eta_{mr} J_{ns} + i \eta_{ns} J_{mr}) - (r \leftrightarrow s) , \tag{22}$$

$$[P_{m}, J_{rs}] = -i \eta_{mr} P_{s} + i \eta_{ms} P_{r} , \tag{23}$$

$$[S_{a}, J_{rs}] = \frac{1}{2} (\sigma_{rs})_{a}{}^{b} S_{b} , \tag{24}$$

$$\{S_{a}, S_{b}\} = -2i (\gamma^{m})_{ab} P_{m} . \tag{25}$$

The generator $J_{rs}$ does not appear on the right hand side except in (22), in which it appears on the left hand side as well. From a mathematical point of view this indicates that the generators $P_m$ and $S_a$ form an <u>ideal</u> sub (graded) Lie algebra. We can choose the parameter of $J_{rs}$ to be global consistently without any interference with the other local parameters of $P_m$ and $S_a$. In fact, this can be seen in the notation of ref.5. The relations $[\delta(\theta_1'), \delta(\theta_2')] = \delta([\theta_1', \theta_2'])$ (14a) and the other relations (14b) $\sim$ (14d) hold under the restirctions $\omega_\mu{}^{rs} = \chi_\alpha{}^{rs} = D^{rs} \equiv 0$ and $\partial_\mu \lambda^{rs} \equiv 0$ by the peculiar property of the generator $J_{rs}$. In superspace, this means the absence of the supercurvature: $R_{MN}{}^{rs}(z) \equiv 0$.[7] The problem of the minimal set of auxiliary fields is still left open. It is obvious, however, that the auxiliary fields of ref.4 are not appropriate, since the local Lorentz transformations with field dependent parameters take part in closure of the gauge algebra.

As for the renormalizability of the theory, it is likely that our theory is less divergent than the ordinary supergravity theory, as new general relativity is less divergent than ordinary general relativity. Notice that, in our theory, there arises no counter term containing the curvatures formed of the Lorentz connexion such as $R_{\mu\nu}{}^{rs}(\omega) R^{\mu\nu}{}_{rs}(e,\omega)$, $R_\mu{}^m(e,\omega) R^\mu{}_m(e,\omega)$, $R(e,\omega)^2$, etc. The only possible counter terms are formed of the torsion tensor $T_{\rho\sigma}{}^\tau$ (or $R_{\rho\sigma}{}^\tau$), the covariant derivative $D_\mu^{(W)}$, and $\psi_\mu$. These problems are yet to be studied in the near future.

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