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METRIC-SPACE TECHNIQUES FOR THE
STUDY OF ASTROPHYSICAL MAPS**

Fred C. Adams
University of Michigan

and

Jennifer J. Wiseman
Harvard-Smithsonian Center for Astrophysics

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Fred C. Adams¹ and Jennifer J. Wiseman²

¹*Physics Department, University of Michigan
Ann Arbor, MI 48109*

²*Harvard-Smithsonian Center for Astrophysics
60 Garden Street, Cambridge, MA 02138*

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Abstract

We extend a newly developed formal system for the description of astrophysical maps. In this formalism, we consider the difference between maps to be the distance between elements of a pseudometric space (the space of all such maps). This ansatz allows us to measure quantitatively the difference between any two maps and to order the space of all maps. For each physical characteristic of interest, this technique assigns an “output” function to each map; the difference between the maps is then determined from the difference between their corresponding output functions. In this present study, we show that the results of this procedure are invariant under a class of transformations of the maps and the domains of the maps. In addition, we study the propagation of errors (observational uncertainties) through this formalism. We show that the uncertainties in the output functions can be controlled provided that the signal to noise ratios in the original astrophysical maps are sufficiently high. The results of this paper thus increase the effectiveness of this formal system for the description, classification, and analysis of astrophysical maps.

Subject headings: methods: analytical – methods: data analysis

1. INTRODUCTION

Although many different types of astrophysical data are found in maps, the science of form description for these types of data structures remains poorly developed (see, e.g., Lord & Wilson 1984). In a previous paper (Adams 1992; hereafter Paper I), we presented a mathematical formalism for the analysis and classification of astrophysical maps. This formalism considers the collection of all maps of a given type as a metric space. By defining distance functions (pseudometrics) on the space of maps, we can measure the difference between any two maps in a quantitative manner. We can also provide an ordering scheme for the space of maps by measuring the “distance” from each map to a well-defined reference map; we can then order the elements of the space (the maps) by the size of these “distances”. In Paper I, we presented the formalism and proved basic results. However, before this formal system becomes a fully functional astrophysical technique, we must show that it can adequately deal with the standard difficulties associated with astronomical data, e.g., observational uncertainties, calibration errors, varying distances to sources, etc. The overall goal of this paper is to prove a series of results which show how this formalism behaves in the presence of the aforementioned astronomical difficulties. Our hope is to place this formal system on a solid theoretical footing which will allow for its effective use in the analysis of astrophysical maps. In a companion paper (Wiseman & Adams 1994; hereafter WA), we illustrate the efficacy of this approach by applying it to a collection of continuum maps of molecular clouds; in particular, we show that this formal system provides an effective means of describing and classifying molecular clouds.

Previous studies involving form description in astrophysics have often been concerned with the large scale structure of the universe (see, e.g., the reviews of Shandarin & Zel’dovich 1989; Melott 1990). For example, Gott, Melott, & Dickinson (1986) have studied large scale structure by calculating the topological genus of the surfaces separating high density regions from those of low density. Elizalde (1987) has presented a metric which measures the distributions of voids and clusters for point distributions (such as distributions of galaxies in the universe). Another topic that has been explored is the possibility that fractal structure appears in observed maps of molecular clouds (e.g., Bazell & Désert 1988; Dickman, Horvath, & Margulis 1990; Falgarone, Phillips, & Walker 1991). Clumps and clump mass spectra in molecular clouds have been studied over the last decade (e.g., Larson 1981, 1985; Williams, de Geus, & Blitz 1994). Recently, more detailed studies of substructure within molecular clouds have begun (Myers 1991; Houlahan & Scalo 1992; Scalo 1990; Veeraraghavan & Fuller 1991; Wood, Myers, & Daugherty 1994; see also our companion paper). However, as emphasized by Elmegreen (1993) in his recent review, a well defined mathematical framework with which to describe molecular clouds (and other astrophysical maps) is badly needed.

The fundamental concept involved in the formal system of Paper I is that we can consider astrophysical maps as elements of a metric space; the difference between any two maps is then the “distance” between two elements of a metric space (the space of all such maps). We measure this “distance” by constructing a distance function (a metric), written as

$$d(\sigma_A, \sigma_B), \tag{1.1}$$

where σ_A, σ_B are maps. As discussed in Paper I, these distance functions are actually *pseudometrics* and they make the space of all maps into a *pseudometric space*. The basic difficulty in this procedure is that the maps are often very complicated structures and hence the interpretation of any distance function becomes complicated. We thus want to simplify our interpretation of distance. We therefore “simplify” each map σ by “measuring” some physical characteristic of the map. For example, we can easily determine what fraction of the map has values above a certain threshold level Σ ; we thus obtain this fraction as a function of the threshold level. If we interpret the maps as a measure of density (or column density), we obtain a function which represents the distribution of density in the map. We denote this profile for a given map as $m(\sigma; \Sigma)$. We then measure the difference between any two maps by measuring the distance between their profiles of density, i.e., we find

$$d[m(\sigma_A), m(\sigma_B)] . \quad (1.2)$$

The distance as given by equation (1.2) now has a straightforward physical interpretation: the distance measures the difference between the distributions of density of the two maps. In addition to the distribution of density, we can construct other profiles (as a function of threshold level) which measure other properties of the maps, e.g., the number of parts the maps break up into, the shape of these parts, the self-gravity of the maps, etc. These profiles (usually one dimensional functions of the threshold level) are denoted as *output functions*. The essential feature of this formalism is that all of the above concepts can be made mathematically precise. We can thus obtain quantitatively meaningful results.

In this paper, we extend our formal system for the study of astrophysical maps in two different ways: (1) We prove a series of results which show how the output functions (and hence the distance functions) vary when we make various transformations of the original maps. These transformations include scalings of the map values, rotations, translations, stretching the maps, and scalings of the map sizes. We also briefly discuss resolution and projection effects. (2) We study the propagation of observational uncertainties through this formalism. In particular, we show how observational uncertainties in the original maps produce corresponding uncertainties in the output functions. We find that the errors can be controlled provided that the signal to noise ratio of the original map is sufficiently large.

This paper is organized as follows. In §2, we review the formal system for measuring the distance between astrophysical maps and ordering the space of all maps. We then extend this formalism in §3, where we prove several new results which quantify the manner in which transformations of the maps affect the output functions. In §4, we discuss the effects of random errors in the maps and other observational effects. We conclude in §5 with a discussion and summary of our results. The mathematical details of the effects of transformations and observational uncertainties on the formalism are presented in the appendices.

2. REVIEW OF THE FORMALISM

In this section, we review the formalism of Paper I. As discussed above, we consider each astrophysical map to be an element of an abstract space – the space of all possible maps (Paper I; see also Elizalde 1987). We thus consider the difference between two maps to be the “distance” between elements of this space. The space of all maps is thus a metric space (actually a pseudometric space) and our goal is to define distance functions (pseudometrics) on the space. †

The procedure we use to define distance functions on the space of all maps contains two steps. For a given physical characteristic of interest, we first determine a one-dimensional function (denoted here as an *output function*) to each map; this output function represents a profile of some physically meaningful quantity (e.g., distribution of density – see below). For the second step of this procedure, we determine the difference between any two maps by finding the difference between their corresponding output functions. This difference, in turn, is measured using a standard distance function (denoted here as d) defined on the space of functions (for further discussion of standard distance functions and metric spaces, see, e.g., Copson 1968; Kelley 1955; Edgar 1990).

This formal system also allows for the ordering of a collection of maps. This ordering is accomplished by assigning a positive real number – a *coordinate* – to each map, where the coordinate is defined to be the distance between the map and a well-defined reference state (or set of states).

This entire procedure can be depicted schematically as:

$$\begin{array}{ccc}
 X = \{ \sigma \mid \sigma \text{ is a map} \} & \xrightarrow{\chi} & Y_\chi \\
 \downarrow d \circ \chi & & \downarrow d \\
 (X, d \circ \chi) & \xrightarrow{I} & (Y_\chi, d) \\
 \downarrow d \circ \chi|_{\sigma_0} & & \\
 \{\text{coordinates}\} \subset \mathbf{R}^+ & &
 \end{array}$$

In the above diagram, we have used the symbol χ to represent the assignment of an output function to a given map. The space Y_χ is the space of output functions (one output function for each astrophysical map in the space X of all maps). We make the space of output functions into a metric space (Y_χ, d) by using a standard metric d (as we discuss below, we generally take d to be the usual L_2 norm). However, we can assign a

† The difference between a metric space and a pseudometric space is that the latter can have two distinct elements with zero distance between them. A distance function that allows two distinct elements to be separated by zero distance is known as a pseudometric. In the study of astrophysical maps, we want to allow two different maps to be “the same” and hence we require a pseudometric space (see Paper I for further discussion).

distance function $d \circ \chi$ (where \circ denotes composition) directly on the original space of maps to produce a pseudometric space $(X, d \circ \chi)$. The two spaces are related through a distance preserving function I known as an isometry. Finally, we assign coordinates to the maps through the operation denoted as $d \circ \chi|_{\sigma_0}$, which measures the distance from the map to the nearest reference map σ_0 (see §2.4). Further details of this procedure are discussed in Paper I. In the following subsections, we describe some of the output functions that can be used for the description and study of astrophysical maps.

2.1 Distributions of Density and Volume

As mentioned above, one approach to characterizing a map to determine how much of the material is at the highest densities [We note that the map need not be a density map – we use this interpretation only as a conceptual guide. In most astrophysical applications, the maps will be of either column density or intensity.]

We can define the fraction of the material at high densities in two different ways. We first determine the fraction m of the mass in the map at densities higher than a given reference Σ :

$$m(\sigma; \Sigma) \equiv \frac{\int d^n \mathbf{x} \sigma(\mathbf{x}) \Theta[\sigma(\mathbf{x}) - \Sigma]}{\int d^n \mathbf{x} \sigma(\mathbf{x})}, \quad (2.1)$$

where Θ is a step function and where the integrals are taken over the (bounded) domain D of the map. Notice that, for a given map σ , m is a function of one variable (namely Σ). We can also define an analogous function $v(\sigma; \Sigma)$ which measures the fraction of the volume (area in a 2-dimensional map) greater than the reference density Σ :

$$v(\sigma; \Sigma) \equiv \frac{\int d^n \mathbf{x} \Theta[\sigma(\mathbf{x}) - \Sigma]}{\int d^n \mathbf{x}}. \quad (2.2)$$

Given these definitions, we can define a distance between two maps by measuring the difference between their corresponding output functions (using either m or v), i.e., we define a pseudometrics d_m and d_v through

$$d_m(\sigma_A, \sigma_B) = \left[\frac{1}{\langle \Sigma \rangle} \int_0^\infty d\Sigma |m(\sigma_A; \Sigma) - m(\sigma_B; \Sigma)|^2 \right]^{1/2}, \quad (2.3)$$

$$d_v(\sigma_A, \sigma_B) = \left[\frac{1}{\langle \Sigma \rangle} \int_0^\infty d\Sigma |v(\sigma_A; \Sigma) - v(\sigma_B; \Sigma)|^2 \right]^{1/2}. \quad (2.4)$$

Notice that in all cases, both $m = 0$ and $v = 0$ at sufficiently large threshold densities Σ ; the integrals in the above equations are thus always convergent. Notice also that we have made the metrics non-dimensional by normalizing the integrals with an appropriate reference density $\langle \Sigma \rangle$.

The output functions $m(\sigma; \Sigma)$ and $v(\sigma; \Sigma)$ have another useful interpretation. Let us define \mathcal{P}_m to be (minus) the derivative of the function m with respect to the variable Σ , i.e.,

$$\mathcal{P}_m(\sigma; \Sigma) = -\frac{dm}{d\Sigma} = \frac{\int d^n \mathbf{x} \sigma(\mathbf{x}) \delta[\sigma(\mathbf{x}) - \Sigma]}{\int d^n \mathbf{x} \sigma(\mathbf{x})}, \quad (2.5)$$

where we have used the fact that the derivative of a step function Θ is a delta function $\delta(\mathbf{x})$. Similarly, we define \mathcal{P}_v via

$$\mathcal{P}_v(\sigma; \Sigma) = -\frac{dv}{d\Sigma} = \frac{\int d^n \mathbf{x} \delta[\sigma(\mathbf{x}) - \Sigma]}{\int d^n \mathbf{x}}. \quad (2.6)$$

The quantity \mathcal{P}_v is the probability (per unit column density) of a point in the map σ having the column density Σ . Similarly, the quantity \mathcal{P}_m is the probability (weighted by the mass) of a point in the map σ having the column density Σ . It is straightforward to show that these probability functions are properly normalized, i.e., $\int \mathcal{P}_v d\Sigma = 1$ and $\int \mathcal{P}_m d\Sigma = 1$. The interpretation of the derivatives of m and v as probability distributions greatly facilitates our understanding of how these output functions behave under various transformations (see below).

2.2 Distribution of Components

We now consider a diagnostic which can discriminate between different geometrical distributions of the high density material. For example, a given map could be composed of N separate “units”, each with mass M and a given distribution of density; alternately, the map could consist of a single unit (with mass NM) with the same distribution of density. Both maps would have the same profile $m(\Sigma)$ as defined above, but would be quite different. One way to discriminate between these two cases is to count the number of topological components as a function of threshold density Σ . To be precise, we first define a reduced space:

$$X_\Sigma^+ \equiv \left\{ \mathbf{x} \in D \mid \sigma(\mathbf{x}) > \Sigma \right\}. \quad (2.7)$$

For a given threshold density, the space (X_Σ^+, d_E) has a well defined number $n(\sigma; \Sigma)$ of topological components (where d_E is the usual Euclidean metric). We can then define a pseudometric d_n on the space X of all maps through

$$d_n(\sigma_A, \sigma_B) = \left[\frac{1}{\langle \Sigma \rangle} \int_0^\infty d\Sigma |n(\sigma_A; \Sigma) - n(\sigma_B; \Sigma)|^2 \right]^{1/2}. \quad (2.8)$$

Since $\sigma(\mathbf{x})$ is always a bounded function $X_\Sigma^+ = \emptyset$ for sufficiently large Σ ; thus, $n = 0$ at sufficiently large values of Σ and the integral is convergent.

Notice that this pseudometric provides us with topological information on two different levels. For a given threshold density Σ , the function $n(\sigma; \Sigma)$ measures a topological property (the number of components) of the *reduced space* X_Σ^+ , which is in turn derived from the original map σ . When combined with our “standard” function space metric d , the component function n defines a pseudometric (namely $d_n = d \circ n$) on the space X of all maps.

2.3 Distribution of Filaments

We also require some description which measures the shapes of the pieces of the map. Given the breakup of a map into components (as described above), we can obtain

a measure of the degree to which the components are filamentary (i.e., stringlike). We begin with the usual definition of the diameter \mathcal{D} of a set A , i.e.,

$$\mathcal{D}(A) \equiv \max\{|\mathbf{x} - \mathbf{y}| \mid \mathbf{x}, \mathbf{y} \in A\}. \quad (2.9)$$

For a given threshold density, an astrophysical map breaks up into components as described in the previous section; each of these components has a well defined diameter. We can also calculate the area \mathcal{A} of a given component. Notice that for a perfectly round (circular) component, the area and the diameter are related by the obvious relation $\mathcal{A} = \pi\mathcal{D}^2/4$. In order to obtain a measure of the departure of a given component from a circular shape, we first define a factor \mathcal{F}_j , which is simply the inverse of the filling factor for a given component, i.e.,

$$\mathcal{F}_j \equiv \frac{\pi\mathcal{D}_j^2}{4\mathcal{A}_j}. \quad (2.10)$$

We denote the quantity \mathcal{F}_j as the “filament index” of the j th component. We also define an average factor f :

$$f(\sigma; \Sigma) = \frac{1}{n(\sigma; \Sigma)} \sum_j w_j \mathcal{F}_j, \quad (2.11)$$

where the sum is taken over all of the components and where f is explicitly written as a function of threshold density Σ . The quantities w_j are weighting values; we consider both an unweighted version of the filament index ($w_j = 1$) and a weighted version in which each \mathcal{F}_j is weighted by the fraction of material in that component (i.e., $w_j \equiv \mathcal{A}_j/\langle A \rangle$, where $\langle A \rangle$ is the average area of the components at the given threshold level). A highly filamentary map will thus have a very large value of f . The pseudometric d_f on the space of maps then can be written

$$d_f(\sigma_A, \sigma_B) = \left[\frac{1}{\langle \Sigma \rangle} \int_0^\infty d\Sigma |f(\sigma_A; \Sigma) - f(\sigma_B; \Sigma)|^2 \right]^{1/2}, \quad (2.12)$$

where f can be either the weighted or unweighted version of the filament function [2.11].

2.4 Assigning Coordinates

In applications, we often require a method of putting astrophysical maps into some kind of well defined order. To order the space of all maps using this formal system, we assign “coordinates” to the maps by measuring the distance from a given map σ to a well-defined reference map σ_0 . The resulting coordinates are thus positive real numbers and the space of maps can be ordered by the size of these numbers. Notice that, in general, there will be different coordinates (and hence a different ordering) for each type of output function considered.

As described in Paper I, we use uniform density maps ($\sigma_0 = \text{constant}$) as reference maps. Notice that there are an infinite number of such reference maps – one for each possible value of the constant. We follow Paper I in defining the coordinate to be the

distance to the *nearest* uniform density map. Specifically, for a given pseudometric d_χ , we define the coordinate η_χ by

$$\eta_\chi \equiv \min \left\{ d_\chi(\sigma, \sigma_0) \mid \sigma_0 \text{ is a uniform density map} \right\}. \quad (2.13)$$

See Paper I for further details on implementing this minimization procedure. We can thus say that map σ_A is “greater than” map σ_B provided that

$$\eta_\chi(\sigma_A) > \eta_\chi(\sigma_B). \quad (2.14)$$

In addition to the coordinates obtained from the output functions described above, we can define other coordinates. For example, in applications (see WA) we sometimes want to consider the “self-gravity” of a map. Thus, for a given map we define the quantity

$$\eta_w \equiv \frac{1}{2} \mathcal{N}_w \int d^n \mathbf{x} \int d^n \mathbf{y} \frac{\sigma(\mathbf{x})\sigma(\mathbf{y})}{|\mathbf{x} - \mathbf{y}|}, \quad (2.15)$$

where \mathbf{x} and \mathbf{y} are (n -dimensional) position vectors and where \mathcal{N}_w is a normalization factor which is chosen to make the coordinate a dimensionless quantity (see Paper I). The integrals are carried out over the entire map. The quantity η_w thus plays the role of an additional coordinate. If the map traces a mass distribution, then this coordinate provides a measure of the self-gravity of the distribution.

3. BEHAVIOR OF THE FORMALISM UNDER TRANSFORMATIONS

In this section, we present formal results which show how the formalism described in the preceding section behaves under various transformations of the maps. In particular, we discuss how the output functions transform under scaling transformations which change the values of the maps, but not the spatial positions of the map pixels in the sky. Next, we study the effects of changing the domains of the maps for a general class of affine transformations (e.g., rotations, translations, stretching of the maps).

3.1 Scaling Transformations

We now discuss what happens to the output functions under various types of scaling transformations. For the sake of definiteness, we begin with the simplest type of scaling transformation, i.e.,

$$\sigma \rightarrow \beta\sigma, \quad (3.1)$$

where β is a constant. We will also consider more general types of rescalings of the maps.

Scaling transformations (e.g., equation [3.1]) can arise in many ways. For example, overall calibration considerations can necessitate an overall scaling transformation of this type. More complicated scaling transformations arise in practice due to calibration problems/uncertainties (see the discussion of WA and Theorem 2 below).

Another example of scaling occurs due to the two dimensional nature of many astrophysical maps (such as the *IRAS* maps considered in our companion paper). These maps only measure the column density $\sigma(\mathbf{x}_2)$, whereas we would like to know the true volume density $\rho(\mathbf{x}_3)$, where we have explicitly used subscripts to refer to the number of spatial dimensions. We are thus limited to considering only the average of the volume density over the line of sight. [†] We can conceptually break up the observed map σ into two parts through the ansatz

$$\sigma(\mathbf{x}_2) = \langle \rho \rangle(\mathbf{x}_2) L(\mathbf{x}_2), \quad (3.2)$$

where $L(\mathbf{x}_2)$ is the depth of the object (along the line of sight) at position \mathbf{x}_2 in the plane of the sky. In many applications, the interpretation of maps is driven by the assumption that the observed maps trace the mean density $\langle \rho \rangle$ and the function $L(\mathbf{x}_2)$ is slowly varying compared to $\langle \rho \rangle$. As we show in WA, this assumption can be justified by the analysis of real data (at least for some cases). In the sample of WA, the observed maps vary by several orders of magnitude in σ . On the other hand, the filament index (which measures the shapes in the plane of the sky – see §2.3) is rarely greater than 3 and is typically ~ 2 . Since we do not expect that the variation in map size along the line of sight is markedly different from the observed variations in the plane of the sky, we expect that the depth function $L(\mathbf{x}_2)$ varies only by factors of 2 or 3 across the face of a typical map. We can thus regard $L(\mathbf{x}_2)$ as a slowly varying function. We can now consider the question: What happens when we want to compare two maps which are essentially the same in all regards except that one map is twice as large as the other? If we can take the extreme limit where $L(\mathbf{x}_2) = L = \text{constant}$ for each map, then we are comparing two maps related by a scaling transformation of the form [3.1] where $\beta = 1/L$.

Thus, it is useful to know how the output functions behave under scaling transformations. For simple transformations of the form of equation [3.1], the answer to this question can be given as follows:

Theorem 1. *Let σ be any map and let $\chi(\sigma; \Sigma)$ be any of the following output functions: distribution of density $m(\sigma; \Sigma)$, volume $v(\sigma; \Sigma)$, components $n(\sigma; \Sigma)$, or filaments $f(\sigma; \Sigma)$. Then the output function scales under the transformation $\sigma \rightarrow \beta\sigma$ according to $\chi(\sigma; \Sigma) \rightarrow \chi(\sigma; \Sigma/\beta)$.*

A proof is given in Appendix A. Theorem 1 provides us with a very useful property of this formal system. For example, if we want to consider an alternate calibration of the original astrophysical maps, we can simply transform the output functions after the fact and need not recalculate them. As discussed above, we can also use Theorem 1 to compare maps of different physical sizes provided that we know the appropriate scaling factor β from independent observations.

[†] We note that the column density is a measure of the average density only when the emissivity of the emitting medium is sufficiently well behaved. In the case of the *IRAS* maps considered in our companion paper (WA), for example, the presence of temperature gradients in the maps can lead to interpretation problems – see Langer et al. (1989).

Given the dependence of the output functions on the scaling transformation [3.1], we can determine the dependence of the coordinates on scaling. This result can be stated as the following corollary, which follows directly from the definition of coordinates and Theorem 1:

Corollary. *Let σ be any map and let $\chi(\sigma; \Sigma)$ be any of the following output functions: $m(\sigma; \Sigma)$, $v(\sigma; \Sigma)$, $n(\sigma; \Sigma)$, or $f(\sigma; \Sigma)$. Then the coordinate η_χ scales under the transformation equation [3.1] according to $\eta_\chi(\beta\sigma) = \sqrt{\beta} \eta_\chi(\sigma)$.*

We now consider a more general type of rescaling law in which the maps are transformed according to

$$\sigma \rightarrow F(\sigma)\sigma, \quad (3.3)$$

where F is a given function of σ . In order for this transformation to be “well behaved”, we must make the further restriction that the quantity $F(\sigma)\sigma$ is a monotonic function of σ . Notice that we have written the transformation in the form $F(\sigma)\sigma$ rather than simply $G(\sigma)$ for convenience; in the transformations of this type encountered for *IRAS* maps (see WA; Jarrett, Dickman, & Herbst 1989), $F = 1$ for small values of σ and only departs significantly from unity at large map values.

We now discuss how the output functions behave under this transformation. Suppose, for example, that we have already computed output functions for a sample of maps. We may want to consider an alternate calibration procedure (which will result in a transformation of the maps according to equation [3.3]). Since the calculation of the output functions is computationally expensive, we would like to be able to simply transform the output functions after the fact.

We first note that, for a given threshold level Σ , the equation

$$F(\sigma)\sigma = \Sigma \quad (3.4)$$

has (at most) one root because of our assumption that the left hand side is monotonic in σ . We denote this root by $\sigma = \tilde{\Sigma}$, which is simply a rescaling of the threshold level under the transformation. Given this definition, we can determine the manner in which the output functions transform under equation [3.3]. We state this result as the following theorem:

Theorem 2. *Under transformations of the form given in equation [3.3], the output functions transform according to $\chi(\sigma; \Sigma) \rightarrow \chi(\sigma; \tilde{\Sigma})$, where χ represents any of the four output functions $m(\sigma; \tilde{\Sigma})$, $v(\sigma; \tilde{\Sigma})$, $n(\sigma; \tilde{\Sigma})$, or $f(\sigma; \tilde{\Sigma})$, and where $\tilde{\Sigma}$ is the root of equation [3.4].*

A proof is given in Appendix B. Theorem 2 shows that even under complicated rescalings of the maps (such rescalings can easily arise in practice due to calibration problems), the output functions transform in a simple manner.

3.2 Transformations of the Domain

In this section, we study the dependence of the output functions under transformations which change the domain of the maps. We concentrate on a general class of

transformations known as affine transformations and briefly consider the effects of resolution transformations.

We begin with the general definition of affine transformations:

Definition. An affine transformation W in two dimensions is a transformation of the form

$$W \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}. \quad (3.5)$$

We denote the matrix in the above definition as A and we assume that this matrix is non-singular, i.e., $\det A \neq 0$. We are also interested in a particular type of affine transformation known as a similitude.

Definition. A similitude is a special type of affine transformation in which the matrix A takes the form

$$A = \begin{bmatrix} r \cos \theta & -r \sin \theta \\ r \sin \theta & r \cos \theta \end{bmatrix} \quad \text{or} \quad A = \begin{bmatrix} r \cos \theta & r \sin \theta \\ r \sin \theta & -r \cos \theta \end{bmatrix}. \quad (3.6)$$

Thus, a similitude corresponds to a rotation (through angle θ), a scaling of the domain (by the factor r), a translation (by $[b_1, b_2]$), and a possible reflection. Affine transformations are more general and include the possibility of stretching the \hat{x} direction and the \hat{y} direction by different amounts (see, e.g., Barnsley 1988 for further discussion of affine transformations).

For astrophysical maps σ , we think of affine transformations as changing the domain D of the map. For example, we generally do not know precisely the distance to a given object. If we want to conceptually “change the distance” to an object, the corresponding change in the map is a similitude (where r is the ratio of the new distance to the old distance, $\theta = 0$, $b_1 = 0 = b_2$). In general, an affine transformation acts on an astrophysical map according to

$$\begin{bmatrix} x \\ y \\ z = \sigma(x, y) \end{bmatrix} \rightarrow \begin{bmatrix} W \begin{bmatrix} x \\ y \end{bmatrix} \\ z = \sigma(x, y) \end{bmatrix}. \quad (3.7)$$

Given these definitions we can state the main result of this section:

Theorem 3. *The output functions $m(\sigma; \Sigma)$, $v(\sigma; \Sigma)$, and $n(\sigma; \Sigma)$ are invariant under general affine transformations, whereas the distribution of filaments $f(\sigma; \Sigma)$ is invariant under similitudes.*

Although the behavior of the output functions under affine transformations is relatively easy to understand, it is somewhat more difficult to actually prove what happens. In any case, the proof is presented in Appendix C. Theorem 3 shows that the output functions transform in a fairly simple manner under transformations which are often encountered

in astrophysics; this result thus provides another justification for using the formalism of Paper I.

3.3 Resolution Transformations

In this section, we briefly discuss the effects of resolution on the maps. We begin by defining a *resolution transformation* \mathcal{R} according to

$$\mathcal{R}[\sigma(\mathbf{x})] = \mathcal{N} \int d^2\mathbf{x}' \exp[-(\mathbf{x}' - \mathbf{x})^2/L^2] \sigma(\mathbf{x}'), \quad (3.8)$$

where \mathcal{N} is a normalization factor for the transformation. If we normalize the transformation such that $\mathcal{R}(1) = 1$, we obtain the normalization factor $\mathcal{N} = 1/\pi L^2$. Notice that we have explicitly written the integral in two spatial dimensions, since the transformation occurs in practice due to telescope beams averaging signals within the plane of the sky. Here, L is the physical length scale corresponding to the given beam size of the telescope. Let us also introduce a constant physical length scale R , which corresponds to the length scale on which the true map σ varies. We then define

$$\lambda \equiv 2R^2/L^2. \quad (3.9)$$

It is easy to see that in the limit $\lambda \ll 1$, the integral in equation [3.8] averages out much of the structure in the map and we are left with the age-old astronomical problem of loss of information. In the opposite limit, $\lambda \gg 1$, we can perform an asymptotic expansion (Appendix D) of the transformation in order to quantify this loss of information. In this limit, the leading order change in the map due to a resolution transformation is given by

$$\mathcal{R}[\sigma] = \sigma + \frac{1}{2\lambda} \Delta_2 \sigma + \mathcal{O}(\lambda^{-2}), \quad (3.10)$$

where Δ_2 is the two-dimensional Laplacian operator in dimensionless units (see equation [D5]). Thus, we see explicitly the expected result that, under a resolution transformation, the peaks of the map become less pronounced ($\Delta_2 \sigma < 0$) whereas the low points (valleys) of the map become filled in ($\Delta_2 \sigma > 0$). Thus, the resolution transformation [3.8] leads to a loss of dynamic range in the map and the size of this effect is $\sim \langle \Delta_2 \sigma \rangle / \lambda$.

4. PROPAGATION OF ERRORS

An important issue which must be addressed is the effect of observational errors on the classification scheme outlined above. In particular, we must determine the effects of observational errors on the output functions and on the corresponding coordinates. For this present discussion we assume that the observed value σ_{obs} of the map at a particular spatial point and the uncertainty $\Delta\sigma$ in the map at that point can be written

$$\sigma_{\text{obs}} = \sigma_R + \Delta\sigma, \quad (4.1)$$

where σ_R is the true value of the map at that point. The uncertainty $\Delta\sigma$ is usually considered to have a Gaussian distribution.

The goal of this section is to show that the output functions used here to describe astrophysical maps are sufficiently “well behaved” in the presence of uncertainties in the maps. In particular, we want to show that small uncertainties $\Delta\sigma$ in the original maps do not lead to large variations in the resulting output functions. Fortunately, by working within the formal system of Paper I, we can provide a mathematically precise answer to this question (see Appendices E, F, and G). As we discuss below, the net result of this analysis is that observational uncertainties can be well controlled within this formalism provided that the signal to noise ratios in the original maps are sufficiently high.

For the distribution of density $m(\sigma; \Sigma)$, it is fairly straightforward to determine the relative error $\Delta m/m$ in the output function in terms of the uncertainty in the map; we find

$$\frac{\Delta m}{m} = \frac{1}{m} \left| \frac{dm}{d\Sigma} \right|_{\Sigma} \Delta\sigma, \quad (4.2)$$

where we have used the results of Appendix E. The errors are thus well controlled: small uncertainties $\Delta\sigma$ in the maps lead to small relative uncertainties $\Delta m/m$ in the output functions. In principle, the slope $dm/d\Sigma$ can be quite large at a particular value of Σ and the corresponding error could be large; in practice, however, the right hand side of equation [4.1] does not become overly large (see WA). Notice that we can average equation [4.2] over all threshold densities Σ and thus obtain the average relative error in the output function; we find

$$\left\langle \frac{\Delta m}{m} \right\rangle = \frac{\log(SNR)}{SNR}, \quad (4.3)$$

where SNR denotes the signal to noise ratio (which we have assumed to be uniform over the map). We thus conclude that the distribution of density is sufficiently well behaved provided that the signal to noise ratio in the map is sufficiently large. The distribution of volume $v(\sigma; \Sigma)$ can be treated similarly.

We now consider possible errors in the distribution of components $n(\sigma; \Sigma)$. This present discussion is complicated by the fact that relatively small errors in the map can, in principle, produce rather large errors in the number of components. For example, the distribution of components $n(\sigma; \Sigma)$ can have extra components due to erroneous pixels sticking up above the threshold level. Notice, however, that the opposite does not generally occur – we seldom get extraneous components due to a pixel being too low. A pixel with an erroneously low value will show up as a “hole” in a component but will not, in general, change the number of components.

We must estimate the probability P_1 that a pixel in the map will be erroneously larger than a given threshold level. This estimate is given in Appendix F and shows that the probability P_1 is bounded to be quite small; very roughly we obtain P_1 in the range 10^{-2} to 10^{-3} . Thus, in a map containing $N \times N$ pixels, the number of pixels erroneously appearing larger than the threshold Σ is bounded to be less than $N^2 P_1$. For the representative values $N = 400$ and $P_1 = 10^{-2}$ (these values are appropriate for the maps considered in WA), we have $N^2 P_1 = 1600$. Although most of these erroneous pixels are not expected to produce erroneous components, this number is uncomfortably large.

In order to control the number of erroneous components, we should eliminate all single-pixel components from the function $n(\sigma; \Sigma)$. The probability that two adjacent pixels are both erroneously above a threshold is proportional to P_1^2 and the number of (possible) erroneous components is bounded by $2N^2 P_1^2 \sim 32$ (for the same values of N and P_1 as before). Although the number of spurious components with two pixels is bounded to be fairly small, we should adopt the conservative approach and also eliminate all two-pixel components from consideration. The smallest components that are kept in the function $n(\sigma; \Sigma)$ are those with three pixels and the probability of obtaining a spurious component is proportional to $P_1^3 \sim 10^{-6}$ and the number of spurious components is expected to be less than $\sim 6N^2 P_1^3 \sim 0.96 < 1$. We therefore conclude that the errors in the distribution of components can be well controlled, provided that we eliminate from consideration all components with only one or two pixels. We note that in practice the values of N and P_1 may differ from those taken here for purposes of illustration, although we have been reasonably conservative here. The exercise outlined above should be performed for each application of the formalism and, in some cases, components with three (or even more) pixels might need to be removed in order to control errors in the distribution of components function $n(\sigma; \Sigma)$.

Another issue which must be mentioned is the choice of the step size of the threshold level Σ used in computing the distribution of components output function. This step size should be related to the noise levels in the maps used. If the step size is much smaller than the RMS noise of the map, then extraneous noise can appear in the output function itself. On the other hand, if the step size is much larger than the RMS noise level, then information will be lost and we will lose some ability to detect structure.

For the distribution of filaments $f(\sigma; \Sigma)$, the relative error is approximately given by

$$\frac{\Delta f}{f_R} = \frac{\Delta n}{n_R} \left\{ 1 - \frac{f_S}{f_R} \right\}, \quad (4.4)$$

where f_S is the average filament index of the spurious components and f_R is the average filament index of the real components (see Appendix G). Here, $\Delta n/n_R$ is the relative error in the distribution of components function. The quantity in brackets in equation [4.4] is expected to be of order unity or smaller (if $f_S \approx f_R$, then the errors nearly cancel). The quantity f_S represents the average shape (filament index) of the spurious components. Since these components are made up of pixels which are erroneously on the wrong side of the threshold level, and since the probability of erroneous pixels occurring is small, these components are generally made up of a small number of pixels. The possible shapes of such components can be computed and the average filament index for such shapes is not far from unity. Thus, the relative error in the distribution of filaments is well controlled provided that $\Delta n/n_R$ is small, i.e., provided that the number of spurious components is small compared to the number of true components. As we have argued above, however, the number of spurious components can be controlled by considering only those components with three or more pixels; thus, Δn can be made small and hence Δf can also be controlled.

Although we have shown that uncertainties $\Delta\sigma$ in the maps do not produce unacceptably large uncertainties in the output functions themselves, we still must show that

the corresponding uncertainties in the coordinates are sufficiently well behaved. Fortunately, Theorem 3 of Paper I shows that when two maps σ_A and σ_B are close together in the space of maps,

$$d_\chi(\sigma_A, \sigma_B) < \epsilon, \quad (4.5)$$

then the corresponding coordinates (η_A and η_B) are close together in the space of real numbers, i.e.,

$$|\eta_A - \eta_B| < \epsilon. \quad (4.6)$$

This result follows directly from the fact that we have built this formalism using pseudometrics (which are *uniformly continuous* when considered as functions of the maps – see Paper I; Copson 1968). Here, we let $\sigma_A = \sigma_{\text{obs}}$ denote the observed map, and we let $\sigma_B = \sigma_R$ denote the true map, i.e., the map we would have if there were no observational uncertainties (see equation [4.1]). For any given type of output function χ , the results of the previous analysis show that the difference between the output function χ_{obs} of the observed map and the output function χ_R of the true map is small, i.e.,

$$\chi_{\text{obs}} = \chi_R + \Delta\chi, \quad (4.7)$$

where $\Delta\chi$ is small compared to χ_R . Using the definitions of the pseudometrics (see equations [2.3], [2.4], [2.8], and [2.12]), we find that the difference between the true map and the observed map is given by

$$d_\chi(\sigma_R, \sigma_{\text{obs}}) = \left[\frac{1}{\langle \Sigma \rangle} \int_0^\infty d\Sigma |\Delta\chi|^2 \right]^{1/2} = \mathcal{O}(|\Delta\chi|). \quad (4.8)$$

Using equation [4.8] and Theorem 3 of Paper I, we find that the coordinates of the true and the observed maps are also “close together” in the sense that

$$|\eta_R - \eta_{\text{obs}}| \leq \mathcal{O}(|\Delta\chi|). \quad (4.9)$$

We thus conclude that the uncertainties in the coordinates are sufficiently well behaved in this formalism.

5. SUMMARY AND DISCUSSION

In this paper, we have extended the formalism for using metric space techniques to study astrophysical maps. In particular, we have proved a number of useful results concerning this method of form description. These results are summarized below.

Our first result applies to scaling of the output functions due to the scaling of the original maps. Such scalings can arise from calibration considerations. In addition, if two maps are similar except for their size, then the corresponding column density maps should show similar structure except for an overall scaling factor β . If we apply a scaling transformation $\sigma \rightarrow \beta\sigma$ to a given map, the output functions transform in a relatively simple manner, i.e., $\chi(\sigma; \Sigma) \rightarrow \chi(\beta\sigma; \Sigma) = \chi(\sigma; \Sigma/\beta)$. This transformation thus allows us

to compare maps which are known to have different sizes. We have also shown that under more general transformations of the form $\sigma \rightarrow F(\sigma)\sigma$, the output functions transform according to $\chi(\sigma; \Sigma) \rightarrow \chi(\sigma; \tilde{\Sigma})$, where $\tilde{\Sigma}$ is the root of equation [3.4].

Our next result shows how the output functions are affected by a general class of transformations of the domain of the maps. This class of affine transformations includes rotations, translations, and an overall stretching of the domain. The output functions for the distributions of density, volume, and components are all invariant under general affine transformations. The distribution of filaments is invariant under a restricted class of transformations known as similitudes, and is not invariant under general affine transformations.

We have found one additional property of this formal system. For any given astrophysical map, the distribution of density output function is always greater than the distribution of volume (we prove this result in Appendix H). This property of the output functions is evident for the maps considered in our companion paper (WA).

We have discussed the propagation of errors in the original maps into the corresponding errors in the output functions. The distribution of density function is well behaved in the sense that small map errors lead to small errors in the output function. The distribution of volume function behaves similarly. The distribution of filaments is well behaved provided that the number of components can be well determined. On the other hand, the distribution of components is potentially more problematic. The function $n(\sigma; \Sigma)$ can have considerable errors due to single erroneous pixels sticking up above the threshold level. However, we can estimate the probability of a spurious component arising with k pixels. This probability is $\propto P_1^k$ (where P_1 is expected to be a small number – see Appendix F) and thus the probability of obtaining a spurious component with a large number k of pixels is quite small. In practice, we expect that by removing from consideration all components with only one or two pixels, we can adequately control the errors in the distribution of components $n(\sigma; \Sigma)$. Finally, we have shown that uncertainties in the coordinates are small as long as the corresponding uncertainties in the output functions are small.

This present paper (in conjunction with Paper I) provides a well-defined formal system for the quantitative description of astrophysical maps and other similar data structures. This formal system can be directly applied to many types of existing astrophysical maps (see, e.g., WA for an application to continuum maps of molecular clouds). In addition, this method of form description can have applications outside the field of astrophysics. For example, studies of computer vision and artificial intelligence require the construction of explicit and meaningful descriptions of physical objects from images (see, e.g., Ballard & Brown 1982).

In addition to direct applications of this formal system, many possibilities for future developmental work remain. So far, we have only used the most basic physical quantities (e.g., fraction of high density material by mass and by volume, number of pieces, and shape of the pieces) as the basis for our output functions. As the study of astrophysical form description becomes better understood, additional output functions should be developed to provide a better description of these maps. These additional output functions

should be constructed so that they are either invariant under the transformations discussed in §3 or they transform in a simple manner. These new output functions should also be “well behaved” in terms of their sensitivity to observational uncertainties in the maps.

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APPENDIX A: PROOF OF THEOREM 1

In this Appendix, we present a proof of Theorem 1, which states that the output functions scale under the transformation $\sigma \rightarrow \beta\sigma$ according to $\chi(\sigma; \Sigma) \rightarrow \chi(\sigma; \Sigma/\beta)$, where $\chi(\sigma; \Sigma)$ represents any of the following output functions: distribution of density $m(\sigma; \Sigma)$, volume $v(\sigma; \Sigma)$, components $n(\sigma; \Sigma)$, or filaments $f(\sigma; \Sigma)$.

Proof: For the distribution of density $m(\sigma; \Sigma)$, we begin with the definition of m , i.e.,

$$m(\sigma; \Sigma) = \frac{\int d^n \mathbf{x} \sigma(\mathbf{x}) \Theta[\sigma(\mathbf{x}) - \Sigma]}{\int d^n \mathbf{x} \sigma(\mathbf{x})}. \quad (\text{A1})$$

Under the transformation [3.1] we obtain

$$m(\beta\sigma; \Sigma) = \frac{\int d^n \mathbf{x} \sigma(\mathbf{x}) \Theta[\beta\sigma(\mathbf{x}) - \Sigma]}{\int d^n \mathbf{x} \sigma(\mathbf{x})}, \quad (\text{A2})$$

where we have canceled the constants β in the numerator and the denominator. If we then invoke the identity for step functions,

$$\Theta[ax - c] = \Theta[x - c/a], \quad (\text{A3})$$

where a and c are constants, we obtain the scaling relation

$$m(\beta\sigma; \Sigma) = m(\sigma; \Sigma/\beta). \quad (\text{A4})$$

Similarly, using the definition [2.2] for the distribution of volume and the scaling transformation [3.1], we can write

$$v(\beta\sigma; \Sigma) = \frac{\int d^n \mathbf{x} \Theta[\beta\sigma(\mathbf{x}) - \Sigma]}{\int d^n \mathbf{x}} = v(\sigma; \Sigma/\beta), \quad (\text{A5})$$

where the second equality follows from applying the identity [A3] for the step function.

Since the distributions of components $n(\sigma; \Sigma)$ and filaments $f(\sigma; \Sigma)$ measure properties of the reduced space X_Σ^+ , we must begin with its definition

$$X_\Sigma^+ \equiv \left\{ \mathbf{x} \in D \mid \sigma(\mathbf{x}) > \Sigma \right\}. \quad (\text{A6})$$

Under the transformation $\sigma \rightarrow \beta\sigma$, the reduced space becomes

$$\begin{aligned} X_\Sigma^+ &\rightarrow \left\{ \mathbf{x} \in D \mid \beta\sigma(\mathbf{x}) > \Sigma \right\} \\ &= \left\{ \mathbf{x} \in D \mid \sigma(\mathbf{x}) > \Sigma/\beta \right\}. \end{aligned} \quad (\text{A7})$$

We thus have the result

$$X_\Sigma^+(\beta\sigma) = X_{\Sigma/\beta}^+(\sigma). \quad (\text{A8})$$

Since both output functions $n(\sigma; \Sigma)$ and $f(\sigma; \Sigma)$ depend only on the properties of the reduced space, the relation [A8] implies that

$$n(\beta\sigma; \Sigma) = n(\sigma; \Sigma/\beta), \quad (\text{A9})$$

$$f(\beta\sigma; \Sigma) = f(\sigma; \Sigma/\beta), \quad (\text{A10})$$

which concludes the proof. •

APPENDIX B: PROOF OF THEOREM 2

In this Appendix, we present a proof of Theorem 2, which describes how the output functions transform when the maps are rescaled as in the nonlinear transformation of equation [3.3].

Proof: To consider the effect of the transformation on the distribution of density function $m(\sigma; \Sigma)$, we use the interpretation that $-dm/d\Sigma$ is the probability that the map has a value Σ . Thus, the probability that the transformed map has a value of some threshold Σ is the probability that

$$F(\sigma)\sigma = \Sigma, \quad (\text{B1})$$

which is simply the probability that $\sigma = \tilde{\Sigma}$, which, in turn, is given by

$$-\frac{dm}{d\Sigma}(\tilde{\Sigma}). \quad (\text{B2})$$

The transformed output function m_T can then be written in terms of the untransformed output function through

$$m_T = - \int_{\Sigma}^{\infty} d\Sigma \frac{dm}{d\Sigma}(\tilde{\Sigma}). \quad (\text{B3})$$

If we evaluate the integral and use the fact that m vanishes as $\Sigma \rightarrow \infty$, then we find

$$m_T(\sigma; \Sigma) = m(\sigma; \tilde{\Sigma}). \quad (\text{B4})$$

Using a similar argument, we can write the transformed output function v_T for the distribution of volume in terms of the original untransformed one via

$$v_T(\sigma; \Sigma) = v(\sigma; \tilde{\Sigma}). \quad (\text{B5})$$

Since the distributions of components $n(\sigma; \Sigma)$ and filaments $f(\sigma; \Sigma)$ measure properties of the reduced space X_Σ^+ , we must consider how the reduced space transforms under equation [3.3]. This argument thus parallels that of Appendix A. Under the transformation of equation [3.3], the reduced space becomes

$$\begin{aligned} X_\Sigma^+ &\rightarrow \left\{ \mathbf{x} \in D \mid F(\sigma)\sigma > \Sigma \right\} \\ &= \left\{ \mathbf{x} \in D \mid \sigma(\mathbf{x}) > \tilde{\Sigma} \right\}, \end{aligned} \quad (\text{B6})$$

where we have used the definition of $\tilde{\Sigma}$. We thus have the result

$$X_\Sigma^+(F(\sigma)\sigma) = X_{\tilde{\Sigma}}^+(\sigma). \quad (\text{B7})$$

Since both output functions $n(\sigma; \Sigma)$ and $f(\sigma; \Sigma)$ depend only on the properties of the reduced space, the relation [B7] implies that

$$n(F(\sigma)\sigma; \Sigma) = n(\sigma; \tilde{\Sigma}), \quad (\text{B8})$$

$$f(F(\sigma)\sigma; \Sigma) = f(\sigma; \tilde{\Sigma}), \quad (\text{B9})$$

which concludes the proof. •

APPENDIX C: PROOF OF THEOREM 3

In this Appendix, we present a proof of Theorem 3, which states that the output functions $m(\sigma; \Sigma)$, $v(\sigma; \Sigma)$, and $n(\sigma; \Sigma)$ are invariant under general affine transformations, whereas the distribution of filaments $f(\sigma; \Sigma)$ is invariant under similitudes.

Proof: We begin by noting that the integral of a function F transforms under an affine transformation according to

$$\int dx dx F(x, y) \rightarrow |\det A| \int dx dy F(x, y), \quad (\text{C1})$$

where A is the matrix of the transformation and where F is constant under the transformation (see, e.g., Barnsley 1988).

Thus, under an affine transformation, the integrals appearing in the definitions of the distributions of density and volume transform according to

$$\int d^n \mathbf{x} \sigma(\mathbf{x}) \Theta[\sigma(\mathbf{x}) - \Sigma] \rightarrow |\det A| \int d^n \mathbf{x} \sigma(\mathbf{x}) \Theta[\sigma(\mathbf{x}) - \Sigma] \quad (\text{C2})$$

$$\int d^n \mathbf{x} \sigma(\mathbf{x}) \rightarrow |\det A| \int d^n \mathbf{x} \sigma(\mathbf{x}), \quad (\text{C3})$$

$$\int d^n \mathbf{x} \Theta[\sigma(\mathbf{x}) - \Sigma] \rightarrow |\det A| \int d^n \mathbf{x} \Theta[\sigma(\mathbf{x}) - \Sigma] \quad (\text{C4})$$

$$\int d^n \mathbf{x} \rightarrow |\det A| \int d^n \mathbf{x}. \quad (\text{C5})$$

Since the distribution of density $m(\sigma; \Sigma)$ is the ratio of the integral appearing in equation [C2] to that in equation [C3], $m(\sigma; \Sigma)$ is invariant under the transformation as long as $|\det A| \neq 0$. Similarly, equations [C4 – C5] imply that the distribution of volume $v(\sigma; \Sigma)$ is also invariant under affine transformations.

To study the effects of affine transformations on the distributions of components $n(\sigma; \Sigma)$ and filaments $f(\sigma; \Sigma)$, we must first consider the effects of the transformation on the reduced space defined by equation [2.7], i.e., we consider the transformed space

$$w(X_\Sigma^+) = \left\{ w(\mathbf{x}) \mid \mathbf{x} \in D, \sigma(\mathbf{x}) > \Sigma \right\}. \quad (\text{C6})$$

If we write the original reduced space as a union of its components

$$X_\Sigma^+ = \bigcup_{j=1}^{n(\sigma; \Sigma)} U_j, \quad (\text{C7})$$

where each U_j is a component, then we can write the transformed space as

$$w(X_\Sigma^+) = \bigcup_{j=1}^{n(\sigma; \Sigma)} w(U_j). \quad (\text{C8})$$

To show that the output function $n(\sigma; \Sigma)$ is invariant under the transformation, we must show that the transformed sets $w(U_j)$ are in fact the components of the transformed space. In order for this statement to be true, we must have:

$$w(U_j) \text{ connected } \forall j, \quad (\text{C9})$$

$$w(U_j) \cap w(U_k) = \emptyset \quad \forall j \neq k. \quad (\text{C10})$$

When the affine transformation w is considered only on a set U_j , i.e.,

$$w : U_j \rightarrow w(U_j), \quad (\text{C11})$$

then w is a homeomorphism. Hence, the first required condition [C9] is automatically satisfied (see, e.g., Kelley 1955; Conover 1975). We now show that the second condition [C10] is also satisfied: Suppose [C10] were false. Then, by supposition, an element \mathbf{x} exists such that

$$\mathbf{x} \in w(U_j) \cap w(U_k) \quad (\text{C12})$$

for some $j \neq k$. Then, because w is a homeomorphism and thus has a unique inverse,

$$w^{-1}(\mathbf{x}) \equiv \mathbf{a} \in U_j, \quad (\text{C13})$$

and also

$$\mathbf{a} \in U_k. \quad (\text{C14})$$

The intersection $U_j \cap U_k$ is thus nonempty. This result contradicts the fact that U_j and U_k are components. Thus, by contradiction, condition [C10] must hold. As a result, the distribution $n(\sigma; \Sigma)$ of components must be invariant under affine transformations.

We now consider how the distribution of filaments $f(\sigma; \Sigma)$ transforms under an affine transformation. Given the definitions [2.9 – 2.11], and given that the distribution of components $n(\sigma; \Sigma)$ is invariant under the transformation, we must calculate how the diameter \mathcal{D} and the area \mathcal{A} of an open set U_j transform. The area \mathcal{A} is simply the integral over the set and the transformation is given by equation [C5] above, i.e.,

$$\mathcal{A}[w(U_j)] = |\det A| \mathcal{A}(U_j), \quad (\text{C15})$$

where w is the affine transformation. Under this same transformation, the diameter of a set becomes

$$\mathcal{D}[w(U_j)] = \max\{|\mathbf{x} - \mathbf{y}| \mid \mathbf{x}, \mathbf{y} \in w(U_j)\}. \quad (\text{C16})$$

Let $\mathbf{d} = \mathbf{x} - \mathbf{y}$ where $\mathbf{x}, \mathbf{y} \in w(U_j)$. Then

$$\mathbf{d} = w(\mathbf{x}_0) - w(\mathbf{y}_0) = A(\mathbf{x}_0 - \mathbf{y}_0), \quad (\text{C17})$$

where $\mathbf{x}_0, \mathbf{y}_0 \in U_j$ and where A is the matrix of the affine transformation. If the transformation w is a similitude (see equation [3.6]), then it follows that

$$|\mathbf{d}|^2 = r^2 |\mathbf{x}_0 - \mathbf{y}_0|^2, \quad (\text{C18})$$

where r is the scaling factor of the transformation. Since equation [C18] holds for all pairs of points, it follows that

$$\mathcal{D}^2[w(U_j)] = r^2 \mathcal{D}^2(U_j) = |\det A| \mathcal{D}^2(U_j), \quad (\text{C19})$$

a result which holds for all similitudes. Taken together, equations [C15] and [C19] imply that the factor $\mathcal{F}_j = \pi \mathcal{D}_j^2 / 4\mathcal{A}$ (see equation [2.10]) is invariant under similitudes. This result, in conjunction with the invariance of $n(\sigma; \Sigma)$, implies that the distribution $f(\sigma; \Sigma)$ of filaments (see equation [2.11]) is also invariant under similitudes. •

We should remark that the distribution of filaments is *not* invariant under a general affine transformation (as in equation [3.5]). This claim is straightforward to verify and will not be discussed here.

APPENDIX D: RESOLUTION TRANSFORMATIONS

In this Appendix we derive an asymptotic expansion for the resolution transformation \mathcal{R} given by

$$\mathcal{R}[\sigma(\mathbf{x})] = \mathcal{N} \int d^2 \mathbf{x}' \exp[-(\mathbf{x}' - \mathbf{x})^2/L^2] \sigma(\mathbf{x}'), \quad (\text{D1})$$

where all of the quantities are the same as those defined in the text. With the definitions

$$\begin{aligned} \xi &= (x' - x)/R, \\ \eta &= (y' - y)/R, \end{aligned} \quad (\text{D2})$$

we can write the resolution transformation in the form

$$\mathcal{R}[\sigma] = \mathcal{N} R^2 \int d\xi d\eta \exp[-\frac{1}{2} \lambda (\xi^2 + \eta^2)] \sigma(\xi, \eta). \quad (\text{D3})$$

Integrals of the above form can be written in terms of an asymptotic expansion when the parameter $\lambda \gg 1$ (see Bleistein & Handelsman 1986; see also Adams 1991). After a considerable amount of algebra, we obtain the expansion

$$\mathcal{R}[\sigma(x, y)] = \sum_{j=0}^{N-1} \frac{1}{(2\lambda)^j j!} \Delta_2^j \sigma \Big|_{(x, y)} + \mathcal{O}(\lambda^{-N}), \quad (\text{D4})$$

where Δ_2 is the two dimensional Laplacian operator, i.e.,

$$\Delta_2 \equiv \frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \eta^2}. \quad (\text{D5})$$

Notice that we have not written the expansion in equation [D4] as an infinite series, but rather as the $N - 1^{\text{th}}$ partial sum with an error term $\mathcal{O}(\lambda^{-N})$. An infinite series is meaningless in this context because the series need not converge (this property is a well known aspect of asymptotic analysis – see, e.g., Bleistein & Handelsman 1986).

APPENDIX E: ERROR ANALYSIS FOR THE DISTRIBUTION OF DENSITY

In this Appendix we estimate how uncertainties in an observed astrophysical map will affect the output function $m(\sigma; \Sigma)$ for the distribution of density. We also discuss how these uncertainties affect the determination of the corresponding coordinate η_m . Using the decomposition [4.1], the function (denoted here as σ_m) which appears in the numerator in equation [2.1] becomes

$$\sigma_m = (\sigma_R + \Delta\sigma)\Theta[\sigma_R + \Delta\sigma - \Sigma], \quad (\text{E1})$$

and the integrals appearing in the definition of the mass fraction m become

$$\int d^n \mathbf{x} \sigma_m = \int d^n \mathbf{x} \sigma_R \Theta[\sigma_R + \Delta\sigma - \Sigma] + \int d^n \mathbf{x} \Delta\sigma \Theta[\sigma_R + \Delta\sigma - \Sigma], \quad (\text{E2})$$

$$\int d^n \mathbf{x} \sigma = \int d^n \mathbf{x} \sigma_R + \int d^n \mathbf{x} \Delta\sigma. \quad (\text{E3})$$

Since the errors $\Delta\sigma$ are randomly distributed (and hence are equally likely to be positive or negative), the integrals of $\Delta\sigma$ tend toward zero. Notice that the step function introduces a different measure of integration, but does not otherwise change this result.[†]

Thus, the distribution of density m_{obs} that is calculated from the observed map σ_{obs} is related to the true distribution m_R through

$$m_{\text{obs}}(\sigma; \Sigma) = m_R(\sigma; \tilde{\Sigma}), \quad (\text{E4})$$

where we have defined

$$\tilde{\Sigma} \equiv \Sigma - \Delta\sigma. \quad (\text{E5})$$

In other words, the net effect of the uncertainty in the observed map is to introduce an uncertainty in the variable Σ of the output function.

Using equations [E4] and [E5], we can calculate the associated error in the output function $m(\Sigma)$ for a give map through a Taylor series expansion:

$$m_R(\Sigma) = m_{\text{obs}}(\tilde{\Sigma}) = m_{\text{obs}}(\Sigma) + \left. \frac{dm}{d\Sigma} \right|_{\Sigma} \Delta\sigma + \mathcal{O}[(\Delta\sigma)^2]. \quad (\text{E6})$$

To leading order the error in the output function is then approximately given by

$$\text{error} = \Delta m \sim \left. \frac{dm}{d\Sigma} \right|_{\Sigma} \Delta\sigma, \quad (\text{E7})$$

[†] This statement is only correct to first order in $\Delta\sigma$. Since the $\Delta\sigma$ integral in equation [E2] contains a step function which includes the error contribution $\Delta\sigma$, noise in the map will push more pixels above the threshold Σ than below. However, this effect is second order in $\Delta\sigma$. To show this, we expand the step function in the integral in equation [E2] to obtain $\Theta[\sigma_R + \Delta\sigma - \Sigma] = \Theta[\sigma_R - \Sigma] - \Delta\sigma \delta[\sigma_R - \Sigma]$. Using this expansion in the original integral, we see that a $\Delta\sigma^2$ term appears. This term is positive definite and hence does not integrate to zero; however, the term is second order and can be ignored.

which is first order in $\Delta\sigma$ and therefore small if the signal to noise ratio in the original map is sufficiently high.

Given the expansion [E6] for the error in the output function, we now calculate the corresponding uncertainty in the coordinate η_m . Using the expansion [E6] in the definition of the coordinate (see Paper I), we can show that

$$\langle \Sigma \rangle (\eta_{mR}^2 - \eta_{m\text{obs}}^2) = - \int_0^{\sigma_C} 2(1 - m_{\text{obs}}) \frac{dm}{d\Sigma} \Delta\sigma d\Sigma + \int_{\sigma_C}^{\infty} 2m_{\text{obs}} \frac{dm}{d\Sigma} \Delta\sigma d\Sigma. \quad (\text{E8})$$

Since the uncertainty $\Delta\sigma$ can be either positive or negative, the integrals in equation [E8] tend to cancel out. If, however, we consider the uncertainty to be bounded $|\Delta\sigma| \leq \epsilon$, we can place corresponding bounds on the integrals:

$$\left| \int_0^{\sigma_C} 2(1 - m_{\text{obs}}) \frac{dm}{d\Sigma} \Delta\sigma d\Sigma \right| \leq \epsilon/4, \quad (\text{E9a})$$

$$\left| \int_{\sigma_C}^{\infty} 2m_{\text{obs}} \frac{dm}{d\Sigma} \Delta\sigma d\Sigma \right| \leq \epsilon/4, \quad (\text{E9b})$$

where we have used the fact that $m(0) = 1$, $m(\sigma_C) = 1/2$, and $m(\infty) = 0$. Combining all of the above results shows that

$$\Delta\eta_m \leq \frac{\epsilon}{4\eta_m \langle \Sigma \rangle} + \mathcal{O}(\epsilon^2) + \mathcal{O}(\Delta\eta^2). \quad (\text{E10})$$

APPENDIX F: ERROR ANALYSIS FOR THE DISTRIBUTION OF COMPONENTS

In this Appendix we discuss how observational errors in the map σ can produce errors in the output function $n(\sigma; \Sigma)$ which determines the number of topological components as a function of threshold density Σ .

As discussed in the text (§4), we must estimate the probability that a given pixel in a map will have an observed value Σ_{obs} which is greater than a given threshold value Σ_0 when in fact the true value Σ_T of the pixel is less than that of the threshold. We first assume that the probability of error in the observed value has a Gaussian distribution so that the probability of having an observed value Σ_{obs} and a true value Σ_T is given by

$$p(\Sigma_{\text{obs}}, \Sigma_T) = \mathcal{N} \exp[-(\Sigma_{\text{obs}} - \Sigma_T)^2 / \tilde{\sigma}^2], \quad (\text{F1})$$

where \mathcal{N} is the normalization factor and where $\tilde{\sigma}$ is the standard deviation. For a given threshold Σ_0 , the probability $P_1(\Sigma_0)$ that an observed value is greater than the threshold and that the true value is less than the threshold is given by

$$P_1(\Sigma_0) = \mathcal{N} \int_{\Sigma_0}^{\infty} d\Sigma_{\text{obs}} \left[-\frac{dm}{d\Sigma} \right]_{\text{obs}} \int_0^{\Sigma_0} d\Sigma_T \exp[-(\Sigma_{\text{obs}} - \Sigma_T)^2 / \tilde{\sigma}^2], \quad (\text{F2})$$

where we have integrated over all possible values $\Sigma_{\text{obs}} \geq \Sigma_0$ and all possible values $\Sigma_T \leq \Sigma_0$ and we have used the probability $-dm/d\Sigma$ that the observed value is Σ_{obs} .

Although the probability P_1 depends on the threshold Σ_0 and on the distribution of density $m(\sigma; \Sigma)$ for the map, we can derive a useful upper bound. We first introduce non-dimensional quantities

$$\begin{aligned} x &= (\Sigma_{\text{obs}} - \Sigma_T)/\tilde{\sigma}, \\ a &= (\Sigma_{\text{obs}} - \Sigma_0)/\tilde{\sigma} \geq 0, \\ b &= \Sigma_{\text{obs}}/\tilde{\sigma} \geq a. \end{aligned} \tag{F3}$$

The integral over Σ_T appearing in equation [F2] can then be written in the form

$$\tilde{\sigma} \int_a^b e^{-x^2} dx \leq \tilde{\sigma} \frac{\sqrt{\pi}}{2} e^{-a^2}, \tag{F4}$$

where the final inequality is straightforward to show. Using the bound of equation [F4] in the expression [F2] for P_1 , we obtain the bound

$$P_1(\Sigma_0) \leq \frac{1}{2} \int_{\Sigma_0}^{\infty} d\Sigma_{\text{obs}} \left[-\frac{dm}{d\Sigma} \right]_{\text{obs}} \exp[-(\Sigma_{\text{obs}} - \Sigma_0)^2/\tilde{\sigma}^2], \tag{F5}$$

where we have used the normalization condition for the Gaussian probability distribution (see equation [F1]), i.e.,

$$2\mathcal{N}\tilde{\sigma} \int_0^{\infty} e^{-v^2} dv = \mathcal{N}\tilde{\sigma}\sqrt{\pi} = 1. \tag{F6}$$

A weaker bound on the probability P_1 can be found by replacing $-dm/d\Sigma$ by its maximum value and evaluating the remaining integral to obtain

$$P_1(\Sigma_0) \leq \frac{\sqrt{\pi}}{4} \tilde{\sigma} \left[-\frac{dm}{d\Sigma} \right]_{\text{max}}. \tag{F7}$$

In practice, we can use the observed distribution of density profile $m(\sigma; \Sigma)$ to find the maximum value of $-dm/d\Sigma$ and thus constrain P_1 . However, we can obtain a crude numerical estimate for P_1 as follows. We write

$$-\tilde{\sigma} \frac{dm}{d\Sigma} \approx \frac{\tilde{\sigma}}{\Delta\Sigma}, \tag{F8}$$

where $\Delta\Sigma$ is the overall dynamic range of the map. The ratio in equation [F8] is thus the inverse of the average signal to noise ratio of the map. We thus obtain the estimate

$$P_1 \approx \frac{\sqrt{\pi}}{4} \frac{\tilde{\sigma}}{\Delta\Sigma} \approx 10^{-3}, 10^{-2}, \tag{F9}$$

where we have used signal to noise ratios of ~ 400 and ~ 40 to obtain the numerical estimates. These values are representative of the *IRAS* maps considered in our companion paper (WA).

Thus, for a map with a large number of pixels, many pixels can be “wrong” at a given threshold in the sense that the pixel value appears to be larger than the threshold value when in fact the true value is less than the threshold. We note that this estimate represents an upper bound on the probability P_1 and we also note that only a small fraction of these “erroneous” pixels will produce spurious topological components. As we discuss in the text, the number of spurious components (those components produced by pixels erroneously appearing larger than a threshold) can be controlled if we restrict our attention to components with three or more pixels (see §4).

APPENDIX G: ERROR ANALYSIS FOR THE DISTRIBUTION OF FILAMENTS

In this Appendix, we consider the effects of observational errors on the output function $f(\sigma; \Sigma)$ which measures the filamentary nature of the map. As defined in §2, the filament function is a sum over all of the components of the map. For a given threshold level Σ , we separate the true components n_R from the spurious ones Δn through

$$n_{\text{obs}} = n_R + \Delta n, \quad (\text{G1})$$

where it is understood that all three quantities are functions of the threshold level Σ . The filament function can then be written in the form

$$f_{\text{obs}} = \frac{1}{n_R + \Delta n} \sum_{j=0}^{n_R} (\mathcal{F}_j + \Delta \mathcal{F}_j) + \frac{1}{n_R + \Delta n} \sum_{k=0}^{\Delta n} (\mathcal{F}_k + \Delta \mathcal{F}_k), \quad (\text{G2})$$

where $\Delta \mathcal{F}_j$ is the uncertainty in the filament index of a given component. If we let f_R denote the true value of the distribution of filaments, the uncertainty can be written in the form

$$\Delta f = f_R - f_{\text{obs}} = \frac{\Delta n}{n_R + \Delta n} \left\{ f_R - \frac{1}{\Delta n} \sum_{k=0}^{\Delta n} \mathcal{F}_k \right\} - \frac{1}{n_R + \Delta n} \sum_{j=0}^{n_R + \Delta n} \Delta \mathcal{F}_j, \quad (\text{G3})$$

where the first term in brackets is due to the uncertainty in the number of components, the second term in brackets is the contribution of the spurious components, and the final sum is the error due to the uncertainty in each individual filament index. Since the uncertainties $\Delta \mathcal{F}_j$ can be either positive or negative, this final sum will tend toward zero. Let us therefore concentrate on the terms in brackets. The sum in brackets is the filament index of the spurious components and we will denote this quantity as f_S . The relative error in the distribution of filaments can then be written

$$\frac{\Delta f}{f_R} = \frac{\Delta n}{n_R + \Delta n} \left\{ 1 - \frac{f_S}{f_R} \right\} \approx \frac{\Delta n}{n_R} \left\{ 1 - \frac{f_S}{f_R} \right\}, \quad (\text{G4})$$

where we have assumed that $\Delta n \ll n_R$ in obtaining the second approximate equality. As we argue in the text (see §4), the term in brackets is expected to be of order unity. Thus, the relative error in the filament index is small if the relative error in the distribution of components can also be made small.

APPENDIX H: RELATIONSHIP BETWEEN THE DISTRIBUTIONS OF DENSITY AND VOLUME

In this appendix, we present a simple relationship between the distribution of density $m(\sigma; \Sigma)$ and the distribution of volume $v(\sigma; \Sigma)$. For any map σ and for any threshold level Σ , the distribution of density $m(\sigma; \Sigma)$ is always greater than the corresponding distribution of volume $v(\sigma; \Sigma)$. We show that this claim is true by the following argument:

We begin by defining an average density $\langle \sigma \rangle$,

$$\langle \sigma \rangle = \frac{1}{A_n} \int d^n \mathbf{x} \sigma(\mathbf{x}) \quad \text{where} \quad A_n = \int d^n \mathbf{x} , \quad (\text{H1})$$

and where the integrals are taken over the entire map. The subscript n denotes the number of spatial dimensions (this argument holds for arbitrary n). The distribution of density can now be written in the form

$$m(\sigma; \Sigma) = \frac{1}{A_n} \int d^n \mathbf{x} \frac{\sigma(\mathbf{x})}{\langle \sigma \rangle} \Theta[\sigma - \Sigma]. \quad (\text{H2})$$

The distribution of volume can be written in the form

$$v(\sigma; \Sigma) = \frac{1}{A_n} \int d^n \mathbf{x} \Theta[\sigma - \Sigma]. \quad (\text{H3})$$

For high threshold levels, i.e., $\Sigma > \langle \sigma \rangle$, the quantity $\sigma(\mathbf{x})/\langle \sigma \rangle$ is greater than unity whenever the integrand of either equation [H2] or [H3] is nonzero. It follows directly that

$$m(\sigma; \Sigma) > v(\sigma; \Sigma) \quad \text{for} \quad \Sigma > \langle \sigma \rangle. \quad (\text{H4})$$

We must now consider the case when the threshold level $\Sigma < \langle \sigma \rangle$. The derivatives of the two distributions (with respect to the threshold Σ) are given by

$$-\frac{dm}{d\Sigma} = \frac{1}{A_n} \int d^n \mathbf{x} \frac{\sigma(\mathbf{x})}{\langle \sigma \rangle} \delta[\sigma - \Sigma], \quad (\text{H5})$$

$$-\frac{dv}{d\Sigma} = \frac{1}{A_n} \int d^n \mathbf{x} \delta[\sigma - \Sigma]. \quad (\text{H6})$$

For the threshold range of interest, $\Sigma < \langle \sigma \rangle$, the quantity $\sigma(\mathbf{x})/\langle \sigma \rangle$ is less than unity whenever the integrand of either equation [H5] or [H6] is nonzero (i.e., whenever $\sigma = \Sigma$). As a result,

$$\left| \frac{dm}{d\Sigma} \right| < \left| \frac{dv}{d\Sigma} \right| \quad (\text{H7})$$

for the threshold range of interest. Since both functions start at unity, $m(\sigma; 0) = 1 = v(\sigma; 0)$, and both functions have negative slopes, equation [H7] implies that

$$m(\sigma; \Sigma) > v(\sigma; \Sigma) \quad \text{for} \quad \Sigma < \langle \sigma \rangle. \quad (\text{H8})$$

Thus, by equations [H4] and [H8], the distribution of density $m(\sigma; \Sigma)$ is always greater than the distribution of volume $v(\sigma; \Sigma)$.

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