Expansion Parameter of Chiral Perturbation Theory

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Abstract

The masses and decay constants of the octet pseudoscalar mesons are calculated to all orders of quark mass in terms of the effective meson theory. Consequently, the expansion parameter of the chiral perturbation theory has been determined to be 0.782GeV which is much greater than the quark masses. It shows that the quark mass expansions of the masses and constants of mesons are convergent and the formulas at the first order of quark mass are very good approximations.

One of the most important features revealed from quantum chromodynamics (QCD) is the existence of an approximate chiral symmetry. The chiral perturbation theory is successful in describing the meson physics[1]. In this theory both momentum expansion and quark mass expansion are used. The expansion parameter has been proposed theoretically to be $\Lambda_\chi \simeq 2\pi F_\pi \sim 1GeV$[2]. In ref.[3] this parameterΛ_χ "the scale of chiral symmetry" has been determined from data on nonleptonic kaon decay to be $\Lambda_\chi \sim 1GeV$ which is consistent with the theoretical speculation[2]. However, in ref.[4] the Numbu-Jona-Lasinio(NJL) model[5] has been used to study the convergence radius of the expansion in the strange-quark mass. The convergence radius is determined to be $30 - 50$MeV which is much less than strange quark mass. It is claimed in ref.[5] that the validity of the first-order mass formula of the octet pseudoscalar mesons is only a accident. Obviously, this is a very serious conclusion. In order to check if this is a model-independent conclusion, in this paper following lagrangian is used to determine the expansion parameter of chiral perturbation theory

$$\mathcal{L} = \bar{\psi}(x)(i\partial\!\!\!/ - mu(x) - M)\psi(x) \tag{1}$$

where ψ are quark fields

$$u(x) = exp\{2i\gamma_5 P\},$$

where P is defined as

$$P = \sqrt{2}\begin{pmatrix} \frac{\pi^0}{\sqrt{2}F_{\pi^0}} + \frac{\eta}{\sqrt{6}F_\eta} & \frac{\pi^+}{F_{\pi^+}} & \frac{K^+}{F_{K^+}} \\ \frac{\pi^-}{F_{\pi^+}} & \frac{-\pi^0}{\sqrt{2}F_{\pi^0}} + \frac{\eta}{\sqrt{6}F_\eta} & \frac{K^0}{F_{K^0}} \\ \frac{K^-}{F_{K^+}} & \frac{\bar{K}^0}{F_{K^0}} & -\frac{2\eta}{\sqrt{6}F_\eta} \end{pmatrix}, \tag{2}$$

and M is the quark mass matrix

$$M = \begin{pmatrix} m_u & 0 & 0 \\ 0 & m_d & 0 \\ 0 & 0 & m_s \end{pmatrix},$$

and m is a parameter. In eq.(1) the pseudoscalar fields are taken as the background fields. This lagrangian has been studied for long time [6] and used to derive the effective lagrangian of the chiral perturbation theory, which is in substantially agreement with data[7]. This is the advantage using this lagrangian to determine the expansion parameter of the chiral perturbation theory. The parameter m in eq.(1) is just the parameter of both momentum expansion and quark mass expansion[6].

In this paper we use lagrangian(1) to calculate the meson masses and decay constants. According to ref.[6c] the effective lagrangian of chiral perturbation theory derived from eq.(1) is composed of two parts: normal part and abnormal part. In this paper the normal part of the lagrangian shown below in Euclidian space(3) is used to calculate the masses and decay constants of the mesons up to one quark loop.

$$\mathcal{L}_E = \frac{1}{2}Tr\int_0^\infty \frac{d\tau}{\tau}(e^{-\tau D^\dagger D} - e^{-\tau\Delta_0}), \tag{3}$$

where $D = \not{\partial} + mu + M$, $D^\dagger = -\not{\partial} + m\hat{u} + M$, Δ_0 has been defined as below. Where $\hat{u} = exp\{-2i\gamma_5 P\}$. In eq.(3) the proper time regularization is exploited[8]. Substituting D and $D^\dagger$ into eq.(2) we obtain

$$\begin{aligned}\mathcal{L}_E &= \frac{1}{2\delta^4(0)}\int d^D x \frac{d^D p}{(2\pi)^D}\int_0^\infty \frac{d\tau}{\tau}(e^{-\tau\{p^2+m^2+M^2-\partial^2-2ip\cdot\partial-m(\not{\partial}u)+m(Mu+\hat{u}M)\}} - e^{-\tau\Delta_0})\delta^4(x-y)|_{y\to x} \\ &= \frac{1}{2\delta^4(0)}\int d^D x \frac{d^D p}{(2\pi)^D}\sum_{n=1}^{\infty}\frac{1}{n!}(-\frac{\partial^2}{\partial m^2})^{n-1}\frac{1}{p^2+m^2} \\ &Tr\{\partial^2 + 2ip\cdot\partial + m(\not{\partial}u) - m(Mu+\hat{u}M) - M^2\}^n \delta^D(x-y)|_{y\to x}\end{aligned} \tag{4}$$

u and $\hat{u}$ can be rewritten as

$$u = \frac{1}{2}(1+\gamma_5)U + \frac{1}{2}(1-\gamma_5)U^\dagger, \tag{5}$$

$$\hat{u} = \frac{1}{2}(1-\gamma_5)U + \frac{1}{2}(1+\gamma_5)U^\dagger. \tag{6}$$

where $U = \exp\{2iP\}$ and $\Delta_0 = p^2 + m^2$.

To determine the decay constants the effective lagrangian(4) has been calculated to the second order derivatives of meson fields and to all orders of quark masses. Comparing with the kinetic term of the lagrangian of the meson theory

$$\mathcal{L}_{KE} = \frac{1}{2}\partial_\mu \pi^i \partial^\mu \pi^i + \partial_\mu K \partial^\mu K + \frac{1}{2}\partial_\mu \eta \partial^\mu \eta \tag{7}$$

and using the scheme of regularization presented in ref.[6c] we obtain

$$\begin{aligned}
\frac{F^2_{\pi^0}}{16} &= \frac{N_c m^2}{(4\pi)^2}(log\frac{\mu^2}{m^2} - \gamma) - \frac{N_c m^2}{2(4\pi)^2} log\{(1+x_u)(1+x_d)\} \\
\frac{F_{\pi^+}}{16} &= \frac{N_c m^2}{(4\pi)^2}(log\frac{\mu^2}{m^2} - \gamma) + \frac{N_c m^2}{(4\pi)^2}(x_u - x_d)^{-1}\{g(x_u) - g(x_d)\} + \frac{N_c m^2}{(4\pi)^2} f(x_u, x_d)(m_d^2 - m_u^2) \\
\frac{F_{K^+}}{16} &= \frac{N_c m^2}{(4\pi)^2}(log\frac{\mu^2}{m^2} - \gamma) + \frac{N_c m^2}{(4\pi)^2}(x_s - x_u)^{-1}\{g(x_s) - g(x_u)\} + \frac{N_c m^2}{(4\pi)^2} f(x_s, x_u)(m_s^2 - m_u^2) \\
\frac{F_{K^0}}{16} &= \frac{N_c m^2}{(4\pi)^2}(log\frac{\mu^2}{m^2} - \gamma) + \frac{N_c m^2}{(4\pi)^2}(x_s - x_d)^{-1}\{g(x_s) - g(x_d)\} + \frac{N_c m^2}{(4\pi)^2} f(x_s, x_d)(m_s^2 - m_d^2) \\
\frac{F^2_{\eta}}{16} &= \frac{N_c m^2}{(4\pi)^2}(log\frac{\mu^2}{m^2} - \gamma) - \frac{N_c m^2}{6(4\pi)^2} log\{(1+x_u)(1+x_d)(1+x_s)^4\}
\end{aligned} \tag{8}$$

where N_c is the number of color,

$$\begin{aligned}
x_q &= \frac{2m_q}{m} + \frac{m_q^2}{m^2}, \\
g(x) &= x - (1+x)log(1+x), \\
f_1(x) &= -\frac{x}{2} - 1 + \frac{1}{x}(1+x)log(1+x), \\
f_2(x) &= \frac{3}{4}x + \frac{1}{2} - \frac{1}{2x}(1+x)^2 log(1+x), \\
f_3(x) &= \frac{x}{6} + \frac{1}{x}(1-x)f_2(x), \\
f(x,y) &= \frac{1}{(x-y)^2}\{f_1(x) + f_1(y)\} - \frac{2}{(x-y)^3}\{y f_2(x) - x f_2(y)\} \\
&+ \frac{1}{2(x-y)^2}\{\frac{1}{6}(x+y) + \frac{f_1(x)}{x} + \frac{f_1(y)}{y}\} + \frac{1}{(x-y)^3}\{\frac{1}{6}(x^2 - y^2) + f_2(x) - f_2(y) \\
&+ \frac{x}{y}(1-y)f_2(y) - \frac{y}{x}(1-x)f_2(x)\}
\end{aligned} \tag{9}$$

From the formulas of the decay constants(8) it can be seen that in the chiral limit all quark masses are zero and we have

$$F_{\pi^+} = F_{\pi^0} = F_{K^+} = F_{K^0} = F_\eta$$

On the other hand, these decay constants can also be calculated in terms of the octet axial-vector current defined as

$$< 0|A^a_\mu(0)|\phi^a >= -ip_\mu \frac{F_{\phi^a}}{\sqrt{2}} \tag{10}$$

where ϕ^a is a octet pseudoscalar meson. In terms of the path integral the effective axial-vector current

$$A^a_\mu(x) =< \bar{\psi}(x)\lambda_a\gamma_\mu\gamma_5\psi(x) > \tag{11}$$

can be expressed as[9]

$$A^a_\mu(x) = -iTr\lambda_a\gamma_\mu\gamma_5 S_F(x,x) \tag{12}$$

where the propagator $S_F(x,y)$ satisfies following equation

$$\{i\not{\partial}_x - mu(x) - M\}S_F(x,y) = \delta^4(x-y). \tag{13}$$

In the momentum picture the eq.(13) becomes

$$\{i\not{\partial}_x - \not{p} - mu(x) - M\}S_F(x,p) = 1,$$
$$S_F(x,y) = \frac{1}{(2\pi)^4}\int d^4p S_F(x,p)e^{ip(x-y)}. \tag{14}$$

At low energy, derivative expansion is used to solve eq.(14). The solution is

$$S_F(x,p) = S^{(0)}_F(x,p)\sum_{n=0}^{\infty}\{(-i\not{\partial}_x + M)S^{(0)}_F(x,p)\}^n,$$
$$S^{(0)}_F = -\frac{\not{p} - m\hat{u}}{p^2 - m^2}, \tag{15}$$

Using eqs.(12,15) we calculate the decay constants to the second order of quark masses. After removing complete derivative terms in the calculation we obtain

$$\frac{F_{\pi^0}^2}{16} = \frac{N_c m^2}{(4\pi)^2}(log\frac{\mu^2}{m^2} - \gamma) + \frac{N_c m^2}{(4\pi)^2}\{-\frac{1}{2}(x_u + x_d) + (\frac{m_u^2}{m^2} + \frac{m_d^2}{m^2})\}$$
$$\frac{F_{\pi^+}}{16} = \frac{N_c m^2}{(4\pi)^2}(log\frac{\mu^2}{m^2} - \gamma) + \frac{N_c m^2}{(4\pi)^2}\{-\frac{1}{2}(x_u + x_d) + \frac{2}{3}(\frac{m_u^2}{m^2} + \frac{m_u m_d}{m^2} + \frac{m_d^2}{m^2}) - \frac{1}{12}(\frac{m_d}{m} - \frac{m_u}{m})^2\}$$
$$\frac{F_{K^+}}{16} = \frac{N_c m^2}{(4\pi)^2}(log\frac{\mu^2}{m^2} - \gamma) + \frac{N_c m^2}{(4\pi)^2}\{-\frac{1}{2}(x_u + x_s) + \frac{2}{3}(\frac{m_u^2}{m^2} + \frac{m_u m_s}{m^2} + \frac{m_s^2}{m^2}) - \frac{1}{12}(\frac{m_s}{m} - \frac{m_u}{m})^2\}$$
$$\frac{F_{K^0}}{16} = \frac{N_c m^2}{(4\pi)^2}(log\frac{\mu^2}{m^2} - \gamma) + \frac{N_c m^2}{(4\pi)^2}\{-\frac{1}{2}(x_d + x_s) + \frac{2}{3}(\frac{m_d^2}{m^2} + \frac{m_d m_s}{m^2} + \frac{m_s^2}{m^2}) - \frac{1}{12}(\frac{m_s}{m} - \frac{m_d}{m})^2\}$$
$$\frac{F_\eta^2}{16} = \frac{N_c m^2}{(4\pi)^2}(log\frac{\mu^2}{m^2} - \gamma) + \frac{N_c m^2}{(4\pi)^2}\{-\frac{1}{6}(x_u + x_d + 4x_s) + \frac{1}{3}(\frac{m_u^2}{m^2} + \frac{m_d^2}{m^2} + 4\frac{m_s^2}{m^2})\}. \quad (16)$$

These formulas(16) are the same with the ones obtained by expanding the formulas(8) to the second orders in quark masses.

The masses of the octet pseudoscalar mesons have been calculated to all orders of quark masses by using the effective lagrangian(4),

$$m_{\pi^0}^2 = \frac{16N_c m^4}{F_{\pi^0}^2(4\pi)^2}(log\frac{\mu^2}{m^2} - \gamma)(y_u + y_d + y_u x_u + y_d x_d)$$
$$+\frac{16N_c m^4}{F_{\pi^0}^2(4\pi)^2}(y_u + y_d + y_u g(x_u) + y_d g(x_d))$$
$$m_{\pi^+}^2 = \frac{16N_c m^4}{F_{\pi^+}^2(4\pi)^2}(log\frac{\mu^2}{m^2} - \gamma)(y_u + y_d + y_u x_u + y_d x_d + (y_d - y_u)^2)$$
$$+\frac{16N_c m^4}{F_{\pi^+}^2(4\pi)^2}(y_u + y_d + y_u g(x_u) + y_d g(x_d) + (x_d - x_u)^{-1}(g(x_d) - g(x_u))(y_d - y_u)^2)$$
$$m_{K^+}^2 = \frac{16N_c m^4}{F_{K^+}^2(4\pi)^2}(log\frac{\mu^2}{m^2} - \gamma)(y_u + y_s + y_u x_u + y_s x_s + (y_s - y_u)^2)$$
$$+\frac{16N_c m^4}{F_{K^+}^2(4\pi)^2}(y_u + y_s + y_u g(x_u) + y_s g(x_s) + (x_s - x_u)^{-1}(g(x_s) - g(x_u))(y_s - y_u)^2)$$
$$m_{K^0}^2 = \frac{16N_c m^4}{F_{K^0}^2(4\pi)^2}(log\frac{\mu^2}{m^2} - \gamma)(y_d + y_s + y_d x_d + y_s x_s + (y_s - y_d)^2)$$

$$+\frac{16N_cm^4}{F_{K^0}^2(4\pi)^2}(y_d + y_s + y_d g(x_d) + y_s g(x_s) + (x_s - x_d)^{-1}(g(x_s) - g(x_d))(y_s - y_d)^2)$$

$$3m_\eta^2 = \frac{16N_cm^4}{F_\eta^2(4\pi)^2}(log\frac{\mu^2}{m^2} - \gamma)(y_u + y_d + 4y_s + y_u x_u + y_d x_d + 4y_s x_s)$$

$$+\frac{16N_cm^4}{F_\eta^2(4\pi)^2}(y_u + y_d + 4y_s + y_u g(x_u) + y_d g(x_d) + 4y_s g(x_s)) \quad (17)$$

where $y_q = m_q/|m|$.

Input the physical values of F_{π^+}, F_{K^+}, m_{π^+}, m_{K^+}, and m_{K^0} to eqs.(8) and (17) we determine

$$m = -782.3MeV, \quad m_u = 1.37MeV, \quad m_d = 2.55MeV, \quad m_s = 73.2MeV. \quad (18)$$

The absolute value of the expansion parameter m is much greater than the values of quark masses. It can be seen from eqs.(8) and (17) that the formulas of F_{π^0} and m_{π^0} are different from F_{π^+} and m_{π^+} respectively. However, it is found that $F_{\pi^0} = F_{\pi^+}$ and $m_{\pi^0} = m_{\pi^+}$ numerically. Using eqs.(8) and (17) we determine

$$F_\eta^2 = 1.69F_\pi^2, \quad m_\eta = 528.5MeV. \quad (19)$$

The decay constants and masses of the octet pseudoscalar mesons (8) (17) can be expanded as series of quark masses. Due to the fact that $|m| \gg m_q$ we expect that the series converge very fast. The decay constants at the first order of quark mass can be obtained from eq.(16) and the masses of the pseudoscalar mesons are obtained from eqs.(17)

$$m_{\pi^0}^2 = \frac{16N_cm^4}{F_{\pi^0}^2(4\pi)^2}(log\frac{\mu^2}{m^2} - \gamma)(y_u + y_d) + \frac{16N_cm^4}{F_{\pi^0}^2(4\pi)^2}(y_u + y_d)$$

$$m_{\pi^+}^2 = \frac{16N_cm^4}{F_{\pi^+}^2(4\pi)^2}(log\frac{\mu^2}{m^2} - \gamma)(y_u + y_d) + \frac{16N_cm^4}{F_{\pi^+}^2(4\pi)^2}(y_u + y_d),$$

$$m_{K^+}^2 = \frac{16N_cm^4}{F_{K^+}^2(4\pi)^2}(log\frac{\mu^2}{m^2} - \gamma)(y_u + y_s) + \frac{16N_cm^4}{F_{K^+}^2(4\pi)^2}(y_u + y_s),$$

$$m_{K^0}^2 = \frac{16N_cm^4}{F_{K^0}^2(4\pi)^2}(log\frac{\mu^2}{m^2} - \gamma)(y_d + y_s) + \frac{16N_cm^4}{F_{K^0}^2(4\pi)^2}(y_d + y_s)$$

$$3m_\eta^2 = \frac{16N_c m^4}{F_\eta^2(4\pi)^2}(log\frac{\mu^2}{m^2} - \gamma)(y_u + y_d + 4y_s) + \frac{16N_c m^4}{F_\eta^2(4\pi)^2}(y_u + y_d + 4y_s). \quad (20)$$

In terms of eq.(8) $\frac{N_c m^4}{(4\pi)^2}(log\frac{\mu^2}{m^2} - \gamma)$ can be determined by $F^2_{K^+} - F^2_{\pi^+}$. Substituting the values of m and quark masses(8) into eq.(20) and eq.(16)(the first orders in quark masses) we obtain

$$m_{\pi^+} = m_{\pi^0} = 139.6MeV, \quad m_{K^+} = 499.4MeV, \quad m_{K^0} = 501.9MeV, \quad m_\eta = 545.3MeV$$

$$F_{\pi^+} = F_{\pi^0} = 186MeV, \quad F^2_{K^+} = 1.486F^2_\pi, \quad F^2_{K^0} = 1.494F^2_\pi, \quad F^2_\eta = 1.653F^2_\pi. (21)$$

Comparison of the results(21) with the results (19) and the experimental values of F_π, F_K, m_π, and m_K shows that for the decay constants and masses of pions the formulas at the first order of quark masses are good enough and for K-meson and η meson the deviations of the first order formulas from eqs. (8,17) are only few percents. Therefore the formulas at first order of quark masses are good approximations. As a matter of fact, the contributions the u and d quark masses to F_π are only 1.4%.

To conclude, the decay constants and masses of the octet pseudoscalar mesons are determined to all orders of quark masses. It is found that the value of the expansion parameter is much greater than the masses of u, d, and s quarks in this effective theory of mesons. On the other hand, this m is also the parameter of the momentum expansion[6] and the numerical value of this parameter is compatible with the values presented in [2,3]. It is found that the quark mass expansions are convergent. The formulas of the decay constants and masses of mesons at the first orders of quark masses are very good approximation.

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