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Comments on Weinberg's second sum rule

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Abstract

A mass relation of ρ and a_1 mesons is obtained from both Weinberg's first sum rule and KSFR sum rule. By applying $SU(2)_L \times SU(2)_R$ chiral symmetry, current algebra, and VMD two new sum rules of the amplitude of $a_1 \to \rho\pi$ are found. The expression of the amplitude of $a_1 \to \rho\pi$ is presented in the limit of $p_\pi \to 0$ in terms of the new mass relation.

One of the most important features revealed from quantum chromodynamics (QCD) is the chiral symmetry in the limit of $m_q \to 0$ ($q = u, d, s$). For a long time, it is well known that the ρ and a_1 mesons are chiral partners[1]. Two Weinberg's sum rules[2] of ρ and a_1 mesons have been established

$$\frac{g_\rho^2}{m_\rho^2} - \frac{g_a^2}{m_a^2} = \frac{F_\pi^2}{4}, \tag{1}$$

$$g_\rho = g_a, \tag{2}$$

where

$$< 0|V_\mu^a|\rho_b^\lambda >= \epsilon_\mu^\lambda \delta_{ab} g_\rho,$$

$$< 0|A_\mu^a|a_b^\lambda >= \epsilon_\mu^\lambda \delta_{ab} g_a,$$

and F_π is pion decay constant, $F_\pi = 186 MeV$. The mass relation

$$m_a^2 = 2m_\rho^2 \tag{3}$$

is the consequence of Eqs.(1),(2), and the KSFR sum rule[3]

$$g_\rho^2 = \frac{1}{2} F_\pi^2 m_\rho^2. \tag{4}$$

The determination of a_1 meson has a long history. It was finally established in diffractive production[4] and charge-exchange production[5]. However, the extraction of the mass and the width of a_1 resonance is troubled by the presence of a coherent background, attributed to the Deck effect[6]. The three -pion final state of τ decay[7] presently offers the best conditions to study the a_1 resonance parameters. The determination of m_{a_1} and $\Gamma(\tau \to a_1 \nu)$ enables us to test the Weinberg's two sum rules directly. Let us test the Weinberg's first sum rule first. g_ρ is the coupling constant between ρ and γ and it has been determined to be

$$g_\rho = 0.116(1 \pm 0.05) GeV^2 \tag{5}$$

from $\Gamma(\rho \to l^+l^-)$. Using $m_a = (1230 \pm 40)MeV$, Weinberg's first sum rule(Eq.(1)) predicts that

$$g_a = 0.145 \pm 0.018 GeV^2. \tag{6}$$

On the other hand, g_a can be determined from

$$\Gamma(\tau \to a_1\nu) = \frac{G^2}{8\pi} cos^2\theta_c g_a^2 \frac{m_\tau^3}{m_a^2}(1 - \frac{m_a^2}{m_\tau^2})^2(1 + 2\frac{m_a^2}{m_\tau^2}). \tag{7}$$

The experimental data of the decay rate is[8] $2.14 \times 10^{-13}(1 \pm 0.32)GeV$ and g_a is determined to be $0.145 \pm 0.033 GeV^2$. Therefore, Weinberg's first sum rule is in good agreement with the data. From Ref.[2] we have learned that Weinberg's first sum rule is a result of chiral symmetry, current algebra, and Vector Meson Dominance(VMD)[2]. We abbreviate chiral symmetry, current algebra, and VMD as CHCV. The agreement evidences that CHCV work well in studying hadron physics at low energies.

On the other hand, the present value of m_a and the values of g_ρ and g_a show that Weinberg's second sum rule(Eqs.(2),(3)) is not in good agreement with the data. However, Eq.(3) is widely quoted in the studies of meson physics. It is known that besides CHCV an additional assumption is invoked in deriving Weinberg's second sum rule. Therefore, it is significant to find the mass relation between the two chiral partners, ρ and a_1 mesons, only based on CHCV. In order to do that let us rewrite Eq.(1) as

$$1 - \frac{g_a^2}{g_\rho^2}\frac{m_\rho^2}{m_a^2} = \frac{F_\pi^2}{4}\frac{m_\rho^2}{g_\rho^2} \tag{8}$$

Substituting KSFR sum rule(Eq.(4)) into eq.(8) we obtain

$$m_a^2 = 2\frac{g_a^2}{g_\rho^2}m_\rho^2 \tag{9}$$

Using the values of g_a and g_ρ, it is found

$$m_a = 1.36(1 \pm 0.17)GeV \tag{10}$$

Theoretical prediction of Eq.(9) fits the data of m_a better. One of the purposes of this short note is to point out that the combination of Weinberg's first sum rule and KSFR sum rule leads to a new mass relation of ρ and a_1 mesons.

Now we can ask a question that if we get rid of the additional assumption made in Ref.[2] what can we learn from Eq.(18) of Ref.[2](see Eq.(17) of this note)? The Eq.(18) of Ref.[2] is used to derive the second sum rule by invoking an additional assumption. In order to show that in the Eq.(18) of Ref.[2] a factor of two has been lost we start from the Ward identity presented in Ref.[2]

$$\frac{1}{2}q_\mu M^{\mu\nu\lambda}(q,p) = \Delta_V^{\nu\lambda}(q+p) - \Delta_A^{\nu\lambda}(p), \tag{11}$$

where

$$-i\epsilon_{abc}M^{\mu\nu\lambda} = \int d^4x d^4y < 0|T\{A_a^\mu(x)A_b^\nu(y)V_c^\lambda(0)\}|0 > exp\{-iqx - ipy\}, \tag{12}$$

$$\delta_{ab}\Delta_V^{\nu\lambda}(p) = i\int d^4y e^{-ipy} < 0|T\{V_a^\nu(y)V_b^\lambda(0)\}|0 > . \tag{13}$$

Following Ref.[2] and setting $q_\mu = 0$ in Eq.(11), on the left hand side of the Eq.(11) only pion poles survive. In this limit, Eq. (11)now reads

$$\begin{aligned}
&-i\epsilon_{abc}q_\mu M^{\mu\nu\lambda}|_{q_\mu\to 0} = \\
&\frac{F_\pi}{2k_0}\int d^4y e^{-ipy} k\cdot q\{\frac{\theta(y_0)}{q_0+k_0-i\varepsilon}e^{-i(q_0+k_0)y_0} < \pi_a(\vec{k}=-\vec{q},k_0)|A_b^\nu(y)V_c^\lambda(0)|0 > \\
&+\frac{\theta(-y_0)}{q_0+k_0-i\varepsilon} < \pi_a(\vec{k}=-\vec{q},k_0)|V_c^\lambda(0)A_b^\nu(y)|0 > \\
&-\frac{\theta(y_0)}{k_0-q_0-i\varepsilon} < 0|A_b^\nu(y)V_c^\lambda(0)|\pi_a(\vec{k}=\vec{q},k_0) > \\
&-\frac{\theta(-y_0)}{k_0-q_0-i\varepsilon}e^{i(k_0-q_0)y_0} < 0|V_c^\lambda(0)A_b^\nu(y)|\pi_a(\vec{k}=\vec{q},k_0) >\}.
\end{aligned} \tag{14}$$

In Eq.(14) the pion is on mass shell and in the chiral limit, $k_0 = |\vec{q}|$. When $\vec{k} = -\vec{q}$ there is

$$\frac{k\cdot q}{2k_0}\frac{1}{q_0+k_0-i\varepsilon} = \frac{1}{2}, \tag{15}$$

and when $\vec{k} = \vec{q}$ we have

$$\frac{k \cdot q}{2k_0} \frac{1}{k_0 - q_0 - i\varepsilon} = -\frac{1}{2}. \tag{16}$$

In the limit of $q_\mu \to 0$, the 4-momenta of the pion states in Eq.(14) are zero. Considering this fact and substituting the two equations(15,16) into Eq.(14), we obtain

$$F_\pi \int d^4y e^{-ipy} < \pi_a|T\{A_b^\nu(y)V_c^\lambda(0)\}|0 >= -\epsilon_{abc}\{\Delta_V^{\nu\lambda}(p) - \Delta_A^{\nu\lambda}(p)\}. \tag{17}$$

Comparing with the Eq.(18) of Ref.[2], there is an additional factor of two in Eq.(17). This new factor does not affect Weinberg's second sum rule. However, this factor is important for the study of this note. In Eq.(17) $\Delta_V^{\nu\lambda}(p)$ has ρ meson pole and $\Delta_A^{\nu\lambda}(p)$ has both pion pole and a_1 meson pole. The left hand side of the equation(17) should have these three poles. Therefore, multiplying both sides of the equation by $p^2 - m^2$ and letting $p^2 \to m^2$, the corresponding pole term can be picked out from the equation, where m^2 is pion mass(in chiral limit pion mass is zero) , ρ meson mass, and a_1 meson mass respectively. In order to find the poles the left hand side of the Eq.(17) is rewritten as

$$\begin{aligned}
&F_\pi \int d^4y e^{-ipy} < \pi_a|T\{A_b^\nu(y)V_c^\lambda(0)\}|0 >= \\
&\int d^4y\{\theta(y_0)e^{-iky}iF_\pi k_\nu < \pi_a\pi_b(k)|V_c^\lambda(0)|0 > +\theta(y_0)e^{iky}(-)iF_\pi k_\nu < \pi_a|V_c^\lambda(0)|\pi_a > \\
&+\theta(y_0)e^{-iky}g_\rho\varepsilon_\lambda^{\sigma*}(k) < \pi_a|A_b^\nu(0)|\rho_c^\sigma(k) > +\theta(y_0)e^{iky}g_\rho\varepsilon_\lambda^\sigma(k) < \pi_a\rho_c^\sigma(k)|A_b^\nu(0)|0 > \\
&+\theta(y_0)e^{-iky}g_a\varepsilon_\nu^\sigma(k) < \pi_a a_b^\sigma(k)|V_c^\lambda(0)|0 > +\theta(y_0)e^{iky}g_a\varepsilon_\nu^{\sigma*}(k) < \pi_a|V_c^\lambda(0)|a_b^\sigma(k) >\}
\end{aligned} \tag{18}$$

After using VMD, the following formulas are obtained

$$\begin{aligned}
< \pi_a|V_c^\lambda(0)|\pi_b(k) >&= i\epsilon_{abc}g_\rho f_{\rho\pi\pi}\frac{k^\lambda}{m_\rho^2}, \\
< \pi_a\pi_b(k)|V_c^\lambda(0)|0 >&= -i\epsilon_{abc}g_\rho f_{\rho\pi\pi}\frac{k^\lambda}{m_\rho^2},
\end{aligned} \tag{19}$$

where $f_{\rho\pi\pi}$ is the coupling constant of the decay of $\rho \to 2\pi$ in the limit of $p_\pi \to 0$. Using VMD, other four matrix elements in Eq.(18) are related to the decay of $a_1 \to \rho\pi$. The vertex

of this decay can be written as

$$\{Ag_{\mu\nu} + Bp_{\pi\mu}p_{\pi\nu}\}\epsilon_{abc}a^a_\mu\rho^b_\nu\pi_c, \tag{20}$$

where A and B are functions of momenta of a_1, ρ, and π. In Eq.(18), the 4-momenta of the state $|\pi_a>$ are zero, therefore only the amplitude A contributes to these matrix elements of Eq.(18) and the amplitude A is defined in the limit of $p_\pi = 0$. Using Eq.(20) and VMD we obtain

$$\begin{aligned} <\pi_a|A^\nu_b(0)|\rho^\sigma_c(k)> &= -\epsilon_{abc}g_a A(m^2_\rho)\varepsilon^\sigma(k)^\nu \frac{1}{m^2_\rho - m^2_a}, \\ <\pi_a\rho^\sigma_c(k)|A^\nu_b(0)|0> &= -\epsilon_{abc}g_a A(m^2_\rho)\varepsilon^{\sigma *}(k)^\nu \frac{1}{m^2_\rho - m^2_a}. \end{aligned} \tag{21}$$

In Eq.(21), due to $k^2 = m^2_\rho$, we have $A(k^2) = A(m^2_\rho)$. In the same way, two other matrix elements of Eq.(18) are written as

$$\begin{aligned} <\pi_a|V^\lambda_c(0)|a^\sigma_b(k)> &= -\epsilon_{abc}g_\rho A(m^2_a)\frac{1}{m^2_a - m^2_\rho}\epsilon^\sigma(k)^\lambda, \\ <\pi_a a^\sigma_b(k)|V^\lambda_c(0)|0> &= -\epsilon_{abc}g_\rho A(m^2_a)\frac{1}{m^2_a - m^2_\rho}\epsilon^{\sigma *}(k)^\lambda. \end{aligned} \tag{22}$$

It is necessary to point out that because of the limit of $p_\pi = 0$ in Eq.(17), the amplitude A is function of k^2 where k is either the momentum of ρ or a_1. It has been found that $k^2 = m^2_\rho$ in Eq.(21) and $k^2 = m^2_a$ in Eq.(22). Substituting the six matrix elements(19,21,22) into Eq.(18), the pole terms on the left hand side of the Eq.(17) are found to be

$$\begin{aligned} F_\pi \int d^4y e^{-ipy} <\pi_a|T\{A^\nu_b(y)V^\lambda_c(0)\}|0>|_{poles} &= i\epsilon_{abc}g_\rho f_{\rho\pi\pi}\frac{F^2_\pi}{m^2_\rho}\frac{p^\nu p^\lambda}{p^2} \\ -i\epsilon_{abc}F_\pi g_\rho g_a \frac{1}{m^2_\rho - m^2_a}\frac{A(m^2_\rho)}{p^2 - m^2_\rho}(-g^{\nu\lambda} &+ \frac{p^\lambda p^\nu}{m^2_\rho}) \\ -i\epsilon_{abc}F_\pi g_\rho g_a \frac{1}{m^2_a - m^2_\rho}\frac{A(m^2_a)}{p^2 - m^2_a}(-g^{\nu\lambda} &+ \frac{p^\nu p^\lambda}{m^2_a}). \end{aligned} \tag{23}$$

By applying the VMD to the right hand side of the Eq.(17), the pole terms are obtained

$$iF_\pi^2\frac{p^\nu p^\lambda}{p^2}+ig_\rho^2\frac{1}{p^2-m_\rho^2}(-g^{\nu\lambda}+\frac{p^\nu p^\lambda}{p^2})+ig_a^2\frac{1}{p^2-m_a^2}(-g^{\nu\lambda}+\frac{p^\nu p^\lambda}{m_a^2}). \tag{24}$$

Multiply both sides of Eq.(17) by p^2 and set $p^2=0$, the pion pole is picked out and the following relation is found

$$g_\rho f_{\rho\pi\pi}=m_\rho^2. \tag{25}$$

Multiply Eq.(17) by $p^2-m_\rho^2$ and set $p^2=m_\rho^2$, the ρ pole is picked out and the second relation is obtained

$$g_a F_\pi A(m_\rho^2)=-g_\rho(m_a^2-m_\rho^2). \tag{26}$$

Multiply Eq.(17) by $p^2-m_a^2$ and set $p^2=m_a^2$, the a_1 pole is picked out and the third relation is derived

$$g_\rho F_\pi A(m_a^2)=-g_a(m_a^2-m_\rho^2). \tag{27}$$

The first relation is well known from VMD[9]. It has been pointed out by Brown and Wess[10], Schnitzer and Weinberg[11] that the amplitude A determined in the soft pion limit($p_\pi\to 0$) is far away from the physical amplitude of the decay $a_1\to\rho\pi$ and they have applied the hard pion approximation[10,11] in this decay. However, because Eqs.(26,27) are based on CHCV and CHCV are very general, then these two equations can serve as the constraints on effective chiral theories in which ρ and a_1 mesons are included. In other words, in the limit $p_\pi\to 0$ the amplitudes of the decay $a_1\to\rho\pi$ derived from an effective chiral meson theory must satisfy the two equations(26,27) obtained from CHCV. Use of the new mass formula(Eq.(9)) leads to the determination of the amplitude A in the limit of $p_\pi\to 0$. Following Schnitzer and Weinberg[11] the amplitude A can be written as a simple polynomial in 4-momenta in the limit of $p_\pi=0$

$$A(k^2)=a+bk^2.$$

By using the new mass formula(Eq.(9)) and the two new sum rules(Eqs.(26),(27)) we obtain

$$a = \frac{2}{f_\pi}\frac{g_a}{g_\rho}m_\rho^2, \quad b = \frac{2}{f_\pi}\frac{g_a}{g_\rho}(1 - \frac{g_\rho^2}{g_a^2}).$$

In the limit of $p_\pi \to 0$, the amplitude A is determined to be

$$A = \frac{2}{f_\pi}\frac{g_a}{g_\rho}[m_\rho^2 + (1 - \frac{g_a^2}{g_\rho^2})k^2]. \tag{28}$$

This amplitude is in unphysical region and cannot be used to calculate the decay width of the a_1 meson. On the other hand, the soft pion approximation($p_\pi \to 0$) is used in deriving the KSFR sum rule (Eq.(4)) and Eq.(25) too and the coupling constant $f_{\rho\pi\pi}$ of the decay $\rho \to \pi\pi$ in the KSFR sum rule and the Eq.(25) is in unphysical region. However, both the KSFR sum rule and the Eq.(25) are in good agreement with the data. In Refs.[12] an effective chiral theory of mesons have been proposed and the physics of the pseudoscalars, vectors, and axial-vector mesons have been studied. The theoretical results are in good agreement with data and the Weinberg's first sum rule is satisfied analytically. This theory provides an explanation to that why soft pion approximation works on $\rho \to \pi\pi$ and doesn't work on $a_1 \to \rho\pi$. In this theory $f_{\rho\pi\pi}$ is almost independent of the momenta of pions, however, the amplitude A of $a_1 \to \rho\pi$ strongly depends on p_π and there is cancellation in the amplitude A. Because of these two factors the soft pion approximation doesn't work on a_1 decay. The theoretical significance of Eqs.(26,27,28) is that in order to satisfy CHCV, the amplitude A derived from any effective chiral theory must satisfy Eqs.(26,27,28) in the limit $p_\pi \to 0$. As an example, the expression of A(Eq.(28)) can be derived from Refs.[12].

To conclude, a new mass relation between ρ and a_1 mesons is presented in terms of both Weinberg's first sum rule and KSFR sum rule. In the limit of $p_\pi \to 0$, two new sum rules of a_1 decay amplitude and the expression of the amplitude of a_1 decay are obtained based on CHCV.

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