Fermilab Library
0 1160 0023683 0

UGVA-DPT-1993-02-806

UNIVERSITÉ DE GENÈVE

SCHOLA GENEVENSIS MDLIX

q–DEFORMATION OF SEMISIMPLE AND NON–SEMISIMPLE LIE ALGEBRAS

HENRI RUEGG

Département de Physique Théorique

Université de Genève

CH-1211 GENEVE 4, Switzerland

UGVA-DPT 1993 / 02-806

ABSTRACT: We give an elementary introduction to the Drinfeld-Jimbo procedure of the quantum deformation $U_q(g)$ of semisimple Lie algebras g. The q–Serre relations are discussed in some detail, in the Chevalley and the Cartan–Weyl basis. We review the real forms of g and $U_q(g)$. The general procedure is illustrated by the examples $U_q(s\ell(2))$ and $U_q(so(5, \mathbf{C}))$. After some general considerations on non-semisimple Lie algebras, we discuss in detail the quantum deformation of the Poincaré algebra, involving the contraction of $U_q(so(3,2))$. Some physical consequences of the resulting κ–Poincaré Hopf algebra are mentioned.

To appear in the Proceedings of the XXIII GIFT seminar *Recent Problems in Mathematical Physics* (June 1992) Salamanca

1. Introduction

The aim of these lectures is to give an elementary introduction to some problems in "quantum groups" and "quantum algebras". These terms do not denote groups and Lie algebras, but particular deformations of Lie groups and Lie algebras, and belong to the more general category of Hopf algebras. We shall speak of "q–deformation" G_q of a Lie group G and q–deformation $U_q(g)$ of a Lie algebra g. The emphasis will be on the latter.

We shall first describe the well established theory of q–deformation of semisimple Lie algebras, particularly $s\ell(2)$. Then we shall discuss real forms of q–algebras. Finally, examples of the less well known quantum deformation of non-semisimple Lie algebras will be given, especially the Poincaré algebra.

Quantum groups were first introduced in physics in two–dimensional integrable systems and soon applied to conformal field theories (see the lectures in these proceedings by L. Faddeev, C. Gomez, G. Sierra, and references [1, 2, 3, 4]).

Another application of q–deformed algebras is to provide a new possibility of symmetry breaking. The first example is the Heisenberg model of a one–dimensional metal, or XXX quantum spin chain, solved by Bethe. It is invariant under $s\ell(2)$, whereas the XXZ chain is only invariant under the classical $U(1)$ but actually also under the quantum $U_q(s\ell(2))$ [5, 6, 7]. In these lectures we shall discuss the possible breaking of Lorentz invariance at high energy by the quantum deformation of the Poincaré algebra in 4 dimensions [8, 9, 10]. Although at present there is no experimental evidence for such violation, it is interesting to get a "figure of merit" of a theory by comparing it to another theory with similar power of prediction, the latter being obtained by continuous deformation of the former. In addition, the corresponding field theory could provide a new way of regularizing the quantum field theories of particle physics. Finally, it is likely that space-time is drastically changed at the Planck mass [11]. For other examples see the Firenze group [12].

As mentioned above the q–deformation theory of semisimple Lie groups G (Faddeev *et al.*[13], Woronowicz [14]) and semisimple Lie algebras g (Drinfeld [15], Jimbo [16]) is well established. They are special examples of Hopf algebras. We shall adopt the procedure given by the two last authors. However, there is no such general scheme for non-semisimple Lie algebras. Examples are known [17, 18, 19] and hopefully they

will be helpful in finding their general features. In these lectures we shall discuss in detail the non-trivial example of the quantum deformation of the Poincaré algebra [8, 9].

1.1. THREE METHODS TO DEFORM INHOMOGENEOUS LIE ALGEBRAS

Most of the above examples are semidirect sums of a semisimple and an abelian algebra, such as rotations and translations in 3 dimensions = $E(3)$ [18], the inhomogeneous Lorentz algebra = Poincaré algebra [8, 9], etc. There exist several methods to quantum deform these algebras :

a) Contraction. One starts with a simple or semisimple Lie algebra, deforms it according to Drinfeld–Jimbo and rescales some of the generators, which in the limit correspond to an abelian invariant translation algebra. It is then shown that the deformed $E(3)$ [18] or Poincaré [8, 9] are Hopf algebras.

b) Subalgebra. The simple algebra $so(4,2)$ contains the Weyl algebra, that is Poincaré plus dilatations. A deformation of $so(4,2)$ yields a q–Weyl Hopf algebra [20, 21]. Unfortunately, the deformation thus obtained of the Poincaré algebra is not a Hopf algebra by itself, *i.e.* the subalgebra hierarchy is not preserved by the deformation.

c) Non-commutative space [22]. As an example consider the q–deformed Lorentz algebra acting on a four–dimensional Manin space. The differential operators in this q–geometry correspond to q–translations. In this way one gets q–Weyl as a Hopf algebra [23].

We shall discuss in detail only method (a), the contraction.

1.1.1. Classical contraction procedure

Let us sketch the contraction procedure for getting Poincaré in the classical case, that is for $q = 1$.

Start with the complex simple algebra $so(5, \mathbf{C})$. It has the three real forms $so(5)$, $so(4,1)$ and $so(3,2)$ corresponding respectively to the diagonal metrics $(+++++)$, $(-++++)$ and $(-+++-)$. The generators

$$M_{AB} = -M_{BA} \quad , \quad A, B = 0, \cdots, 4 \tag{1.1}$$

satisfy the commutation relations

$$[M_{AB}, M_{CD}] = g_{AD}M_{BC} + g_{BC}M_{AD} - g_{AC}M_{BD} - g_{BD}M_{AC}. \tag{1.2}$$

involving the metric tensor

$$g_{AB} = g_{BA} \tag{1.3}$$

with ± 1 on the diagonal as above. The de Sitter and anti–de Sitter algebras $so(4,1)$ and $so(3,2)$ contain a Lorentz subalgebra with generators

$$M_{\mu\nu} \quad , \quad \mu, \nu = 0, \cdots, 3 \ . \tag{1.4}$$

Putting

$$M_{\mu 4} = RP_\mu \quad , \tag{1.5}$$

and letting the "radius of the fifth dimension" R tend to infinity, $R \to \infty$, it follows that

$$\begin{aligned} [P_\mu, P_\nu] &= \frac{1}{R^2}[M_{\mu 4}, M_{\nu 4}] \to 0 \\ [M_{\mu\nu}, P_\lambda] &= g_{\mu\lambda}P_\nu - g_{\nu\lambda}P_\mu \ . \end{aligned} \tag{1.6}$$

which is the customary Poincaré algebra.

1.1.2. Quantum contraction

The actual practical program for obtaining the q–deformation of Poincaré can be summarized in the following six steps :

1) For $q = 1$, start with the simple Lie algebra $so(5, \mathbf{C})$ and its 10 generators M_{AB}. It is convenient to introduce right away some nomenclature. We shall see below the exact relationship between the usual M_{AB} and the generators h_i, e_i which we shall refer to as follows:

$$\begin{array}{ccc} h_1, h_2 & : & \text{Cartan basis} \\ h_1, h_2, e_{\pm 1}, e_{\pm 2} & : & \text{Chevalley basis} \\ h_1, h_2, e_{\pm 1}, e_{\pm 2}, e_{\pm 3}, e_{\pm 4} & : & \text{Cartan} - \text{Weyl basis} \end{array} \tag{1.7}$$

2) To introduce the quantum deformation, q–deform $so(5, \mathbf{C})$ à la Drinfeld–Jimbo in the Chevalley basis to get $U_q(so(5, \mathbf{C}))$. Introduce the more cumbersome but necessary Cartan–Weyl basis.

3) Find the real form of $U_q(so(5, \mathbf{C}))$ corresponding to $U_q(so(3,2))$ or $U_q(so(4,1))$. (The "real form" is a fancy name for some specific linear combinations of the Cartan–Weyl generators with special properties under complex conjugation.) Calculate the (anti) hermitean "physical" generators M_{AB}, which provide the unitary representation.

4) Compute the q–deformation of $so(3,2)$ in the "physical" basis.

5) Contract to get the quantum Poincaré algebra.

6) Check that quantum Poincaré is indeed a Hopf algebra.

1.2. PLAN OF THESE LECTURES

The above introduction motivates the course. My main goal is to describe the above six steps. In order to do so, I will first introduce some notation and basic formalism about quantum groups. Then I will develop the construction program applied to the de Sitter algebra and obtain the quantum Poincaré Hopf algebra. The reader is invited to check the assertions pencil in hand to acquire the necessary technical skill in algebraic manipulations. At the end, our reward will be to explore some physical consequences of the quantum–deformed Poincaré algebra. Accordingly, the lectures are organized as follows:

- Drinfeld–Jimbo deformation in the Chevalley and Cartan–Weyl basis (chapter 2).
- Real forms of (q–deformed) semisimple Lie algebras (chapter 3).
- quantum deformation of the Poincaré algebra using contraction (chapter 4).
- Physical elucubrations (chapter 5).

2. q–Deformation of semisimple Lie algebras

2.1. THE PARADIGMATIC QUANTUM GROUP $U_q(s\ell(2))$

We start with the q–deformation of $s\ell(2)$, which has already many of the essential features of the general case.

The simple Lie algebra $s\ell(2)$, the symmetry algebra of spin $\frac{1}{2}$ systems, has three generators with commutation relations

$$\begin{aligned} [h, e_\pm] &= \pm 2 e_\pm \\ [e_+, e_-] &= h \end{aligned} \tag{2.1}$$

We shall see further ahead (chapter 3) that by taking linear combinations of e_+ and e_- one gets the two real forms $su(2)$ and $su(1,1)$, but for the time being let us remain general and talk about $s\ell(2)$.

The q–deformation of $s\ell(2)$ was proposed long ago [24]; only the third commutation relation is changed :

$$\begin{aligned} [h, e_\pm] &= \pm 2 e_\pm \\ [e_+, e_-] &= [h]_q \end{aligned} \tag{2.2}$$

where the square bracket (with a subindex q, often omitted) stands for the following q–deformation of any quantity (number or operator)

$$[x]_q \equiv \frac{q^x - q^{-x}}{q - q^{-1}} \tag{2.3}$$

We understand the formal parameter q as a complex number. When $q \to 1$, $[x]_q \to x$: the limit $q \to 1$ will often be referred to as classical, in the sense that there is no deformation whatsoever.

Remarkably, in equation (2.2) all positive and negative powers of h appear. Hence (2.2) does not define a Lie algebra in the purest sense, but rather the q–deformation of the universal enveloping algebra of $s\ell(2)$. This q–deformation of $U(s\ell(2))$ is noted, not suprisingly, $U_q(s\ell(2))$. We shall see that it is necessary to include also powers of e_+ and e_-, so that $U_q(s\ell(2))$ contains all the formal power series in h, e_+ and e_- modulo (2.2).

It is useful to introduce

$$k = q^{h/2} \tag{2.4}$$

and replace the first equation (2.2) by the equivalent "commutation" relation

$$k e_\pm k^{-1} = q^{\pm 1} e_\pm \ . \tag{2.5}$$

For most purposes it is enough to consider only polynomials in k, k^{-1}, e_+, e_-.

The fundamental representation of $s\ell(2)$, corresponding to spin $\frac{1}{2}$, is identical to the fundamental representation of $U_q(s\ell(2))$. One obtains higher spin states (where q–dependent terms will appear explicitly) through tensoring, so we must specify how

to introduce the tensor product of representations. For $s\ell(2)$, the generators act trivially on one factor (like a derivative) :

$$\Delta(h) = h \otimes I + I \otimes h \tag{2.6a}$$

$$\Delta(e_\pm) = e_\pm \otimes I + I \otimes e_\pm \tag{2.6b}$$

This operation Δ, which tells us how to act with the generators of $s\ell(2)$ on the tensor product of two representations, is called the co-product or co-multiplication, and I is the identity. The reader is familiar with this co-product from quantum mechanics, where the action of an angular momentum operator on a many-particle system is the sum of its actions on the one-particle states.

For k defined by (2.4), the operation (2.6a) implies

$$\Delta(k) = k \otimes k \tag{2.6c}$$

The co-product Δ must be a homomorphism of Lie algebras, meaning that it must preserve the commutation relations (2.1). Since the commutation relations of $U_q(s\ell(2))$ are different from those of $s\ell(2)$, we should expect that the co-product for $U_q(s\ell(2))$ is also different from that for $s\ell(2)$. This is indeed the case. For $U_q(s\ell(2))$, (2.6a), or equivalently (2.6c), is retained, but (2.6b) is replaced by

$$\Delta(e_\pm) = e_\pm \otimes k + k^{-1} \otimes e_\pm. \tag{2.7}$$

This co-product respects the commutation relations (2.2), i.e. it is a homomorphism of $U_q(s\ell(2))$:

$$\Delta(ab) = \Delta(a)\Delta(b) \qquad \forall a, b \in U_q(s\ell(2)). \tag{2.8}$$

Perhaps the most noteworthy feature of the co-product Δ for a q-deformed algebra $U_q(s\ell(2))$ is that it is not symmetric, *i.e.* that $\Delta(e_\pm)$ acts non-symmetrically on the two factors. Formally, the co-product is a map

$$\Delta : U_q(s\ell(2)) \to U_q(s\ell(2)) \otimes U_q(s\ell(2)). \tag{2.9}$$

Motivated by the concept of Hopf algebra (see next paragraph) we define the co-unit map ε :

$$\begin{aligned} \varepsilon :\ & U_q \to \mathbf{C} \\ & \varepsilon(ab) = \varepsilon(a)\varepsilon(b) \qquad \forall a, b \in U_q(s\ell(2)) \\ & \varepsilon(I) = 1 \\ & \varepsilon(k) = 1 \\ & \varepsilon(h) = 0 \\ & \varepsilon(e_\pm) = 0 \end{aligned} \tag{2.10}$$

and the antipode map S (which at the group level G_q is the q-deformation of the inverse) :

$$\begin{aligned} S :\ & U_q(s\ell(2)) \to U_q(s\ell(2)) \\ & S(ab) = S(b)S(a) \quad \forall a, b \in U_q(s\ell(2)) \\ & S(k) = k^{-1} \\ & S(h) = -h \\ & S(e_\pm) = -q^{\pm 1} e_\pm \ . \end{aligned} \tag{2.11}$$

Because the co-product Δ is not symmetric, one can define a second co-product Δ' as its transpose

$$\begin{aligned} \Delta' &= P \circ \Delta \\ P(a \otimes b) &= b \otimes a \quad a, b \in U_q \end{aligned} \tag{2.12}$$

Explicitly, Δ' of the generators of $U_q(s\ell(2))$ reads as follows:

$$\begin{aligned} \Delta'(k) &= k \otimes k \\ \Delta'(e_\pm) &= e_\pm \otimes k^{-1} + k \otimes e_\pm \end{aligned} \tag{2.13}$$

The two co-products Δ and Δ' are related through a "similarity transformation" by the so-called R-matrix, $R \in U_q(s\ell(2)) \otimes U_q(s\ell(2))$

$$R \Delta R^{-1} = \Delta' \tag{2.14}$$

It is often useful to use the "braid-group $\hat{R}$-matrix" which differs from the above by a permutation:

$$\begin{aligned} \hat{R} &= P \circ R \\ \hat{R} \Delta \hat{R}^{-1} &= \Delta \ . \end{aligned} \tag{2.15}$$

An explicit expression for R can be given [25] in terms of $e_\pm$ and h (but not k) by

$$R = q^{\frac{1}{2}h\otimes h} \sum_{n\geq 0} \frac{(1-q^{-2})^n}{[n]_q!} q^{\frac{(n-n^2)}{2}} q^{\frac{n}{2}h}(e_+)^n \otimes q^{-\frac{n}{2}h}(e_-)^n \tag{2.16}$$

This universal R–matrix can be shown [25] to satisfy the famous Yang–Baxter equation, a key property of two–dimensional integrable models [1]

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12} \tag{2.17}$$

or equivalently

$$\hat{R}_{12}\hat{R}_{23}\hat{R}_{12} = \hat{R}_{23}\hat{R}_{12}\hat{R}_{23} \tag{2.18}$$

where, if the general form of R is

$$R = \sum_i x_i \otimes y^i \quad , \quad x_i, y^i \in U_q(s\ell(2)) \tag{2.19}$$

then the subindexed notation means

$$\begin{aligned} R_{12} &= \sum_i x_i \otimes y^i \otimes I \\ R_{13} &= \sum_i x_i \otimes I \otimes y^i \\ R_{23} &= \sum_i I \otimes x_i \otimes y^i \end{aligned} \tag{2.20}$$

Note that the indices i go up or down without any implication about a dual.

2.2. HOPF ALGEBRA

We now have all the ingredients necessary for defining the general concept of a Hopf algebra, of which $U_q(s\ell(2))$ is a particular example. We shall do it in several steps [26,27].

Let A be an associative algebra with unity I over $\mathbf{C}$. This means that the product

$$\begin{aligned} m: \quad & U_q(s\ell(2)) \otimes U_q(s\ell(2)) \to U_q(s\ell(2)) \\ & (x,y) \mapsto m(x,y) = xy \\ & (x,I) \mapsto x \\ & (I,y) \mapsto y \end{aligned} \tag{2.21}$$

is associative,

$$(xy)z = x(yz) \tag{2.22}$$

A bi-algebra is a set of two algebras sharing the same base set: the first algebra proper (A, m, i) with product m and unit i, the second algebra or co-algebra (A, Δ, ε) with co-product Δ and co-unit ε, defined by :

$$\begin{aligned} m: & A \otimes A \to A \\ & x \otimes y \mapsto xy \\ i: & \mathbf{C} \to A \\ & \lambda \mapsto \lambda I \end{aligned} \tag{2.23}$$

and

$$\begin{aligned} \Delta: & A \to A \otimes A \\ & x \mapsto \sum_i x_i \otimes x^i \\ \varepsilon: & A \to \mathbf{C} \\ & I \mapsto 1 \end{aligned} \tag{2.24}$$

Again, the short–hand above does not mean that x^i belongs to the dual of A.

Just like the product m is taken to be associative, the co-product Δ is assumed to be co-associative, namely

$$(\Delta \otimes I)\Delta = (I \otimes \Delta)\Delta \tag{2.25}$$

or more explicitly

$$\sum_i \Delta(x_i) \otimes x^i = \sum_i x_i \otimes \Delta(x^i) \tag{2.26}$$

The co-product Δ, and by extension the co-algebra (A, Δ, ε), is called co-commutative if $\Delta = \Delta'$.

The two algebras are related if Δ and ε are homomorphisms of the first algebra, that is of (A, m, i):

$$\begin{aligned} & \Delta(xy) = \Delta(x)\Delta(y) \\ & \Delta(I) = I \otimes I \\ & \varepsilon(xy) = \varepsilon(x)\varepsilon(y) \\ & (I \otimes \varepsilon) \circ \Delta = I \\ & (\varepsilon \otimes I) \circ \Delta = I \ . \end{aligned} \tag{2.27}$$

In this case the algebra and co-algebra assemble into a bi-algebra $(A, m, i, \Delta, \varepsilon)$.

A Hopf algebra is a bi-algebra with an antipode S, *i.e.* it is given by the set $(A, m, i, \Delta, \varepsilon, S)$ with the bi-algebra conditions above and furthermore :

$$m \circ (S \otimes I) \circ \Delta(x) = m \circ (I \otimes S) \circ \Delta(x) = i \circ \varepsilon(x) \tag{2.28}$$

which relate all elements of the set. The antipode can be thought of as a co-inverse.

We can verify (2.28) on the example of $U_q(s\ell(2))$. Using (2.7) for Δ and (2.11) for S we first get

$$\begin{aligned} m \circ (S \otimes I) \circ \Delta(e_\pm) &= m \circ (S \otimes I) \circ (e_\pm \otimes k + k^{-1} \otimes e_\pm) \\ &= m \circ (S(e_\pm) \otimes k + S(k^{-1}) \otimes e_\pm) \\ &= m \circ (-q^{\pm 1} e_\pm \otimes k + k \otimes e_\pm) \\ &= -q^{\pm 1} e_\pm k + k e_\pm \ . \end{aligned} \tag{2.29}$$

Similarly, we obtain

$$m \circ (I \otimes S) \circ \Delta(e_\pm) = e_\pm k^{-1} - k^{-1} q^{\pm 1} e_\pm \tag{2.30}$$

Using now Eq. (2.10) and (2.23) for ε and i, we get

$$i \circ \varepsilon(e_\pm) = 0 \ . \tag{2.31}$$

With the help of the commutation relations (2.5) we find the equality of (2.29), (2.30) and (2.31).

Notice that we could have used (2.29) and (2.31) to calculate S from Δ, m, i and ε, and use (2.30) to check the consistency of the Hopf algebra structure.

Finally, one defines the concept of quasi-triangular Hopf algebra to be a Hopf algebra $(A, m, i, \Delta, \varepsilon, S, R)$ with an R–matrix satisfying (2.14), *i.e.*

$$R\Delta = \Delta' R \tag{2.32}$$

and [*cf.* (2.20)]

$$\begin{aligned} (\Delta \otimes I)R &= R_{13} R_{23} \\ (I \otimes \Delta)R &= R_{13} R_{12} \\ (S \otimes I)R &= R^{-1} \ . \end{aligned} \tag{2.33}$$

One can verify [25] that $U_q(s\ell(2))$ is indeed a quasi-triangular Hopf algebra.

Crucially, the Yang–Baxter equation (2.17) for R follows from the requirements of quasi-triangularity, namely equations (2.32) and (2.33).

Quite often, a quasi-triangular Hopf algebra is referred to simply as a quantum group.

2.3. $U_q(g)$ IN THE CHEVALLEY AND CARTAN–WEYL BASIS

Let us sketch the general procedure due to Drinfeld [15] and Jimbo [16] for q–deforming a semisimple Lie algebra g. It also applies to Kac-Moody algebras [28] and, with some minor and obvious modifications, to superalgebras [28, 29].

2.3.1. Lie algebra g In the Chevalley basis, one uses only the Cartan subalgebra $h_1, \cdots, h_\ell$, $\ell =$ rank of g, and the raising and lowering operators $e_{\pm 1}, \cdots, e_{\pm \ell}$ corresponding to the simple roots $\alpha_1, \cdots, \alpha_\ell$. Nevertheless, the commutation relations in the set $(h_i, e_{\pm i})$ must be supplemented by the so-called Serre relations which ensure that g is semisimple [30]: this is the price to pay for using a small basis.

The main new ingredient is the Cartan matrix a_{ij} which gives the scalar products of the simple roots α_i. For the simply-laced algebras A_ℓ, D_ℓ and E_ℓ, the Cartan matrix a_{ij} is symmetric. This is not the case for the non-simply laced algebras B_ℓ, C_ℓ, F_4 and G_2. In the latter case it is convenient to introduce the symmetrized Cartan matrix a^S_{ij}.

The commutation relations for any semisimple Lie algebra or Kac–Moody algebra with symmetrizable Cartan matrix [31] are, in the Chevalley basis,

$$\begin{aligned} [h_i, h_j] &= 0 \\ [h_i, e_{\pm j}] &= \pm a^S_{ij} e_{\pm j} \end{aligned} \tag{2.34}$$

$$[e_{+i}, e_{-j}] = \delta_{ij} h_i \tag{2.35}$$

where

$$i, j = 1, \cdots, \ell = \text{ rank of } g \qquad e_{\pm i} \equiv e_{\pm \alpha_i} \tag{2.36}$$

Remark : One can write the commutation relations in terms of the non-symmetrized Cartan by a simple rescaling of the generators.

Note that equations (2.34) and (2.35) define a semisimple Lie algebra iff the following Serre relations are satisfied :

$$(\mathrm{ad}\, e_{\pm i})^{1-a_{ij}} e_{\pm j} = 0 \quad i \neq j \tag{2.37}$$

where

$$(\mathrm{ad}\, x)y = xy - yx = [x, y] \tag{2.38}$$

and a_{ij} is the non-symmetrized Cartan matrix.

2.3.2. An example: C_2

We shall illustrate the general procedure on the example of the simple algebra $B_2 = so(5) = sp(4) = C_2$, which will be used for the quantum deformation of the Poincaré algebra via contraction (see the Introduction and chapter 4).

The Lie algebra C_2 has two simple roots α_1, α_2 with lengths squared one and two:

$$\alpha_1^2 = 1 \quad , \quad \alpha_2^2 = 2 \ . \tag{2.39}$$

The Cartan matrix is

$$a_{ij} = \frac{2(\alpha_i, \alpha_j)}{\alpha_i^2} = \begin{pmatrix} 2 & -2 \\ -1 & 2 \end{pmatrix} \tag{2.40}$$

The notation (α_i, α_j) stands, as usual, for the scalar product of the simple roots. The symmetrized Cartan matrix is obtained from a_{ij} by left multiplication with the diagonal matrix D:

$$a_{ij}^S = D a_{ij} \quad ; \quad D = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{pmatrix} \tag{2.41}$$

$$a_{ij}^S = (\alpha_i, \alpha_j) \equiv \alpha_{ij} = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} \tag{2.42}$$

The Cartan subalgebra is h_1, h_2 and the "simple" generators are $e_{\pm 1}, e_{\pm 2}$.

The Serre relations for C_2 are the following:

$$\begin{aligned} (\mathrm{ad}\, e_1)^{1-a_{12}} e_2 &= (\mathrm{ad}\, e_1)^3 e_2 = [e_1, [e_1[e_1, e_2]]] \\ &= e_1^3 e_2 - 3e_1^2 e_2 e_1 + 3e_1 e_2 e_1^2 - e_2 e_1^3 = 0 \end{aligned} \tag{2.43}$$

$$\begin{aligned} (\mathrm{ad}\, e_2)^{1-a_{21}} e_1 &= (\mathrm{ad}\, e_2)^2 e_1 = [e_2[e_2, e_1]] \\ &= e_2^2 e_1 - 2e_2 e_1 e_2 + e_1 e_2^2 = 0 \ . \end{aligned} \tag{2.44}$$

The Serre relations have a simple interpretation in the Cartan–Weyl basis where one considers all the generators corresponding to the root diagram, and not only those associated with simple roots. For C_2 we have, in addition to h_i and $e_{\pm i}$ $(i = 1, 2)$, the four generators $e_{\pm 3}$ and $e_{\pm 4}$ corresponding to the remaining non-simple roots $\pm\alpha_3, \pm\alpha_4$. From the root relations

$$\begin{aligned} \alpha_3 &= \alpha_1 + \alpha_2 \\ \alpha_4 &= \alpha_1 + \alpha_3 \end{aligned} \tag{2.45}$$

one defines the associated generators

$$\begin{aligned} e_3 &= [e_1, e_2] \\ e_4 &= [e_1, e_3] \ . \end{aligned} \tag{2.46}$$

The important observation is that $\alpha_1 + \alpha_4$ and $\alpha_2 + \alpha_3$ are not roots, hence

$$\begin{aligned} [e_1, e_4] &= 0 \\ [e_3, e_2] &= 0 \end{aligned} \tag{2.47}$$

which are precisely the Serre relations (2.43) and (2.44), as the reader will check with the help of (2.46).

2.3.3. The quantum group $U_q(g)$

We now come to the q–deformation of g à la Drinfeld–Jimbo. The scheme is a simple generalization of $U_q(sl(2))$.

The commutation relations (2.34) are unchanged :

$$\begin{aligned} [h_i, h_j] &= 0 \\ [h_i, e_{\pm j}] &= \pm a_{ij}^S e_{\pm j} \equiv \pm \alpha_{ij} e_{\pm j} \ . \end{aligned} \tag{2.48}$$

Defining $k_i = q^{\frac{1}{2}h_i}$ these relation can be rewritten as

$$\begin{aligned} k_i k_j &= k_j k_i \\ k_i e_{\pm j} k_i^{-1} &= q^{\pm\frac{1}{2}\alpha_{ij}} e_{\pm j} \end{aligned} \tag{2.49}$$

On the other hand, the commutators (2.35) between raising and lowering operators do change, and they become

$$[e_i, e_{-j}] = \delta_{ij}[h_i]_q \ . \tag{2.50}$$

The technical problem is now to q–deform the Serre relations. Recall that in the limit $q = 1$, that is for classical g, we used the <u>adjoint representation</u> in equations (2.37) and (2.38). So we should try to q–deform the adjoint operator <u>ad</u>. The main property we want to maintain is associativity. The following definitions, rather involved, do the trick [17, 32, 33]:

$$\begin{aligned} \mathrm{ad}^+(x) &\stackrel{\mathrm{def}}{=} (m_L \otimes m_R) \circ (I \otimes S) \circ \Delta(x) \\ \mathrm{ad}^-(x) &\stackrel{\mathrm{def}}{=} (m_L \otimes m_R) \circ (S \otimes I) \circ \Delta(x) \ . \end{aligned} \tag{2.51}$$

Here, the left and right multiplications m_L resp. m_R are given by

$$(m_L \otimes m_R) \circ (x \otimes y)(z) \stackrel{\mathrm{def}}{=} xzy \ . \tag{2.52}$$

In general, the co-product Δ is a sum of terms, as in equations (2.6) and (2.7) for the simplest case of $U_q(s\ell(2))$, which can be formally written as

$$\Delta(x) = \sum_\mu x_\mu \otimes x^\mu \equiv x_\mu \otimes x^\mu \ . \tag{2.53}$$

Using (2.52) and (2.53), ad^+x applied on y gives

$$((m_L \otimes m_R)(I \otimes S)\Delta(x))(y) = (m_L \otimes m_R)(x_\mu \otimes S(x^\mu))(y) = x_\mu y S(x^\mu) \tag{2.54}$$

Hence

$$\begin{aligned} (\mathrm{ad}^+x)(y) &= x_\mu y S(x^\mu) \\ (\mathrm{ad}^-x)(y) &= S(x_\mu) y x^\mu \ . \end{aligned} \tag{2.55}$$

<u>Associativity</u> of $\mathrm{ad}^\pm$ is defined by

$$\begin{aligned} (\mathrm{ad}^+xy)(z) &= (\mathrm{ad}^+x)((\mathrm{ad}^+y)(z)) \\ (\mathrm{ad}^-xy)(z) &= (\mathrm{ad}^-y)((\mathrm{ad}^-x)(z)) \end{aligned} \tag{2.56}$$

from which follows the <u>representation property</u> :

$$(\mathrm{ad}^\pm[x,y])(z) = \pm\{(\mathrm{ad}^\pm x)((\mathrm{ad}^\pm y)(z)) - (\mathrm{ad}^\pm y)((\mathrm{ad}^\pm x)(z))\} \ . \tag{2.57}$$

The proof of the associativity of the adjoint goes as follows :

$$\begin{aligned} (\mathrm{ad}^+xy)(z) &= (m_L \otimes m_R) \circ (I \otimes S) \circ \Delta(xy)(z) \\ \Delta(xy) &= \Delta(x) \circ \Delta(y) = (x_\mu \otimes x^\mu) \circ (y_\nu \otimes y^\nu) = x_\mu y_\nu \otimes x^\mu y^\nu \\ (I \otimes S)\Delta(xy) &= x_\mu y_\nu \otimes S(x^\mu y^\nu) = x_\mu y_\nu \otimes S(y^\nu) S(x^\mu) \\ (\mathrm{ad}^+xy)(z) &= x_\mu y_\nu z S(y^\nu) S(x^\mu) = x_\mu (\mathrm{ad}^+y)(z) S(x^\mu) = \\ &= (\mathrm{ad}^+x)((\mathrm{ad}^+y)(z)) \ . \end{aligned} \tag{2.58}$$

Clearly, in order to compute the adjoint action we need the explicit forms of the co-product and the antipode. For g semisimple, in the Chevalley basis, the <u>co-product</u> Δ of $U_q(g)$ is very similar to the Δ of $U_q(s\ell(2))$:

$$\begin{aligned} \Delta(h_i) &= h_i \otimes I + I \otimes h_i \\ \Delta(k_i) &= k_i \otimes k_i \\ \Delta(e_{\pm i}) &= e_{\pm i} \otimes k_i + k_i^{-1} \otimes e_{\pm i} \ . \end{aligned} \tag{2.59}$$

The <u>co-unit</u> is again

$$\begin{aligned} \varepsilon(h_i) &= \varepsilon(e_{\pm i}) = 0 \\ \varepsilon(k_i) &= \varepsilon(I) = 1 \ . \end{aligned} \tag{2.60}$$

The relation $m \circ (S \otimes I) \circ \Delta(x) = m \circ (I \otimes S) \circ \Delta(x) = i \circ \varepsilon(x)$ gives for the <u>antipode</u> S :

$$\begin{aligned} S(h_i) &= -h_i \\ S(k_i) &= k_i^{-1} \\ S(e_{\pm i}) &= -k_i e_{\pm i} k_i^{-1} = -q^{\pm\frac{1}{2}\alpha_{ii}} e_{\pm i} \ . \end{aligned} \tag{2.61}$$

We now have all the ingredients to calculate $\mathrm{ad}^\pm$. Using (2.55), (2.59) and (2.61) we get

$$\begin{aligned} (\mathrm{ad}^\pm h_i)(y) &= [h_i, y] \\ (\mathrm{ad}^+e_{\pm i})(y) &= (e_{\pm i}y - k_i^{-1} y k_i e_{\pm i}) k_i^{-1} \\ (\mathrm{ad}^-e_{\pm i})(y) &= k_i(y e_{\pm i} - e_{\pm i} k_i^{-1} y k_i) \ . \end{aligned} \tag{2.62}$$

Let us specialize (2.62) to $y = e_{\pm j}$:

$$\begin{aligned}(\mathrm{ad}^+ e_{\pm i})(e_{\pm j}) &= (e_{\pm i}e_{\pm j} - q^{\mp\frac{1}{2}\alpha_{ij}} e_{\pm j}e_{\pm i})k_i^{-1} \\ (\mathrm{ad}^+ e_{\pm i})(e_{\mp j}) &= (e_{\pm i}e_{\mp j} - q^{\pm\frac{1}{2}\alpha_{ij}} e_{\mp j}e_{\pm i})k_i^{-1} \\ (\mathrm{ad}^- e_{\pm i})(e_{\pm j}) &= k_i(e_{\pm j}e_{\pm i} - q^{\mp\frac{1}{2}\alpha_{ij}} e_{\pm i}e_{\pm j}) \ .\end{aligned} \tag{2.63}$$

In order to get the q–Serre relations, we could try to replace in Eq. (2.37) ad by $\mathrm{ad}^{\pm}$. It turns out that the factors k_i and k_i^{-1} in (2.62) complicate matters. We can eliminate them by a change of variables which leaves the algebra untouched but modifies the co-product and the antipode :

$$\begin{aligned}E_i &\stackrel{\mathrm{def}}{=} e_i k_i^{-1} \\ E_{-i} &\stackrel{\mathrm{def}}{=} k_i e_{-i} \\ H_i &\stackrel{\mathrm{def}}{=} h_i \ .\end{aligned} \tag{2.64}$$

Hence, from (2.49) and (2.50) we get

$$\begin{aligned}k_i E_{\pm j} k_i^{-1} &= q^{\pm\frac{1}{2}\alpha_{ij}} E_{\pm j} \\ [E_i, E_{-j}] &= \delta_{ij}[H_i]_q\end{aligned} \tag{2.65}$$

which have the same structure as the commutation relations (2.49) and (2.50) for $e_{\pm i}$. The generators $E_{\pm i}$ are the natural ones in the contour representation of quantum groups [3, 4].

From (2.59) and (2.61) we obtain

$$\begin{aligned}\Delta(E_i) &= E_i \otimes I + k_i^{-2} \otimes E_i \\ \Delta(E_{-i}) &= E_{-i} \otimes k_i^2 + I \otimes E_{-i} \\ S(E_i) &= -k_i^2 E_i \\ S(E_{-i}) &= -E_{-i}k_i^{-2} \ .\end{aligned} \tag{2.66}$$

From (2.53), (2.55) and (2.66) we calculate

$$\begin{aligned}(\mathrm{ad}^+ E_i)(E_{\pm j}) &= E_i E_{\pm j} - q^{\mp\alpha_{ij}} E_{\pm j} E_i \equiv [E_i, E_{\pm j}]_{q'} \\ (\mathrm{ad}^- E_{-i})(E_{\pm j}) &= E_{\pm j} E_{-i} - q^{\mp\alpha_{ij}} E_{-i} E_{\pm j} \equiv [E_{\pm j}, E_{-i}]_{q'}\end{aligned} \tag{2.67}$$

where $q' = q^{\pm 1}$. The right-hand side is called a q-commutator.

Thanks to the above change of basis, one can write the <u>q–Serre</u> relations in the simple form

$$(\mathrm{ad}^+ E_{\pm i})^{1-a_{ij}}(E_{\pm j}) = 0 \quad i \neq j \tag{2.68}$$

For conciseness, from now on we display the explicit expressions only for ad^+; it is left as an exerciser for the reader to carry on with ad^-. For actual computations, the following iterative formula is useful:

$$(\mathrm{ad}^+ E_i)^{P+1}(E_j) = E_i(\mathrm{ad}^+(E_i))^P(E_j) - q^{-P\alpha_{ii}-\alpha_{ij}}(\mathrm{ad}^+(E_i))^P(E_j)E_i \tag{2.69}$$

so that, for instance,

$$\begin{aligned}(\mathrm{ad}^+ E_i)^2(E_j) &= [E_i[E_i, E_j]_q]_q = \\ &= E_i[E_i, E_j]_q - q^{-\alpha_{ii}}q^{-\alpha_{ij}}[E_i, E_j]_q E_i \ .\end{aligned} \tag{2.70}$$

To establish these formulae, we use (2.55),(2.66) and (2.67) to find

$$(\mathrm{ad}^+ E_i)(E_j E_k) = E_i(E_j E_k) - q^{-\alpha_{ij}}q^{-\alpha_{ik}}(E_j E_k)E_i \ . \tag{2.71}$$

Notice that this can also be written :

$$(\mathrm{ad}^+ E_i)(E_j E_k) = (\mathrm{ad}^+ E_i)(E_j)E_k + q^{-\alpha_{ij}} E_j(\mathrm{ad}^+ E_i)(E_k) \ . \tag{2.72}$$

Hence, ad^+ can be considered as a <u>q–derivation</u> obeying a <u>q–Leibniz</u> rule [34, 35].

For the simple Lie algebra C_2 we have seen [recall equations (2.40) and (2.42)]

$$a_{ij} = \begin{pmatrix} 2 & -2 \\ -1 & 2 \end{pmatrix} \qquad \alpha_{ij} = a_{ij}^S = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} . \tag{2.73}$$

Hence, the q–Serre relations for $U_q(C_2)$ are, using (2.67) to (2.70) :

$$\begin{aligned}E_2^2 E_1 - (q + q^{-1})E_2 E_1 E_2 + E_1 E_1^2 &= 0 \\ E_1^3 E_2 - (q + 1 + q^{-1})(E_1^2 E_2 E_1 - E_1 E_2 E_1^2) - E_2 E_1^3 &= 0\end{aligned} \tag{2.74}$$

which reduce to (2.43) and (2.44) for $q = 1$.

Notice that the q–Serre relations are invariant under $q \leftrightarrow q^{-1}$. The same is true for the whole algebra (see Eq. (2.65) and also (2.48) to (2.50)), but not for the co-algebra.

2.3.4. Cartan–Weyl basis

As was the case for $q = 1$, the Serre relations have a much simpler interpretation in the Cartan-Weyl basis. This is defined in a similar way for $q = 1$ and $q \neq 1$, replacing the operator ad by $\mathrm{ad}^{\pm}$. But since q enters in a non-symmetric way into the definition of $\mathrm{ad}^{\pm}$, it is necessary to order the positive (negative) roots in a specific way [36, 28]. Namely :

- Definition 1 : The system Δ_+ of positive roots is in normal–order if each non-simple root $\gamma = \alpha + \beta \in \Delta_+$, where $\alpha \neq \lambda\beta$, $\alpha, \beta \in \Delta_+$, is written between α and β.

The q–analog of the Cartan-Weyl basis is constructed using the following inductive algorithm [37].

- Definition 2 : Fix some normal ordering in Δ_+. Let $\alpha, \beta, \gamma \in \Delta_+$ be pairwise non-collinear roots, such that $\gamma = \alpha + \beta$. Suppose, moreover, that between α and β (in the normal ordering at hand) there are no other roots α' and β' such that $\alpha' + \beta' = \gamma$.

Then, if $E_{\pm\alpha}$ and $E_{\pm\beta}$ have already been constructed, we set (cf. (2.82))

$$E_\gamma = [E_\alpha, E_\beta]_q \quad , \quad E_{-\gamma} = [E_{-\beta}, E_{-\alpha}]_{q^{-1}} \tag{2.75}$$

and we get the commutation relations

$$\begin{aligned} [h_i, E_\gamma] &= (\alpha_i, \gamma) E_\gamma \\ [E_\gamma, E_{-\gamma}] &= a_\gamma [H_\gamma]_q \\ H_\gamma &= H_\alpha + H_\beta \end{aligned} \tag{2.76}$$

where a_γ is a function of q. We say that $\alpha < \beta$ if α is located to the left of β in the normal–ordered Δ_+.

Now the q–Serre relations are equivalent to the statement that, if $\alpha + \beta$ is not a root, the q–commutator (2.82) is zero :

$$[E_\alpha, E_\beta]_q = 0 \quad , \quad \alpha + \beta \notin \Delta_+ \quad , \quad \alpha < \beta \ . \tag{2.77}$$

To complete the Cartan–Weyl basis in the co-algebra sector, one uses the homomorphism and antihomomorphism properties applied to (2.75) :

$$\begin{aligned} \Delta(xy) &= \Delta(x)\Delta(y) \\ \varepsilon(xy) &= \varepsilon(x)\varepsilon(y) \\ S(xy) &= S(y)S(x) \ . \end{aligned} \tag{2.78}$$

The diligent reader will check that

$$m \circ (S \otimes I) \circ \Delta = m \circ (I \otimes S) \circ \Delta = i \circ \varepsilon \tag{2.79}$$

2.3.5. The example of C_2

As an example, consider again $U_q(C_2)$. Choose the normal order : α_1, α_4, α_3, α_2 where $\alpha_4 = \alpha_1 + \alpha_3$, $\alpha_3 = \alpha_1 + \alpha_2$.

First define, according to (2.67) and (2.75),

$$\begin{aligned} E_3 &\stackrel{\text{def}}{=} [E_1, E_2]_q = E_1E_2 - q^{-\alpha_{12}} E_2E_1 = E_1E_2 - qE_2E_1 \\ E_4 &\stackrel{\text{def}}{=} [E_1, E_3]_q = E_1E_3 - q^{-\alpha_{13}} E_3E_1 = [E_1, E_3] \end{aligned} \tag{2.80}$$

recalling that $\alpha_{ij} = (\alpha_i, \alpha_j)$.

Then, since $\alpha_1 + \alpha_4$, $\alpha_4 + \alpha_3$, $\alpha_3 + \alpha_2$ are not roots :

$$\begin{aligned} [E_1, E_4]_q &= [E_1[E_1, [E_1, E_2]_q]_q]_q = 0 \\ [E_4, E_3]_q &= 0 \\ [E_3, E_2]_q &= [[E_1, E_2]_q, E_2]_q = [E_2, [E_2, E_1]]_q]_q = 0 \ . \end{aligned} \tag{2.81}$$

these are the same equations as (2.74), (the second equation is a consequence), where the q-commutator is defined by

$$[E_\alpha, E_\beta]_q = E_\alpha E_\beta - q^{-(\alpha,\beta)} E_\beta E_\alpha \ . \tag{2.82}$$

Similarly,

$$\begin{aligned} E_{-3} &= [E_{-2}, E_{-1}]_{q^{-1}} \\ E_{-4} &= [E_{-3}, E_{-1}]_{q^{-1}} \end{aligned} \tag{2.83}$$

and

$$[E_{-2}, E_{-3}]_{q^{-1}} = [E_{-3}, E_{-4}]_{q^{-1}} = [E_{-4}, E_{-1}]_{q^{-1}} = 0 \ . \tag{2.84}$$

It turns out that it is possible to obtain the q–Serre relations and the Cartan–Weyl basis in terms of the generators e_i substituting E_i by e_i in the equations (2.74) to (2.84). The reader is invited to verify this.

3. Real forms

3.1. DEFINITION OF REAL FORMS

The simple Lie algebra $s\ell(2)$ is generated by h and $e_\pm$. It is a vector space over complex numbers. In physics one is often interested in the spin operators $S_i (i = 1, 2, 3)$ which are hermitean ($S^+ = S$) and generate the unitary representations (finite rotations) $U = \exp(iS)$. In order to get S_i in the framework of $s\ell(2)$, one should first define the hermitean conjugation $+$ acting on the generators of $s\ell(2)$ and then find the complex linear combinations of $h, e_\pm$ which are hermitean. The result is called a real form of $s\ell(2)$ [38]. A real form is a vector space over real numbers. (The representation matrices may be, however, complex). There are different choices of conjugations, and therefore different real forms. One of them is compact ($su(2)$ or $so(3)$ in our example) the others are non-compact ($su(1,1)$ or $so(2,1)$ or $s\ell(2,\mathbb{R})$). The numbers inside the parentheses (,) refer to the signature of the metric.

The situation is similar for the q–deformation $U_q(g)$, with the difference that some real forms which are equivalent for g, are not so for $U_q(g)$. For example $U_q(su(1,1)) \neq U_q(s\ell(2,\mathbb{R}))$, as shown below. This is due to the fact that the co-product Δ now contains the complex number q. Two main choices are possible : $q \in \mathbb{R}$ or $|q| = 1$. (The choice $q \in i\mathbb{R}$ is not essentially new [39]). We shall show that $U_q(su(1,1))$ corresponds to $q \in \mathbb{R}$, whereas $U_q(s\ell(2,\mathbb{R}))$ corresponds to $|q| = 1$.

We define the conjugation $+$ as a morphism of the algebra, by its action on the generators x, y, etc., which satisfies

(i) It is an involution:

$$(x^+)^+ = I \ . \tag{3.1}$$

(ii) It leaves the commutation relations invariant.

(iii) It is an (anti) involution:

$$(xy)^+ = y^+x^+ \tag{3.2}$$

like the hermitean conjugation and is called standard.

One can also define a non-standard conjugation $*$ by

(iii')

$$(xy)^* = x^*y^* \tag{3.3}$$

This is like the ordinary complex conjugation.

(iv) It acts like a complex conjugation on the number field of the Lie algebra, resp. enveloping algebra vector space.

For $U_q(g)$ one requires in addition for the co-product the following

(v) Standard conjugation

$$\Delta(x)^+ = \Delta(x^+) \ . \tag{3.4}$$

(v') Non-standard conjugation

$$\Delta(x)^\oplus = \Delta'(x^\oplus) \ . \tag{3.5}$$

The choices (iii), (iii') and (v), (v') allow four possibilities [40, 41, 42]. The action on the antipode S is then fixed, namely :

$$(xy)^+ = y^+x^+ \quad ; \quad \Delta(x)^+ = \Delta(x^+) \quad ; \quad (S \circ +)^2 = I \tag{3.6}$$

$$\begin{array}{lll} (xy)^* = x^*y^* & \Delta(x)^* = \Delta'(x^*) & (S \circ *)^2 = I \\ (xy)^\oplus = y^\oplus x^\oplus & \Delta(x)^\oplus = \Delta'(x^\oplus) & S \circ \oplus = \oplus \circ S \\ (xy)^\circledast = x^\circledast y^\circledast & \Delta(x)^\circledast = \Delta(x^\circledast) & S \circ \circledast = \circledast \circ S \ . \end{array} \tag{3.7}$$

The conjugation (3.6) will be called standard, the others, non-standard.

The standard involution $+$ is used by [13, 14, 39] and in the second version of quantum Poincaré [9]. Non-standard conjugations $\oplus$ are natural in conformal field theory when q is a root of unity [43, 44, 45], and they were also used in the first version of quantum Poincaré [8, 46].

3.2. STANDARD REAL FORMS OF $s\ell(2)$ AND $U_q(s\ell(2))$

Recall the commutation relations (2.2) of $U_q(s\ell(2))$:

$$[h, e_\pm] = \pm 2 e_\pm$$
$$[e_+, e_-] = [h]_q = \frac{q^h - q^{-h}}{q - q^{-1}} \ . \tag{3.8}$$

For the standard involution $+$,

$$[x, y]^+ = -[x^+, y^+] \ . \tag{3.9}$$

Therefore, the following conjugations leave (3.8) invariant, for $q = 1$ and $q \neq 1$

I

$$h^+ = h \Rightarrow (e_\pm)^+ = \lambda e_\mp \ , \ \lambda = \pm 1 \tag{3.10}$$

II

$$h^+ = -h \Rightarrow (e_\pm)^+ = \varepsilon e_\pm \ , \ \ \varepsilon = -1 \tag{3.11}$$

whereas $\varepsilon = +1$ gives nothing new. For $|q| = 1$, we get $q^* = q^{-1}$, which does not affect $[h]_q$.

It is elementary to find the linear combinations y which are antihermitean, such that $U = \exp(y)$ is unitary.

Next, we shall give the action of the conjugation on the generators $e_{\pm i}$ and h_i, and we shall display the antihermitean generators y_i. We shall also compute the quadratic Casimir C_2 in the limit $q = 1$, from which we shall derive the signature of the metric, and thereby deduce whether the real form is compact or non-compact. In this way we get [47] the three real forms of $U_q(s\ell(2))$:

$\mathbf{I}_1$

$$h^+ = h \ ; \ (e_\pm)^+ = e_\mp \tag{3.12}$$

$$y_1 = -\frac{i}{2}(e_+ + e_-) \ ; \ y_2 = \frac{1}{2}(e_+ - e_-) \ ; \ y_3 = \frac{i}{2} h \tag{3.13}$$

$$\begin{aligned} [y_1, y_2] &= \frac{i}{2}[-2iy_3]_q \\ [y_2, y_3] &= y_1 \\ [y_3, y_1] &= y_2 \end{aligned} \tag{3.14}$$

Since we define the signature of the metric according to the form the quadratic Casimir takes in the limit $q \to 1$, and in this case, $C_2 = y_1^2 + y_2^2 + y_3^2$ is invariant for $q = 1$ (remember $[x]_q = x$ for $q = 1$), we say that the metric in case I_1 is $(+++)$, and it follows that all the generators are compact. For $q \in \mathbb{R}$ the algebra is $U_q(su(2)) = U_q(so(3))$ (see equ. (3.22)).

$\mathbf{I}_2$

$$h^+ = h \ \ ; \ \ (e_\pm)^+ = -e_\mp \tag{3.15}$$

$$y_1 = \frac{1}{2}(e_+ + e_-) \ ; \ y_2 = \frac{-i}{2}(e_+ - e_-) \ ; \ y_3 = \frac{i}{2} h \tag{3.16}$$

$$\begin{aligned} [y_1, y_2] &= \frac{i}{2}[-2iy_3]_q \\ [y_2, y_3] &= -y_1 \\ [y_3, y_1] &= -y_2 \ . \end{aligned} \tag{3.17}$$

The invariant for $q = 1$ is $-y_1^2 - y_2^2 + y_3^2$, with metric $(--+)$, the algebra is non-compact. It is called, for $q \in \mathbb{R}$, $U_q(su(1,1))$.

II

$$h^+ = -h \ \ ; \ \ (e_\pm)^+ = -e_\pm \tag{3.18}$$

$$y_1 = \frac{1}{2}(e_+ + e_-) \ , \ y_2 = \frac{1}{2}(e_+ - e_-) \ ; \ y_3 = \frac{1}{2} h \ . \tag{3.19}$$

All the coefficients are real. The algebra is called $U_q(s\ell(2, \mathbb{R}))$ and $|q| = 1$ (see equ. (3.23))

$$\begin{aligned} [y_1, y_2] &= -\frac{1}{2}[2y_3]_q \\ [y_2, y_3] &= -y_1 \\ [y_3, y_1] &= y_2 \ . \end{aligned} \tag{3.20}$$

The invariant for $q = 1$ is $-y_1^2 + y_2^2 - y_3^2$, the metric is thus $(-+-)$, and the algebra is non-compact.

In the three situations above, the numbering of the generators is arbitrary, therefore $s\ell(2, \mathbb{R})$ is equivalent to $su(1,1)$. But for $q \neq 1$, the co-product will distinguish between the two real forms. Actually, even for $q = 1$, there is a slight difference

between cases I_2 and II, namely in the former y_3 is compact, and y_1 and y_2 non-compact whereas in the latter y_2 is compact [20, 48]. Consider the standard action on the co-product, where we require $\Delta(x)^+ = \Delta(x^+)$. Since

$$\Delta(e_\pm) = e_\pm \otimes q^{h/2} + q^{-h/2} \otimes e_\pm \tag{3.21}$$

there are, again, two cases

I

$$h^+ = h \Rightarrow q \in \mathbb{R} \tag{3.22}$$

II

$$h^+ = -h \Rightarrow |q| = 1 \tag{3.23}$$

This shows that $U_q(su(1,1)) \neq U_q(s\ell(2,\mathbb{R}))$ for $q \neq 1$.

The choices (3.22) and (3.23) describe standard real Hopf algebras. The Hopf algebras with reality conditions using non standard involutions (see (3.7)) define non standard real Hopf algebras.

3.3. STANDARD REAL FORMS FOR $U_q(g)$

For g semisimple, the general discussion in the Chevalley basis has been given by Twietmeyer [39]. Dobrev [20] gives a canonical procedure in the Cartan–Weyl basis, with emphasis on the (non)-compactness of the real forms.It is different from the one followed by [39]. One may check if he gets standard real forms in all cases.

The features are very similar to $U_q(s\ell(2))$, with the additional freedom of choosing an involutive automorphism η of the Dynkin diagram D. For example for $A_2 = su(3)$, one can interchange the two simple roots, *i.e.* $\eta(1) = 2$, $\eta(2) = 1$. For $A_3 = su(4)$, $\eta(1) = 3$, $\eta(2) = 2$, $\eta(3) = 1$. For $D_4 = so(8)$, three roots can be interchanged (triality). For B_n and C_n there is no such symmetry.

The result is the following, in the Chevalley basis [39]. There are two main categories of real forms

I

$$q \in \mathbb{R}\ ;\ h_i^+ = h_{\eta(i)}\ ;\ e_{\pm i}^+ = \lambda_i e_{\mp\eta(i)} \tag{3.24}$$

where $\eta \in Aut(D)$, $\eta^2 = I$, $\lambda_i = 1$ for $\eta(i) \neq i$, $\lambda_i = \pm 1$ for $\eta(i) = i$.

II

$$|q| = 1\ ;\ h_i^+ = -h_{\eta(i)}\ ;\ e_{\pm i}^+ = -e_{\pm\eta(i)} \tag{3.25}$$

where $\eta \in Aut(D)$, $\eta^2 = I$ for q not a root of unity, whereas if q is an ℓ-th root of unity, then η is allowed to be a permutation of the Dynkin diagram (provided ℓ divides $\alpha_{ij} - \alpha_{\eta(i)\eta(j)}$ for all i and j).

A third possible case, with $q \in \sqrt{-1}\mathbb{R}$, is essentially equivalent to I [39].

For more details on the special role played by the non-equivalent Cartan subalgebras see [39, 48] and [20]. The latter classifies the (non-) compact Cartan generators.

3.4. REAL FORMS OF $U_q(so(5,\mathbb{C})) = U_q(C_2)$

We shall need these real forms for the discussion of quantum Poincaré. We shall also consider non-standard conjugations and state which metrics one obtains. For details see [40, 9, 49].

There are again two standard forms, for which

$$\begin{aligned} (xy)^+ &= y^+x^+ \\ \Delta(x)^+ &= \Delta(x^+) \\ (S \circ +)^2 &= I \end{aligned} \tag{3.26}$$

In the Chevalley basis :

I

$$q \in \mathbb{R}\ ;\ h_i^+ = h_i\ ;\ e_{\pm i}^+ = \lambda_i e_{\mp i}\ ;\ \lambda_i = \pm 1\ . \tag{3.27}$$

For different values of λ_i one gets the metrics related to $U_q(so(5))$, $U_q(so(4,1))$ and $U_q(so(3,2))$. The last one will be contracted to quantum Poincaré (see chapter 4 where the hermitean linear combinations are explicitly given). This conjugation leads outside the Cartan-Weyl basis [40, 9, 49] but of course inside $U_q(g)$.

II

$$|q| = 1\ ;\ h_i^+ = -h_i\ ;\ e_{\pm i}^+ = \lambda_i e_{\pm i}\ . \tag{3.28}$$

Whatever values $\lambda_i = \pm 1$ are chosen, one always gets $U_q(so(3,2))$. This conjugation can be extended to the Cartan-Weyl basis.

We now list three non-standard conjugations which can all be extended to the Cartan-Weyl basis. For notations see paragraph (3.1)

a)

$$q \in \mathbb{R} \; ; \; h_i^* = -h_i \; ; \; e_{\pm i}^* = \lambda_i e_{\mp i} \tag{3.29}$$

b)

$$|q| = 1 \; ; \; h_i^{\oplus} = h_i \; ; e_{\pm i}^{\oplus} = \lambda_i e_{\mp i} \tag{3.30}$$

c)

$$q \in \mathbb{R} \; ; \; h_i^{\circledast} = h_i \; ; \; e_{\pm i}^{\circledast} = \lambda_i e_{\pm i} \; . \tag{3.31}$$

It turns out that if one plays with the different values $\lambda_i = \pm 1$ one gets the same three metrics in cases (a) and (b) as in I. The common features of these conjugations is to interchange raising and lowering operators. In cases (c) and II one only gets $U_q(so(3,2))$. Case (b) was used in [8](see also [45, 46]).

4. Deformation of the Poincaré algebra

We now want to give a physically interesting non-trivial example of a deformation of a non-semisimple Lie algebra, the Poincaré (inhomogeneous Lorentz) algebra. The procedure was announced in chapter 1. We recall the main steps :

1) q–deformation of $so(5, \mathbf{C})$, following Drinfeld-Jimbo.
2) Cartan Weyl-basis.
3) Choice of the real form $U_q(so(3,2))$.
4) Choice of physical generators $\bar{M}_{AB}$.
5) Contraction.
6) Proof that we obtained a Hopf algebra.
7) Casimir operators and representations.

The contraction procedure will force us to replace the dimension-less deformation parameter q by the parameter κ, with the dimension of an energy. Physically this will imply that Lorentz invariance is broken. Rotation and translation invariance will

be maintained at the algebra level, however the co-product for space translations will be non-trivial.

For a fixed value of κ, the amount of Poincaré invariance breaking will increase with energy. A new κ–Poincaré invariance will emerge. For $\kappa \to \infty$, the usual relativistic (classical) invariance is restored.

A realization of κ–Poincaré with derivatives acting on a space-time with commutative co-ordinates will be given. The energy–dependent breaking of Lorentz invariance manifests itself in the appearance of finite–difference time operators. Finite κ–Lorentz boosts are given by elliptic, instead of hyperbolic, functions. The κ–deformed Klein-Gordon and Dirac equations will be displayed.

4.1. q–DEFORMATION OF $so(5, \mathbf{C})$

We start from the commutation relations (2.49),(2.50) for $U_q(so(5, \mathbf{C}))$ in the Chevalley basis, with $(i,j) = (1,2)$:

$$\begin{aligned} k_i e_{\pm j} k_i^{-1} &= q^{\pm \frac{1}{2} \alpha_{ij}} e_{\pm j} \\ [e_i, e_{-j}] &= \delta_{ij} [h_i]_q \; . \end{aligned} \tag{4.1}$$

where $k_i = q^{\frac{1}{2} h_i}$. The Cartan matrix a_{ij} and its symmetrized form $\alpha_{ij} = (\alpha_i, \alpha_j)$, *i.e.* the scalar product of simple roots, are given by

$$a_{ij} = \begin{pmatrix} 2 & -2 \\ -1 & 2 \end{pmatrix} \qquad \alpha_{ij} = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} . \tag{4.2}$$

The non-trivial co-product Δ and the antipode S are (see Eqs. (2.59) and (2.61)

$$\begin{aligned} \Delta(e_{\pm i}) &= e_{\pm i} \otimes k_i + k_i^{-1} \otimes e_{\pm i} \\ S(e_{\pm i}) &= -q^{\frac{\pm \alpha_{ii}}{2}} e_{\pm i} \end{aligned} \tag{4.3}$$

It turns out that the Cartan–Weyl basis can be defined as in (2.80) by

$$\begin{aligned} e_3 &\stackrel{\text{def}}{=} [e_1, e_2]_q = e_1 e_2 - q^{-\alpha_{12}} e_2 e_1 = e_1 e_2 - q e_2 e_1 \\ e_4 &\stackrel{\text{def}}{=} [e_1, e_3]_q = e_1 e_3 - q^{-\alpha_{13}} e_3 e_1 = [e_1, e_3] \end{aligned} \tag{4.4}$$

and

$$\begin{aligned} e_{-3} &\stackrel{\text{def}}{=} [e_{-2}, e_{-1}]_{q^{-1}} = e_{-2} e_{-1} - q^{-1} e_{-1} e_{-2} \\ e_{-4} &\stackrel{\text{def}}{=} [e_{-3}, e_{-1}]_{q^{-1}} = [e_{-3}, e_{-1}] \end{aligned} \tag{4.5}$$

Notice the opposite normal order for the lowering operators.

The q–Serre relations are given by the analogue of (2.81) by

$$[e_1, e_4]_q = [e_3, e_2]_q = 0 \tag{4.6}$$

which agrees with (2.74) using the definitions (2.64).

Similarly.

$$[e_{-2}, e_{-3}]_{q^{-1}} = [e_{-3}, e_{-4}]_{q^{-1}} = 0 \tag{4.7}$$

4.2. REAL FORMS OF $U_q(so(5, \mathbf{C}))$

There are different real forms which can, a priori, be contracted to Poincaré. One needs at least one Minkowski metric, which still leaves the choice of $so(3,2)$ or $so(4,1)$. Furthermore, one can permute the + and − signs in the metric, for example $(+---+)$ or $(--+-+)$. For $q = 1$, these permutations give essentially the same result, except for some subtleties [48].

For $q \neq 1$ this is not the case. Indeed, the Cartan subalgebra generators h_i play a special role since they have undeformed commutation relations and furthermore enter in the definition of the co-product. The relation between Cartan subalgebra and physical generators will depend on a given permutation. In [8, 9] we imposed the requirement that the rotation subalgebra $so(3)$ and its co–product remain undeformed after contraction. This is satisfied by the choice of the real form $so(3,2)$ with signature $(+---+)$, and the selection of h_1 and $e_{\pm 1}$ for the generators of space rotations $so(3)$, as will be shown below.

There are still several possibilities for obtaining this real form [40, 49]. In [8] and [9] we chose the hermitean conjugation +, which is standard in the algebra sector. In [8] we chose $|q| = 1$, which is non-standard in the co-algebra sector, while the conjugate generators remained inside the Cartan–Weyl basis. More explicitly, we chose

$$\begin{aligned} &|q| = 1 \quad ; \quad h_i^{\oplus} = h_i \ , \ i = 1,2 \ ; \\ &e_1^{\oplus} = e_{-1} \quad ; \quad e_2^{\oplus} = -e_{-2} \\ &e_3^{\oplus} = -e_{-3} \quad ; \quad e_4^{\oplus} = -e_{-4} \end{aligned} \tag{4.8}$$

In [9] we chose $q \in \mathbb{R}$, which is standard in both sectors, *i.e.*

$$\begin{aligned} &q \in \mathbb{R} \quad ; \quad h_i^{+} = h_i \ , \ i = 1,2 \\ &e_1^{+} = e_{-1} \quad ; \quad e_2^{+} = -e_{-2} \\ &e_3^{+} = q\tilde{e}_{-3} \quad ; \quad e_4^{+} = -q\tilde{e}_{-4} \end{aligned} \tag{4.9}$$

the generators $\tilde{e}_i$ being outside the Cartan-Weyl basis for $q \neq 1$

$$\begin{aligned} &\tilde{e}_3 = e_2 e_1 - q e_1 e_2 \ ; \ \tilde{e}_4 = [\tilde{e}_3, e_1] \\ &\tilde{e}_{-3} = e_{-1} e_{-2} - q^{-1} e_{-2} e_{-1} \ ; \ \tilde{e}_{-4} = [e_{-1}, \tilde{e}_{-3}] \end{aligned} \tag{4.10}$$

Notice that (4.8) and (4.9) agree in the Chevalley basis, but not in the Cartan-Weyl basis, because q^* distinguishes the two conjugations.

In these lectures we shall pursue only the choice of a standard real Hopf algebra (4.9), which satisfies all the stringent physical requirements.

4.3. PHYSICAL GENERATORS AND CONTRACTION

We want to express the physical generators M_{AB} of $U_q(so(3,2))$ in terms of the Cartan–Weyl generators h, e of $U_q(so(5, \mathbf{C}))$, where

$$M_{AB} = -M_{BA} = M_{AB}^{+} \ ; \ A, B = 0, \ldots, 4 \tag{4.11}$$

and the metric tensor

$$g_{AB} = \text{ diag } (+---+) \ . \tag{4.12}$$

For $q = 1$, there are two Lorentz subalgebras (see Eq. (1.2)).

The Cartan generators h_1 and h_2 commute, which allows for a certain freedom of choice. The choice $h_1 = M_{12}$ and $h_2 = M_{34}$ would be a good candidate for a q–deformation of the Lorentz algebra (without contraction) [50, 3, 51, 52, 20]. For the deformation of the Poincaré algebra (with contraction), the following choice [9] (q real) satisfies the Jacobi identities and the standard reality condition (4.9) :

space rotations :

$$\begin{aligned} &M_3 = M_{12} = h_1 \\ &M_1 = M_{23} = \frac{1}{\sqrt{2}}(e_1 + e_{-1}) \\ &M_2 = M_{31} = \frac{-i}{\sqrt{2}}(e_1 - e_{-1}) \end{aligned} \tag{4.13}$$

boosts :

$$
\begin{aligned}
L_1 &= M_{14} = \frac{1}{2}(e_4 - q\tilde{e}_{-4} - e_2 + e_{-2}) \\
L_2 &= M_{24} = \frac{-i}{2}(e_4 + q\tilde{e}_{-4} + e_2 + e_{-2}) \\
L_3 &= M_{34} = -\frac{1}{\sqrt{2}}(q^{-\frac{i}{2}}e_3 + q^{\frac{i}{2}}q\tilde{e}_{-3})
\end{aligned}
\tag{4.14}
$$

translations :

$$
\begin{aligned}
RP_0 &= M_{04} = h_3 = h_1 + h_2 \\
RP_1 &= M_{01} = -\frac{1}{2}(-e_4 - q\tilde{e}_{-4} + e_2 + e_{-2}) \\
RP_2 &= M_{02} = \frac{1}{2}(e_4 - q\tilde{e}_{-4} + e_2 - e_{-2}) \\
RP_3 &= M_{03} = -\frac{i}{\sqrt{2}}(q^{-\frac{i}{2}}e_3 - q^{\frac{i}{2}}q\tilde{e}_{-3})
\end{aligned}
\tag{4.15}
$$

For $q = 1$, the operators $M_i, i = 1, 2, 3$ generate rotations and L_i generate Lorentz boosts. In view of the contraction we have introduced the notation ($P_0 = P_4$)

$$
RP_\mu = M_{0\mu} \quad \mu = 1, \cdots, 4 \tag{4.16}
$$

Contraction means $R \to \infty$, so that the P_μ's become the translation operators, since for $q = 1$ this limit corresponds to the Poincaré algebra (see Eq. (1.6)).

However, for $q \neq 1$, we run into trouble with the co-product. For example

$$
\begin{aligned}
\Delta(e_2) &= e_2 \otimes q^{\frac{1}{2}h_2} + q^{-\frac{1}{2}h_2} \otimes e_2 \\
&= e_2 \otimes q^{\frac{1}{2}(RP_0 - h_1)} + q^{-\frac{1}{2}(RP_0 - h_1)} \otimes e_2
\end{aligned}
\tag{4.17}
$$

In the limit $R \to \infty$, $\Delta(e_2)$, and therefore $\Delta(L_2)$, become infinite. The same problem arose when the Firenze group [18] tried to obtain the q–deformation of the Euclidean algebra $E(3)$ from the contraction of $U_q(s\ell(2)) \otimes U_q(s\ell(2))$. We follow their solution to the conundrum and put

$$
q = e^{\frac{1}{\kappa R}} \tag{4.18}
$$

where κ is a new parameter with the dimension of an inverse length, *i.e.* an energy (in the usual units with $\hbar = c = 1$).

We now get, for $R \to \infty$

$$
q^{h_3} = e^{\frac{P_0}{\kappa}} \quad ; \quad q^{h_1} = e^{\frac{h_1}{\kappa R}} \to 1 \ . \tag{4.19}
$$

From this follows that the rotation subalgebra (4.13) is not deformed. The physical relevance of this result deserves some comment: usual intuition would immediately throw out any deformation of the Poincaré algebra which blew off the local spatial isotropy. The κ–deformation we propose preserves the sacrosanct three–dimensional spatial rotational symmetry but distinguishes the dimension of time, so that the boosts are not what Lorentz imagined.

4.4. κ–POINCARÉ ALGEBRA

The procedure is now obvious. Use the commutation relations (4.1), (4.4) to (4.7), to calculate the κ–Poincaré algebra with the help of the definitions (4.13) to (4.15) and the contraction (4.16), (4.18) and (4.19), taking the limit $R \to \infty$.

The commutation relations simplify considerably with the following non linear change of variables for the deformed boosts :

$$
\begin{aligned}
N_1 &= L_1 - \frac{1}{4\kappa}P_2 - \frac{1}{4\kappa}(M_2P_3 + P_3M_2) \\
N_2 &= L_2 + \frac{1}{4\kappa}P_1 + \frac{1}{4\kappa}(M_1P_3 + P_3M_1) \\
N_3 &= L_3 + \frac{1}{2\kappa}P_3 - \frac{1}{4\kappa}(M_1P_2 + P_2M_1 - M_2P_1 - P_1M_2) \\
&= L_3 + \frac{1}{2\kappa}P_3 - \frac{1}{2\kappa}(P_2M_1 - M_2P_1)
\end{aligned}
\tag{4.20}
$$

A similar change of variables was proposed by Giller *et al.* [53] in order to simplify the κ–Poincaré algebra of [8].

The κ–Poincaré algebra finally reads most appeallingly as [9]

$$
\begin{aligned}
[M_i, M_j] &= i\varepsilon_{ijk}M_k \quad i, j, k = 1, 2, 3 \\
[M_i, P_j] &= i\varepsilon_{ijk}P_k \\
[M_i, P_0] &= 0 \\
[P_\mu, P_\nu] &= 0 \quad \mu, \nu = 0, \cdots, 3
\end{aligned}
\tag{4.21}
$$

the rotation and translation algebra is not deformed.

$$
\begin{aligned}
[M_i, N_j] &= i\varepsilon_{ijk} N_k \\
[P_0, N_k] &= -iP_k \\
[P_i, N_j] &= -i\kappa\delta_{ij} \sinh \frac{P_0}{\kappa} \\
[N_i, N_j] &= -i\varepsilon_{ijk}(M_k \cosh \frac{P_0}{\kappa} - \frac{1}{4\kappa^2} P_k(\vec{P}\vec{M}))
\end{aligned}
\tag{4.22}
$$

the boosts are deformed.

The usual Jacobi identities and the standard reality conditions are satisfied. For $\kappa \to \infty$ one recovers the usual Poincaré algebra.

One verifies that the κ–deformation of the quadratic Casimir, describing the quantum relativistic mass square operator is

$$
\begin{aligned}
C_1 &= 2\kappa^2(\cosh \frac{P_0}{\kappa} - 1) - P_1^2 - P_2^2 - P_3^2 \\
&= (2\kappa \sinh \frac{P_0}{2\kappa})^2 - \vec{P}^2
\end{aligned}
\tag{4.23}
$$

Expanding in power of $\frac{1}{\kappa^2}$ one gets

$$
C_1 = -\vec{P}^2 + P_0^2 + \frac{1}{12\kappa^2} P_0^4 + O(\kappa^{-4}) \tag{4.24}
$$

so that the energy-momentum relation is modified at high energy for fixed κ. This could be checked by experiment, or alternatively, it could be used to give a lower experimental limit on κ (see the conclusions).

The second Casimir can be obtained by introducing the κ–deformed Pauli–Lubanski four–vector

$$
\begin{aligned}
W_0 &= \vec{P} \cdot \vec{M} \\
W_k &= \kappa M_k \sinh \frac{P_0}{\kappa} + \varepsilon_{ijk} P_i N_j
\end{aligned}
\tag{4.25}
$$

so that

$$
C_2 = (\cosh \frac{P_0}{\kappa} - \frac{\vec{P}^2}{4\kappa^2}) W_0^2 - \vec{W}^2 \tag{4.26}
$$

commutes with everything in the algebra and it provides us with a good quantum number. This curious expression can be obtained from the one given by Giller *et al.* [53] for the non standard κ -Poincaré algebra [8] by the replacement $\kappa \to i\kappa$.

4.5. κ–POINCARÉ HOPF ALGEBRA

To get the co-product Δ, co-unit ε and antipode S for the quantum Poincaré algebra, we start from $U_q(so(3,2))$ and perform the contraction (4.19). Because of (4.19) and (4.13), Δ, ε and S are trivial for the rotation subalgebra

$$
\begin{aligned}
\Delta(M_i) &= M_i \otimes I + I \otimes M_i \\
\varepsilon(M_i) &= 0 \\
S(M_i) &= -M_i
\end{aligned}
\tag{4.27}
$$

The same is true for P_0 because, before the contraction, P_0 is proportional to the Cartan generator $h_3 = h_1 + h_2$

$$
\begin{aligned}
\Delta(P_0) &= P_0 \otimes I + I \otimes P_0 \\
\varepsilon(P_0) &= 0 \\
S(P_0) &= -P_0
\end{aligned}
\tag{4.28}
$$

From (4.15), (4.10), (4.3) to (4.5), it is clear that the co-product for P_i is non-trivial. After contraction one gets

$$
\begin{aligned}
\Delta(P_i) &= P_i \otimes \exp(\frac{P_0}{2\kappa}) + \exp(-\frac{P_0}{2\kappa}) \otimes P_i \\
\varepsilon(P_i) &= 0 \\
S(P_i) &= -P_i
\end{aligned}
\tag{4.29}
$$

The expressions for Δ, ε and S of the boost generators N_i follow from (4.20), (4.14), (4.10), (4.3) to (4.5). After some algebra one gets, after contraction :

$$
\begin{aligned}
\Delta(N_i) &= N_i \otimes \exp(\frac{P_0}{2\kappa}) + \exp(\frac{-P_0}{2\kappa}) \otimes N_i + \\
&\quad + \frac{1}{2\kappa}\varepsilon_{ijk}[P_j \otimes M_k \exp(\frac{P_0}{2\kappa}) + \exp(-\frac{P_0}{2\kappa}) M_j \otimes P_k] \\
\varepsilon(N_i) &= 0 \\
S(N_i) &= -N_i + \frac{3i}{2\kappa} P_i
\end{aligned}
\tag{4.30}
$$

The coproduct and the antipode satisfy the standard reality conditions (see Eq. (3.6)).

In order to prove that the κ–Poincaré algebra is a Hopf algebra one has to show that

1) the co-product is co-associative and it is a homomorphism of the algebra as in (2.27),

$$\begin{aligned} &(\Delta \otimes I)\Delta = (I \otimes \Delta)\Delta \\ &\Delta(xy) = \Delta(x)\Delta(y) \qquad \forall x, y \in \kappa\text{–Poincaré} \end{aligned} \tag{4.31}$$

2) The maps $m, i, \Delta, \varepsilon, S$ satisfy (2.28) :

$$m \circ (S \otimes I) \circ \Delta(x) = m \circ (I \otimes S) \circ \Delta(x) = i \circ \varepsilon(x) \tag{4.32}$$

as well as $(I \otimes \varepsilon) \circ \Delta = I$ and $(\varepsilon \otimes I) \circ \Delta = I$.

Conditions (1) and (2) involve some algebra and have been checked both for the sets of operators (M_i, L_i, P_μ) and (M_i, N_i, P_μ) (see Eqs (4.13), (4.14), (4.15) and (4.20)). Hence the two versions of κ–Poincaré algebras given in [8, 9] are real Hopf algebras, a standard one [9] and a non -standard one [8].

Up to now we did not find a universal R–matrix, so we cannot decide whether the Hopf algebra is quasi-triangular or not. We have found, nevertheless, R–matrices in the four– and five–dimensional representations of κ–Poincaré[54].

4.6. REALIZATIONS OF κ–POINCARÉ

The following realization of κ–Poincaré using derivatives acting on a commutative spacetime was proposed for spin zero systems by Zaugg [55] and also, independently, by Giller *et al.* [53]; the latter also proposed for arbitrary spin s [56]:

$$\begin{aligned} P_\mu &= -i\frac{\partial}{\partial x^\mu} \\ M_i &= \varepsilon_{ijk} x_j P_k + m_i \\ N_i &= x_0 P_i - \kappa x_i \sinh\frac{P_0}{\kappa} + \exp[\mp\frac{P_0}{2\kappa}]n_i \pm \frac{1}{2\kappa}\varepsilon_{ijk} P_j m_k \end{aligned} \tag{4.33}$$

where $(\vec{m}, \vec{n})$ is a standard finite-dimensional representation of the Lorentz algebra

$$\begin{aligned} [m_i, m_j] &= i\varepsilon_{ijk} m_k \\ [n_i, n_j] &= -i\varepsilon_{ijk} m_k \\ [m_i, n_j] &= i\varepsilon_{ijk} n_k \end{aligned} \tag{4.34}$$

Acting on scalar fields $\phi(\vec{x}, t)$ the κ–deformed boosts N_i are realized putting $m_i = n_i = 0$ [10]

$$N_i\phi(\vec{x}, t) = -ix_o\frac{\partial}{\partial x_i}\phi(\vec{x}, t) + i\kappa x_i[\phi(\vec{x}, t + \frac{i}{\kappa}) - \phi(\vec{x}, t - \frac{i}{\kappa})] \,. \tag{4.35}$$

This introduces imaginary finite time–differences, which are the trademark for the passage from trigonometric to elliptic functions.

From the Casimir operator C_1 (4.23) and the realization (4.33) one deduces the κ–deformed Klein-Gordon equation

$$\left[\Delta - 2\kappa^2\left(1 - \cos\frac{\partial_t}{\kappa}\right)\right]\phi(\vec{x}, t) = \left[\Delta - \left(2\kappa\sin\frac{\partial_t}{2\kappa}\right)^2\right]\phi(\vec{x}, t) = M_0^2\phi(\vec{x}, t) \,. \tag{4.36}$$

where Δ is the Laplacian. This should be the starting point of a κ–deformed field theory with regularized Feynman propagators [9].

After setting $P_1 = P_2 = 0$ it is possible [10] to integrate the equation

$$\begin{aligned} [N_1, P_0] &= P_3 \\ [N_1, P_3] &= \kappa\sinh\frac{P_0}{\kappa} \end{aligned} \tag{4.37}$$

with the Casimir operator

$$C_1 = 4\kappa^2\left(\sinh\frac{P_0}{2\kappa}\right)^2 - P_3^2 \equiv M_0^2 \,. \tag{4.38}$$

For $\kappa \to \infty$, we get the usual result, if we first define the initial value

$$\begin{aligned} P_0(\eta_0) &= M\,\cosh\eta_0 \\ P_3(\eta_0) &= M\,\sinh\eta_0 \end{aligned} \tag{4.39}$$

with

$$P_0^2(\eta) - P_3^2(\eta) = M^2 \tag{4.40}$$

and then use the addition formula for hyperbolic functions to get

$$\begin{aligned} P_0(\eta) &= P_0(\eta_0)\,\cosh(\eta - \eta_0) + P_3(\eta_0)\,\sinh(\eta - \eta_0) \\ P_3(\eta) &= P_0(\eta_0)\,\sinh(\eta - \eta_0) + P_3(\eta_0)\,\cosh(\eta - \eta_0) \end{aligned} \tag{4.41}$$

For κ–Poincaré, the <u>rapidity relations</u> (4.39) are replaced by

$$\begin{aligned} 2\kappa\sinh\frac{P_0}{2\kappa} &= M_0\,\mathrm{nc}\,(u|m) \\ P_3 &= M_0\,\mathrm{sc}\,(u|m) \end{aligned} \tag{4.42}$$

with

$$u = \sqrt{1 + \frac{M_0^2}{4\kappa^2}}\,\eta_0$$
$$m = \left(1 + \frac{M_0^2}{4\kappa^2}\right)^{-1} \tag{4.43}$$

and η_0 the rapidity. In these expressions, nc(u|m) and sc(u|m) are Jacobi elliptic functions [57]. As one should expect, (4.42) reduces to (4.39) when $\kappa \to \infty$.

The κ–deformation of (4.41) is now obtained by using the addition formula for elliptic functions [57, 10].

One can use the action of the operators (4.33) on $2s+1$ component Weyl spinor fields and calculate their finite κ–Lorentz transformation[10]. For s =1/2 one gets the κ–Dirac equation, which reads as follows [10]:

$$\begin{pmatrix} 0 & \mathcal{P}_{rr'} \\ \mathcal{P}^{-1}_{rr'} & 0 \end{pmatrix} \begin{pmatrix} \Psi^{(+)}_{r'} \\ \Psi^{(-)}_{r'} \end{pmatrix} = \begin{pmatrix} \Psi^{(+)}_{r} \\ \Psi^{(-)}_{r} \end{pmatrix} \qquad r, r' = \pm\frac{1}{2} \tag{4.44}$$

where

$$\mathcal{P}^{\pm 1} = \frac{1}{M_0\sqrt{1+\frac{M_0^2}{4\kappa^2}}}\left[2\kappa \sinh\left(\frac{P_0}{2\kappa}\right) \pm \vec{P}\cdot\vec{\sigma}\right]\left[\cosh\left(\frac{P_0}{2\kappa}\right) \pm \frac{1}{2\kappa}\vec{P}\cdot\vec{\sigma}\right] \tag{4.45}$$

and $\Psi^{(\pm)}$ are two–component Weyl spinors in momentum space.

This agrees with [58].

The κ Dirac equation is therefore:

$$\left[(\kappa \sinh\left(\frac{P_0}{\kappa}\right) + \frac{1}{2\kappa}\vec{P}^2)\gamma_0 + \exp\left(\frac{P_0}{2\kappa}\right)\vec{P}\vec{\gamma} - M_0\sqrt{1+\frac{M_0^2}{4\kappa^2}}\right]\Psi = 0 \tag{4.46}$$

γ_0 and $\vec{\gamma}$ are the usual Dirac matrices. The reader is invited to check that the four-component spinor Ψ satisfies the κ-Klein–Gordon equation. Introducing the electromagnetic field by the minimal substitution $P_\mu \to P_\mu - eA_\mu$ one finds [10] that the κ–corrections to the fine structure of the energy levels of the hydrogen atom are of order κ^{-2}, whereas the "classical" g-factor of the electron becomes

$$g = 2[1 + \frac{m}{\kappa}] + 0(\kappa^{-2})\ . \tag{4.47}$$

with m the mass of the electron. This value of course does not include radiative corrections, but it signals the crucial modification of our usual space-time concepts arising from a new version of relativistic invariance, in full agreement with the hamiltonian description of quantum field theory which clearly distinguishes between time and space.

5. Conclusions

The mathematical theory of the q–deformation of semisimple Lie algebras is well developed [15, 16, 50]. The deformed commutation relations in the Chevalley basis look rather simple, the new feature being exponentials of the Cartan generators (see Eq. (2.50)). They have to be supplemented by the highly non-linear q-Serre relations ((2.68),(2.74)), except for $s\ell(2)$. These relations are best expressed in terms of the q–adjoint operator whose general definition is rather involved ((2.51),(2.55)). The more cumbersome Cartan–Weyl basis allows a much simpler interpretation of the q–Serre relations ((2.77)). This basis is also more convenient, although not always sufficient, for defining a "physical" basis ((4.13)).

The deformed co-product is non symmetric ((2.59)). This of course has its influence on the tensor product of representations([25, 2, 35, 59] and references therein). The real forms of q–algebras follow a pattern similar to that of classical Lie algebras. However, the reality of the co-product imposes new restrictions so that equivalent Lie algebras differ by the allowed values of q once they get deformed (see end of paragraph 3.2).

There exists up to now no general theory for the quantum deformation of non semisimple Lie algebras. A certain number of examples have been given (see the introduction). The deformation of inhomogeneous algebras (e.g. the semi–direct product of rotations and translations) seems particularly interesting. Three main directions are being explored. 1) Subalgebras of semisimple Lie algebras [20, 21]. 2) Differential calculus on noncommutative space [23]. 3) Contraction of semisimple Lie algebras [18, 8, 9].

In these lectures we have followed this last track, in particular the quantum deformation of the Poincaré algebra, obtained by contraction of the q–deformed simple algebra $so(3,2)$[8, 9]. The final expression is a rather simple example of a

Hopf algebra ((4.21),(4.22),(4.27) to (4.30)). Abstracting some general features from this result, taking also into account quantum E(3) [18], one may hope to find a canonical way to quantise inhomogeneous Lie algebras without going through the cumbersome procedure of compactification. We now list some features which could have a more general signification.

One of the generators (P_0) plays the role of a Cartan generator in the coproduct ((4.29),(4.30)). As a consequence, the deformation parameter q is replaced by the dimensionful parameter κ ((4.19)) [18, 8, 9]. The rotation subalgebra remains classical ((4.21)(4.27)). The deformation of the boosts leads to new expressions for the two Casimir operators ((4.23),(4.26)).

The algebra can be realized in terms of partial derivatives on a commuting manifold, supplemented with finite–dimensional representations of the classical Lorentz algebra for arbitrary spin ((4.33),(4.34)). We can apply these operators on a field and thus easely get the κ–Klein–Gordon equation ((4.36)). Because P_0 is in the exponential one gets a finite time difference equation ((4.35)).

We have been able to integrate the deformed boosts for finite rapidities. The usual hyperbolic functions are replaced by elliptic functions ((4.42)). It has also been possible to give the explicit relation between Weyl wave functions at rest and Weyl wave functions boosted to the momentum p, for any spin s [10]. This allows to obtain the Wigner representations for the quantum Poincaré. One can also deduce the κ covariant Dirac equation ((4.44) to (4.46)).

An open problem is to formulate a field theory with interactions. Another interesting topic are the mathematical and physical properties of finite time difference equations. We have mainly worked in the momentum representation. The structure of space–time compatible with quantum Poincaré invariance is not yet understood[10].

The physical applications of q-semisimple algebras are numerous, especially for two-dimensional problems : integrable systems, conformal field theory, statistical models.

The Poincaré, or inhomogeneous Lorentz invariance, is fundamental for physics. It is worthwhile to compare the standard relativistic algebra with another consistent invariance algebra, depending on a dimensionful parameter κ, which has the Poincaré invariance as a limiting case.

The first item to compare is the quadratic Casimir operator C_1 ((4.24)). From κ-Poincaré one gets a dispersion relation for light in vacuum and a modified relation between energy and momentum for massive particles [60]. An amusing effect appears if one calculates the modified classical partition function : one finds a limiting Hagedorn temperature proportional to κ analogous to the one obtained in string theory [54]. A more mondane piece of data is the measurement of the energy and velocity of 15 to 20 GeV electrons and γ–rays at SLAC [61, 62]. Taking into account the experimental errors, one finds a lower limit for κ of about 10^4 GeV. A more stringent test, because the energies are much higher, comes from astrophysical data concerning the detection time of 10^5 to 10^6 GeV γ -rays from the pulsar in Hercules X–1. This requires a value of κ larger than at least 10^{12} GeV [63, 64].

Once a κ–field theory with interactions is available, one may hope that the dimensionful parameter κ will serve as a natural cut–off. Further tests are then possible. One may also speculate on the modification of space–time at the Planck mass. An unsolved problem is the pertinence of a nonsymmetric coproduct. Further mathematical work and new physical insight are needed. It is a meager consolation that this difficulty does not arise in two–dimensional systems, because, (un)fortunately, the real world has four dimensions.

Acknowledgments

This work was partially supported by the Swiss National Science Foundation. I thank Jerzy Lukierski and Philippe Zaugg for judicious advice and Martí Ruiz–Altaba for his help in considerably improving these notes.

References

[1] L. Faddeev and L. Takhtajan. Preprint Université Paris VI, 1985; In : *Lect. Notes in Physics* **246** (1986) 166-179.

[2] L. Alvarez-Gaumé, C. Gomez and G. Sierra. *Phys. Lett.* **220** (1989) 142; *Nucl. Phys.* **B319** (1989) 155; **B330** (1990) 347.

[3] C. Gomez and G. Sierra. *Phys. Lett.* **255B** (1991) 51.

[4] C. Ramírez, H. Ruegg and M. Ruiz-Altaba. *Nucl. Phys.* **B364** (1991) 195.

[5] V. Pasquier. *Comm. Math. Phys.* **118** (1988) 355; V. Pasquier and H. Saleur. *Nucl. Phys.* **B330** (1990) 523.

[6] M.T. Batchelor, L. Mezinescu, R.I. Nepomechie and V. Rittenberg. *J. Phys. A : Math. Gen.* **23L** (1990) 141.

[7] S. Meljanac, M. Milekovic' and S. Pallua. *J. Phys. A : Math. Gen.* **21** (1991) 581.

[8] J. Lukierski, A. Nowicki, H. Ruegg and V.N. Tolstoy. *Phys. Lett.* **B264** (1991) 331.

[9] J. Lukierski, A. Nowicki and H. Ruegg. *Phys. Lett.* **B293** (1992) 344.

[10] J. Lukierski, H. Ruegg and W. Rühl. *From κ–Poincaré Algebra to κ–Lorentz Quasigroup : A Deformation of Relativistic Symmetry,* Kaiserslautern Preprint KL-TH-92/22, Dec. 1992.

[11] C.W. Misner, K.S.Thorne and J.A.Wheeler. *Gravitation,* Freeman and Company, San Francisco, 1973.

[12] E. Celeghini, R. Giachetti, E. Sorace and M. Tarlini. *Phys. Lett.* **B280** (1992) 180.
F. Bonechi, E. Celeghini, R. Giachetti, E. Sorace and M. Tarlini. *Phys. Rev. Lett.* **68** (1992) 3718.
F. Bonechi, E. Celeghini, R. Giachetti, E. Sorace and M. Tarlini. *Quantum Galilei Group as Symmetry of Magnons,* University of Florence Preprint, DFF 156/3/92.

[13] L. Faddeev, N. Reshetikhin and L. Takhtajan. *Alg. Anal.* **1** (1989) 178. [*Leningrad Math. Journ.* **1** (1990) 193].

[14] L. Woronowicz. *Comm. Math. Phys.* **111** (1987) 613.

[15] W.G. Drinfeld. "Quantum Groups", Proc. Internat. Congress of Mathematics (Berkeley, USA, 1986); p. 70.

[16] M. Jimbo. *Lett. Math. Phys.* **10** (1985) 63 and **11** (1986) 247.

[17] M. Chaichian and P. Kulish. *Phys. Lett.* **B234** (1990) 72.

[18] E. Celeghini, R. Giachetti, E. Sorace and M. Tarlini. *J. Math. Phys.* **31** (1990) 2548; *J. Math. Phys.* **32** (1991) 1155; *J. Math. Phys.* **32** (1991) 1159.

[19] D. Arnaudon and A. Chakrabarti. *Phys. Lett.* **B255** (1991) 242.
D. Arnaudon. *Comm. Math. Phys.* **134** (1990) 523.
A. Chakrabarti. *J. Math. Phys.* **32** (1991) 1227.

[20] V. Dobrev. *Canonical q-Deformations of Noncompact Lie (Super) Algebra,* Göttingen Preprint July 1991; *q-Deformations of Noncompact Lie (Super-) Algebras,* Trieste Preprint IC/92/13, January 1992, to be published in the Proceedings of II-nd Wigner Symposium (Clausthal, July 1991).

[21] J. Lukierski and A. Nowicki, *Phys. Lett.* **B279** (1992) 299.

[22] Yu.T. Manin. "Quantum Groups and Noncommutative Geometry", Publ. Centre de Recherches Math., Univ. Montreal, 1989.

[23] O. Ogievetsky, W.B. Schmidtke, J. Wess and B. Zumino. *Comm. Math. Phys.* **150** (1992) 495.

[24] P.P. Kulish and N. Reshetikhin. *J. Sov. Math.* **23** (1983) 2435. [Russian original in : *Zapiski Nauch Semin LOMI* **101** (1981) 101].

[25] A.N. Kirillov and N. Reshetikhin. LOMI preprint E-9-88 (1988). In : *Infinite Dimensional Lie Algebras and Groups,* Marseille 1988 Meeting, V. Kac (ed.), World Scientific, 1989.

[26] H.-D. Doebner, J.D. Hennig and W. Lücke. *Lecture Notes in Physics* **370** (1990) 29-63.

[27] S. Majid. "Quasitriangular Hopf Algebras and Yang-Baxter Equations", to appear in *Int. J. Mod. Phys.* **A5** (1990) 1.

[28] S.M. Khoroshkin and V.N. Tolstoy. *Comm. Math. Phys.* **141** (1991) 599.

[29] R. Floreanini, D.A. Leites and L. Vinet. *Lett. Math. Phys.* **23** (1991) 127.

[30] J.E. Humphrey. *Introduction to Lie Algebras and Representation Theory,* Springer, New York, 1972, p. 99.

[31] V.G. Kac. *Infinite Dimensional Lie Algebras,* Birkhäuser, Boston, 1983.

[32] M. Rosso. *Comm. Math. Phys.* **124** (1989) 307.

[33] P. Truini and V.S. Varadarajan. *Lett. Math. Phys.* **26** (1992) 53; *Quantization of Reductive Lie Algebras : Construction and Universality,* Genoa Preprint GEF-Th-13/1992.

[34] F.N. Jackson. *Q.J. Pure. Appl. Math.* **41** (1910) 143.

[35] H. Ruegg. *J. Math. Phys.* **31** (1990) 1085.

[36] N. Burroughs. *Comm. Math. Phys.* **127** (1990) 109.

[37] V.N. Tolstoy. *Lecture Notes in Physics* **370** (1990) 188.

[38] R. Gilmore."*Lie Groups, Lie Algebras and Some of Their Applications*",John Wiley, New York 1974.

[39] E. Twietmeyer. *Lett. Math. Phys.* **24** (1992) 49.

[40] J. Lukierski, A. Nowicki and H. Ruegg. *Phys. Lett.* **B271** (1991) 321.

[41] M. Scheunert. Bonn University preprint Bonn, HE-92-13.

[42] M. Mozrzymas. Bordeaux University preprint LPTB 92-2, February 1992.

[43] B. Durhuus, H.P. Jakobsen and R. Nest. *Nucl. Phys. (Proc. Suppl.)* **B666** (1991) 1.

[44] G. Mack and V. Schomerus. *Nucl. Phys.* **B370** (1992) 185.

[45] M. Flato, L.K. Hadjiivanov and I.T. Todorov.*Quantum Deformations of Singletons and of Free Zero-Mass Fields,* Université de Bourgogne and Bulgarian Academy of Sciences preprint, Dec. 1992.

[46] M. Chaichian, J. de Azcarraga and P. Presnajder. *Phys. Lett.* **B291** (1992) 441

[47] T. Masuda, K. Mimachi, Y. Nakagani, M. Noumi, Y. Saburi and K. Ueno. *Lett. Math. Phys.* **19** (1990) 187.

[48] N. Bourbaki. *Groupes et algèbres de Lie* , Chap. 4, 5 et 6, Hermann, Paris 1968.

[49] J. Lukierski, A. Nowicki and H. Ruegg. Wroclaw Univ. preprint, ITP UWr 810/92, July 1992, to be published in the Proceedings of Karpacz Winter School (February 1992); Elsevier Science Publishers B.V.,Amsterdam,1993.

[50] P. Podles and S.L. Woronowicz. *Comm. Math. Phys.* **130** (1990) 381

[51] U. Carow-Watamura, M.Schlieker, M. Scholl and S. Watamura, *Z. Phys.* **C48** (1990) 159.

[52] O. Ogievetsky, W.B. Schmidke, J. Wess and B. Zumino. *Lett. Math. Phys.* **23** (1991) 233.

[53] S. Giller, P. Kosinski, M. Majewski, P. Maslanka and J. Kunz. *Phys. Lett.* **B286** (1992) 57.

[54] M. Ruiz-Altaba. Private communication.

[55] Ph. Zaugg. Private communication.

[56] S. Giller, P. Kosinski, M. Majewski, P. Maslanka and J. Kunz. *On q-covariant wave functions,* University of Lodz preprint, August 1992.

[57] M. Abramowitz and I.A. Stegun. *Handbook of Mathematical Functions,* National Bureau of Standards, Applied Mathematics Series **55**, Washington 1972.

[58] A. Nowicki, E. Sorace and M. Tarlini. "The Quantum Deformed Dirac Equation from the κ-Poincaré Algebra", INFN Preprint DFF 177/12/92 Firenze, Dec. 1992.

[59] M. Gould and L.C. Biedenharn. *J. Math. Phys.* **33** (1992) 3613.

[60] J. Lukierski, A. Nowicki and H. Ruegg. *Quantum Poincaré Algebra and some Physical Consequences,* Geneva University Preprint UGVA-DPT 1992/12-795 Dec. 1992, to be published in the Proceedings of the International Symposium "Symmetry VI, Bregenz, 2.-7.08.92, B. Gruber ed., Plenum Press.

[61] Z.G.T. Guiragossian *et al.* *Phys. Rev. Lett.* **34** (1975) 335.

[62] D.W. MacArthur. *Phys. Rev.* **A33** (1986) 1.

[63] T.J. Haines *et al.* *Phys. Rev.* **D41** (1990) 692.

[64] G. Domokos. *Astrophysical Limits on the Deformation of the Poincaré Group,* The Johns Hopkins University Preprint, JHU-TIPAC-920027, Oct. 1992.