UNIVERSITÉ DE GENÈVE

SCHOLA GENEVENSIS MDLIX

25

Exact Correlation Function at the Lifshitz Points of the Spherical Model [1]

Laurent Frachebourg and Malte Henkel

Département de Physique Théorique, Université de Genève
24 quai Ernest Ansermet, CH - 1211 Genève 4, Switzerland

UGVA-DPT 1992/12-796

The spin-spin correlation function of the spherical model being precisely at an anisotropic Lifshitz point of arbitrary order is calculated exactly. The results are in agreement with scaling. The scaling function is shown to be universal. The direction-dependent long-range correlations may change from ferromagnetic to antiferromagnetic behaviour and back as the dimension is varied. The form of the scaling function is compared to predictions following from local scale invariance for strongly anisotropic critical systems.

[1]Supported in part by the Swiss National Science Foundation

UGVA-DPT-1992-12-796

Fermilab Library

0 1160 0022118 8

1 Introduction

In magnetism, a Lifshitz point (of first order) is defined as the meeting point of the transitions between a paramagnetic, a ferromagnetic and an ordered incommensurate phase [1]. Lifshitz points have been observed experimentally, a familiar example being MnP [2]. They can be realized in lattice models by considering anisotropic competing interactions extending beyond the conventional nearest-neighbour interactions. Models of this kind have been investigated extensively, see [3] for a recent review. A well-studied example is provided by the ANNNI model [4]. While the ANNNI model only contains next-to-nearest neighbour interactions, the extension to more general interaction types with Hamiltonian $\mathcal{H} = \sum_{\vec{a}} \mathcal{H}(\vec{a})$ where

$$\mathcal{H}(\vec{a}) = -J \left(\sum_{j=1}^{d} s_{\vec{a}} s_{\vec{a}+\vec{e}_j} + \sum_{j=1}^{m} \sum_{i=1}^{n} \kappa_n s_{\vec{a}} s_{\vec{a}+(i+1)\vec{e}_j} \right) \tag{1.1}$$

has been studied at length as well. Here $s_{\vec{a}}$ are the spin variables at site $\vec{a}$ of a hypercubic lattice of dimension d, m is the number of directions in which the competing interactions are taken, n is the number of interacting neighbours in these directions and κ_i parametrize the competing interaction terms. We take J to be positive.

The definition of Lifshitz points [5, 6] is quite analogous to the definition of multicritical points. The phase diagramm of the model eq. (1.1) can be quite complex and contains both ordered ferromagnetic and helicoidal phases. A Lifshitz point of first arises when these two ordered phases meet with the disordered one [1]. If several of the κ_i are non-vanishing, the phase diagram may contain a line of Lifshitz points of first order. This line terminates in a Lifshitz point of second order. Lifshitz points of higher order can be defined analogously.

The anisotropies introduced in the lattice model eq. (1.1) may survive in the thermodynamic limit. The critical behaviour of the correlation functions in a system being exactly at the Lifshitz point becomes dependent on the space direction. The correlation function $C(\vec{r}_{\|}, \vec{r}_{\perp})$ depends on whether correlations are studied along those $d-m$ directions with only nearest neighbour interactions (referred as to $\perp$) or those m directions where competing interactions are present (referred to as $\|$). At criticality, one has [1]:

$$\begin{aligned} C(0, \vec{r}_{\perp}) &\sim r_{\perp}^{-(d-d_- + \eta_{\perp})} \\ C(\vec{r}_{\|}, 0) &\sim r_{\|}^{-(\frac{d-d_-}{\theta} + \eta_{\|})} \end{aligned} \tag{1.2}$$

where d_- is the lower critical dimension and θ is defined below. This defines the direction-dependent exponents $\eta_{\|,\perp}$ which for Lifshitz points of first order are also referred to as $\eta_{\ell 4, \ell 2}$, respectively, in the literature. Similarly, two types of correlation length $\xi_{\|}, \xi_{\perp}$ are defined and are direction-dependent as well:

$$\xi_{\|} \sim (T - T_c)^{-\nu_{\|}}, \qquad \xi_{\perp} \sim (T - T_c)^{-\nu_{\perp}} \tag{1.3}$$

while the critical exponents α, β, γ can be defined as usual from the specific heat, the order parameter and the susceptibility. The scaling relation among the exponents for

isotropic systems are replaced by anisotropic scaling relations [1]

$$\begin{aligned} 2-\alpha &= m\nu_{\|} + (d-m)\nu_{\perp} \\ \gamma &= (2-\eta_{\perp})\nu_{\perp} = \left(\frac{2}{\theta} - \eta_{\|}\right)\nu_{\|} \end{aligned} \tag{1.4}$$

where

$$\theta = \frac{\nu_{\|}}{\nu_{\perp}} \tag{1.5}$$

is the anisotropy exponent. Equations (1.4) replace the conventional scaling relations involving ν and η. Consequently, there are three independent critical exponents which describe the leading bulk critical behaviour.

The strong anisotropy of a system being at the Lifshitz point leads, via standard renormalization group arguments [1], to the following well-known scaling of the correlation function

$$C(\lambda\vec{r}_{\|}, \lambda^{1/\theta}\vec{r}_{\perp}) = \lambda^{-(\frac{d-d_-}{\theta}+\eta_{\|})} C(\vec{r}_{\|}, \vec{r}_{\perp}) . \tag{1.6}$$

This is equivalent to the scaling form:

$$C(\vec{r}_{\|}, \vec{r}_{\perp}) \sim r_{\perp}^{-(d-d_-+\eta_{\perp})} \Phi\left(\frac{r_{\|}}{r_{\perp}^{\theta}}\right) \tag{1.7}$$

which defines the scaling function $\Phi(x)$ where

$$x = \frac{r_{\|}}{r_{\perp}^{\theta}} \tag{1.8}$$

is the scaling variable. Note that any attempt to calculate Φ from a lattice model requires that the scaling limit $r_{\|} \to \infty$, $r_{\perp} \to \infty$ such that x is kept fixed has to be taken.

In this paper, we consider Lifshitz points of arbitrary order in the spherical model with additional competing interactions in m directions. In the literature, this system is known as the ANNNS model [3, 5] or the R-S model [7]. Throughout this paper, we restrict attention to anisotropic Lifshitz points, that is we only consider the situation where $1 \leq m \leq d-1$. Although this model is of no direct experimental relevance, it may provide useful insight since all physical quantities of interest can be evaluated exactly. In this respect, the spherical model has been quite a useful tool in providing explicit checks on general concepts in critical phenomena, see [8, 9, 10, 11, 12, 13, 14]. The critical exponents of the ANNNS model are for dimensions between the lower critical dimension d_- and the upper critical dimension

$$d_+ = d_- + 2 = 4 + m - m/L \tag{1.9}$$

given by [5]

$$\eta_{\|} = \eta_{\perp} = 0 \quad , \quad \nu_{\|} = \frac{\gamma}{2L} \quad , \quad \nu_{\perp} = \frac{\gamma}{2} \tag{1.10}$$

$$\begin{aligned} \gamma &= \frac{2L}{(d-2-m)L+m} \\ \alpha &= \frac{m+L(d-4-m)}{m+L(d-2-m)} \end{aligned} \tag{1.11}$$

for a Lifshitz point of order $L-1$. Here we obtain the exact scaling function $\Phi(x)$ in eq. (1.7) for anisotropic Lifshitz points of arbitrary order.

Finding the exact scaling function of a system with a strongly anisotropic scaling, see eq (1.6), is of interest by itself. In particular, we shall find regions of values of the scaling variable x, where the long-range correlations become antiferromagnetic. However, our main motivation for undertaking this work is as follows. Given the fact that local scale invariance has lead to an enormous increase of understanding the critical behaviour of static, isotropic systems using conformal invariance, at least in two dimensions (see e.g. [15]), one may wonder whether at least some of these ideas might be useful in more general situations. In fact, Cardy [16] had proposed to use conformal invariance in the context of critical dynamics, starting from the hypothesis of dynamical scaling of the time-dependent correlation function $\langle\phi(r,t)\phi(0,t)\rangle = t^{-2x/z}\Phi(r^2/t)$. He considered time-dependent systems in two space dimensions where the static system by itself is conformal invariant, this is, at the critical point. Using conformal transformations, the problem is mapped from the two-dimensional infinite plane to a strip of finite width and it is argued, since the strip is finite that it were permissible to use van Hove theory. For a purely relaxational dynamics without any macroscopic conservation law, this leads to [16]

$$\Phi(y) = \mathrm{e}^{-\mu y} \tag{1.12}$$

where μ is a non-universal constant. Note that this result is apparently independent of the dynamical exponent z.

On the other hand, for $z = 2$ but for an arbitrary number of space dimensions, the global scale invariance eq. (1.6) can be generalized to a local one [17]. Then the coordinate transformations to be considered are those given by the Schrödinger group. Then for example the two-point time-delayed correlation function $\langle\phi(r,t)\phi(0,0)\rangle = t^{-2x/z}\Phi(r^2/t)$ is fixed completely where $\Phi(y)$ is in turn given by eq. (1.12) [17], but now without the restriction to two space dimensions and without having to appeal to van Hove theory. We stress that $z = 2$ does not need imply that the system is described by van Hove (mean field) theory, the best-known example probably being the one-dimensional Ising model with Glauber dynamics [18]. The form of the three-point correlation functions was also found.

We consider the spherical model at a Lifshitz point of order $L-1$ as a convenient tool to test these general ideas using the following analogy. In critical dynamics, we have d space dimensions and one time direction, the rescaling of which is described by the dynamical exponent z. This situation can be mimicked by considering either the case of just $m = 1$ direction with additioned competing interactions which leads to the analogy $z \equiv \theta = \frac{1}{L}$, or the case $m = d-1$ which would be analogous to $z \equiv \frac{1}{\theta} = L$. As we shall show, there exist examples confirming the hypothesis of local scale invariance.

The paper is organized as follows. In section 2 we give the general procedure to calculate the correlation function. The case of a Lifshitz point of first order is described in section 3, while Lifshitz points of higher order are treated in section 4. In section 5 we compare the exact results obtained with the expectations from local scaling and conclude.

2 Correlation functions

The correlation function for the (mean) spherical model is defined as

$$C_N\left(\vec{l},T\right) \equiv < \sigma_{\vec{0}}\sigma_{\vec{l}} >_N - < \sigma_{\vec{0}} >_N^2 \tag{2.1}$$

As is well known, see e.g. [8, 9, 10, 14], it is given in the thermodynamic limit ($N \to \infty$) by

$$C(\vec{l};T) = \frac{kT}{(2\pi)^d}\int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi} d\phi_1...d\phi_d \frac{\cos\left(\vec{l}\vec{\phi}\right)}{\tilde{J}(0)\zeta - \tilde{J}(\vec{\phi})} \tag{2.2}$$

where $\tilde{J}(\vec{\phi})$ is the Fourier transform of the exchange integral

$$\tilde{J}(\vec{\phi}) = \sum_{\vec{l}} J(\vec{l})\exp\left(i\vec{l}\vec{\phi}\right) \tag{2.3}$$

and ζ is given by the spherical constraint. It can be shown that $\zeta = 1$ for $T \leq T_c$. We are interested in the behaviour of eq. (2.2) at the critical point where $\zeta = 1$ and we restrict ourselves to this case throughout the paper. In fact, the case we are going to consider is technically the hardest one and all other situations of physical interest are easily obtainable from our results.

For a m-axial Lifshitz point of order $L-1$, $\tilde{J}(\vec{\phi})$ is, see [5, 6]

$$\tilde{J}(\vec{\phi}) = 2J\left(\sum_{j=1}^{d}\cos\phi_j + \sum_{j=1}^{m}\sum_{i=1}^{n}\kappa_i\cos((i+1)\phi_j)\right) \tag{2.4}$$

which can be expanded

$$\tilde{J}(\vec{\phi}) = 2Jd + m\left(\sum_{i=1}^{n}\kappa_i\right) - \frac{1}{2}\sum_{j=m+1}^{d}\phi_j^2 - c_L\sum_{j=1}^{m}\phi_j^{2L} + ... \tag{2.5}$$

Consider for a moment the case $n = 2$. A line of first-order Lifshitz points is obtained for $\kappa_2 = -(1+4\kappa_1)/9$ where $\kappa_1 > -2/5$ is a free parameter and $c_2 = (2+5\kappa_1)/6 > 0$. A second-order Lifshitz point is found if $\kappa_1 = -2/5$ and $\kappa_2 = 1/15$ with $c_3 = 1/30$. The phase diagram is known exactly [19] and we do not repeat this calculation here. Similar results could be obtained without much effort for n arbitrary and yield explicit expressions for c_L. If the κ_i's were chosen such that c_L would become negative, the correlation function becomes modulated and the simple expansion around the assumed ground state given by $\vec{\phi} = 0$ is no longer applicable.

The evaluation of the correlation function closely follows techniques which go back at least to the classic work [20] on random walks on the lattice. As shown in [20], for $|\vec{l}|$ big enough the principal contribution to the integrals comes from $|\vec{\phi}| \sim 0$, and the leading singular behaviour of $C(\vec{l};T_c)$ is contained in

$$C_1(\vec{l};T_c) = \frac{kT_c}{2J(2\pi)^d}\int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi} d\phi_1\ldots d\phi_d \frac{\cos(\vec{l}\vec{\phi})}{\sum_{j=m+1}^{d}\frac{1}{2}\phi_j^2 + c_L\sum_{j=1}^{m}\phi_j^{2L}}. \tag{2.6}$$

It is here that the scaling limit mentioned in the introduction is taken. In this limit, it is enough to restrict attention to C_1, since all other terms contributing to C can be made arbitrarily small [20]. In the sequel, we always suppress the distinction between C and C_1. Using the identity

$$a^{-1} = \int_0^\infty e^{-au}du \tag{2.7}$$

we can rewrite $C(\vec{l};T_c)$ as

$$C(\vec{l};T_c) = \frac{kT_c}{2J(2\pi)^d}\int_0^\infty du F(u) \tag{2.8}$$

with

$$F(u) = \Re\left\{\prod_{j=1}^{m}\left[\int_{-\pi}^{\pi}\exp\left(i\phi_j l_j - c_L\phi_j^{2L}u\right)d\phi_j\right]\prod_{j=m+1}^{d}\left[\int_{-\pi}^{\pi}\exp\left(i\phi_j l_j - \frac{1}{2}\phi_j^2 u\right)d\phi_j\right]\right\} \tag{2.9}$$

where $\Re$ denotes the real part. An Abelian theorem for the Laplace transform [20] states that if $F(u)$ is analytic, the behaviour of $C(\vec{l};T_c)$ is determined by the behaviour of $F(u)$ at $u \to \infty$. For $u \gg 1$ we can therefore replace the range of integration $(-\pi,\pi)$ on the ϕ_j integrals by $(-\infty,\infty)$.

The correlation function now reads

$$\begin{aligned} C(\vec{l};T_c) &= \frac{kT_c}{2J(2\pi)^d}\int_0^\infty du\left(\Re\left\{\prod_{j=1}^{m}\left[\int_{-\infty}^{\infty}\exp\left(i\phi_j l_j - c_L\phi_j^{2L}u\right)d\phi_j\right]\right\}\right. \\ &\left.\times\left(\frac{2\pi}{u}\right)^{\frac{d-m}{2}}\exp\left(-\frac{1}{2u}\sum_{j=m+1}^{d}l_j^2\right)\right) \end{aligned} \tag{2.10}$$

Expanding the cosine and integrating term by term we get

$$\int_{-\infty}^{\infty} e^{-c_L\phi_j^{2L}u}\cos(l_j\phi_j)d\phi_j = \frac{1}{L}\left(\frac{1}{c_L u}\right)^{\frac{1}{2L}}\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k)!}\Gamma\left(\frac{k}{L}+\frac{1}{2L}\right)\left(\frac{l_j^2}{(c_L u)^{\frac{1}{L}}}\right)^k \tag{2.11}$$

In the sequel we specialize to the choice

$$\begin{aligned} r_{\|} &= l_1 \neq 0\ ,\ l_i = 0\ (i = 2,..,m), \\ r_\perp^2 &= \sum_{j=m+1}^{d} l_j^2 \end{aligned} \tag{2.12}$$

and the correlation function becomes

$$\begin{aligned} C(r_{\|},r_\perp;T_c) &= \frac{kT_c}{2J(2\pi)^{\frac{d+m}{2}}}\left(\frac{1}{L}\right)^m\Gamma\left(\frac{1}{2L}\right)^{m-1}\left(\frac{1}{c_L}\right)^{\frac{m}{2L}} \\ &\times\int_0^\infty du\left\{u^{\frac{m}{2}(1-\frac{1}{L})-\frac{d}{2}}\exp\left(-\frac{r_\perp^2}{2u}\right)\left(\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k)!}\Gamma\left(\frac{k}{L}+\frac{1}{2L}\right)\left(\frac{r_{\|}^2}{(c_L u)^{\frac{1}{L}}}\right)^k\right)\right\} \end{aligned} \tag{2.13}$$

The integral converges if

$$d > 2 + m - \frac{m}{L} = d_- \tag{2.14}$$

and the sum is absolutely convergent. Exchanging the sum with the integral, we thus obtain an exact expression for the correlation function

$$\begin{aligned} C(r_\parallel, r_\perp; T_c) &= \frac{kT_c}{2J(2\pi)^{\frac{d+m}{2}}} \left(\frac{1}{L}\right)^m \Gamma\left(\frac{1}{2L}\right)^{m-1} \left(\frac{1}{c_L}\right)^{\frac{m}{2L}} \left(\frac{r_\perp^2}{2}\right)^{1+\frac{m}{2}\left(1-\frac{1}{L}\right)-\frac{d}{2}} \\ &\times \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} \Gamma\left(\frac{k}{L} + \frac{1}{2L}\right) \Gamma\left(\frac{k}{L} + \frac{d-d_-}{2}\right) \left(\left(\frac{2}{c_L}\right)^{1/L} \frac{r_\parallel^2}{r_\perp^{2/L}}\right)^k \end{aligned} \tag{2.15}$$

We remark that the series is absolutely convergent on the whole real axis and that the correlation function is of the form

$$C(r_\parallel, r_\perp; T_c) \sim A(L,m,d) r_\perp^{-(d-d_-)} \Phi\left(\left(\frac{2}{c_L}\right)^{\theta/2} \frac{r_\parallel}{r_\perp^\theta}; L, m, d\right) \tag{2.16}$$

where $A(L,m,d)$ is a constant and $\Phi(x; L, m, d)$ the desired scaling function. This is in agreement with the expected scaling form eq. (1.6) and serves as a useful check of our calculation. Furthermore, since the scaling form of the correlation function only depends through the constant c_L on the details of the lattice structure, we verify the universality of the scaling function.

While this already an exact and complete answer to our problem, it is useful to rewrite this result in a more handy form. This will be done in the next sections. The general strategy is as follows. We use first the identity

$$(2k)! = k!(2\pi)^{-\frac{L}{2}} 2^{2k+\frac{1}{2}} L^k \prod_{n=0}^{L-1} \Gamma\left(\frac{k}{L} + \frac{1}{2L} + \frac{n}{L}\right) \tag{2.17}$$

and the correlation function finally becomes

$$C(r_\parallel, r_\perp; T_c) = B(L,m,d) r_\perp^{-(d-d_-)} \Xi\left(L, \frac{d-d_-}{2}; \frac{2^{\frac{1}{L}}}{4Lc_L^{\frac{1}{L}}} \frac{r_\parallel^2}{r_\perp^{\frac{2}{L}}}\right) \tag{2.18}$$

where the constant $B(L,m,d)$ is given by

$$B(L,m,d) = \frac{kT_c}{J(2\pi)^{\frac{d+m-L}{2}}} \left(\frac{1}{L}\right)^m \Gamma\left(\frac{1}{2L}\right)^{m-1} \left(\frac{1}{c_L}\right)^{\frac{m}{2L}} 2^{\frac{d-d_-}{2}-\frac{3}{2}} \tag{2.19}$$

All properties of $C(r_\parallel, r_\perp; T_c)$ are contained within the series

$$\Xi(L,a;x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \frac{\Gamma\left(\frac{k}{L} + a\right)}{\prod_{n=1}^{L-1} \Gamma\left(\frac{k}{L} + \frac{1}{2L} + \frac{n}{L}\right)} x^k \tag{2.20}$$

which we proceed to study in the following. It is sufficient to consider only those cases where $0 < a \leq 1$, since the other cases can be found from the recursion

$$\Xi(L, a+1; x) = a\Xi(L,a;x) + \frac{1}{L} x \frac{\partial}{\partial x} \Xi(L,a;x) \tag{2.21}$$

In the sequel, we shall use the abbreviation

$$a = \frac{d}{2} - \frac{m}{4} - 1 = \frac{d-d_-}{2} \tag{2.22}$$

For $a = 0$, the model is at the lower critical dimension d_-, while for $a = 1$, it is at the upper critical dimension d_+. For the convenience of the reader, we give in table 1 the values of a if both d and m are integers for the cases $L = 2$ and $L = 3$.

$L = 2$	m					
d	1	2	3	4	5	6
3	1/4	0				
4	3/4	1/2	1/4			
5	5/4	1	3/4	1/2		
6	7/4	3/2	5/4	1	3/4	
7	9/4	2	7/4	3/2	5/4	1

$L = 3$	m							
d	1	2	3	4	5	6	7	8
3	1/6							
4	2/3	1/3	0					
5	7/6	5/6	1/2	1/6				
6	5/3	4/3	1	2/3	1/3			
7	13/6	11/6	3/2	7/6	5/6	1/2		
8	8/3	7/3	2	5/3	4/3	1	2/3	
9	19/6	17/6	5/2	13/6	11/6	3/2	7/6	5/6

Table 1: Some values of the parameter $a = \frac{1}{2}(d - d_-)$ for $L = 2$ and $L = 3$ as a function of d and $m \leq d - 1$.

If d and m are integers, $\Xi(L,a;x)$ can be reexpressed in terms of well-known transcendental functions. We shall derive these expressions for $L = 2$ and $L = 3$ below. In particular, we shall be interested in deriving the behaviour of the correlation function for large values of the scaling variable x and we shall obtain explicit expressions for any L.

3 Lifshitz points of first order

We first study the case of a conventional Lifshitz point, also referred to as a Lifshitz point of first order [5]. This corresponds in the above equations to have $L = 2$. As we have seen above, the correlation function becomes

$$C(r_{\|}, r_{\perp}; T_c) = B(2, m, d) r_{\perp}^{-(d-d_-)} \Psi\left(\frac{d-d_-}{2}, \sqrt{\frac{1}{32c_2}\frac{r_{\|}^2}{r_{\perp}}}\right) \tag{3.1}$$

where $\Psi(a, x)$ is defined by the series

$$\Psi(a, x) = \Xi(2, a; x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \frac{\Gamma\left(\frac{k}{2}+a\right)}{\Gamma\left(\frac{k}{2}+\frac{3}{4}\right)} x^k \tag{3.2}$$

and a is defined in eq. (2.22). Because d and m are integers, we have $a = n/4$ where n is a positive integer. It is sufficient to distinguish between the four cases $n = 1, 2, 3, 4$, which we shall examine below. Indeed, the other cases can be easily found from the recursion relation eq. (2.21). In figure 1, we display the normalized scaling functions $\Psi(a, x)/\Psi(a, 0)$ for the cases we now proceed to study.

3.1 $a = \frac{1}{4}$

For this case we separate our absolutely convergent series $\Psi(a, x)$ in two series for the odd and even terms and use eq. (2.17) (with $L = 1$)

$$\begin{aligned}
\Psi\left(\frac{1}{4}, x\right) &= \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!} \frac{\Gamma\left(k+\frac{1}{4}\right)}{\Gamma\left(k+\frac{3}{4}\right)} - x \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k+1)!} \frac{\Gamma\left(k+\frac{3}{4}\right)}{\Gamma\left(k+\frac{5}{4}\right)} \\
&= \sqrt{\pi}\left[\sum_{k=0}^{\infty} \frac{1}{k!} \frac{\Gamma\left(k+\frac{1}{4}\right)}{\Gamma\left(k+\frac{1}{2}\right)\Gamma\left(k+\frac{3}{4}\right)} \left(\frac{x}{2}\right)^{2k}\right. \\
&\qquad \left. - \frac{x}{2} \sum_{k=0}^{\infty} \frac{1}{k!} \frac{\Gamma\left(k+\frac{3}{4}\right)}{\Gamma\left(k+\frac{5}{4}\right)\Gamma\left(k+\frac{3}{2}\right)} \left(\frac{x}{2}\right)^{2k}\right]
\end{aligned} \tag{3.3}$$

Then we use the identity eq. (10.37.7) from [21]

$$\sum_{k=0}^{\infty} \frac{\Gamma(k+a)}{k!\Gamma(k+2a)\Gamma\left(k+a+\frac{1}{2}\right)} x^{2k} = \sqrt{\pi} x^{1-2a} \mathrm{I}^2_{a-\frac{1}{2}}(x) \tag{3.4}$$

and find

$$\Psi\left(\frac{1}{4}, x\right) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \frac{\Gamma\left(\frac{k}{2}+\frac{1}{4}\right)}{\Gamma\left(\frac{k}{2}+\frac{3}{4}\right)} x^k = x^{\frac{1}{2}}\left[\mathrm{I}_{-\frac{1}{4}}\left(\frac{x}{2}\right) + \mathrm{I}_{\frac{1}{4}}\left(\frac{x}{2}\right)\right] \mathrm{K}_{\frac{1}{4}}\left(\frac{x}{2}\right) \tag{3.5}$$

where I_ν and K_ν are modified Bessel functions. This gives the exact correlation function. Using the known asymptotic form of the Bessel functions [22], the asymptotic behaviour is, as $x \to \infty$

$$\Psi\left(\frac{1}{4}, x\right) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \frac{\Gamma\left(\frac{k}{2}+\frac{1}{4}\right)}{\Gamma\left(\frac{k}{2}+\frac{3}{4}\right)} x^k \simeq \frac{2}{\sqrt{x}}\left\{1 + \mathcal{O}\left(\frac{1}{x^2}\right)\right\} \tag{3.6}$$

3.2 $a = \frac{1}{2}$

We again split the expression for $\Psi(a, x)$ as before and get

$$\begin{aligned}
\Psi\left(\frac{1}{2}, x\right) &= \sqrt{\pi}\left[\sum_{k=0}^{\infty} \frac{1}{k!\Gamma\left(k+\frac{3}{4}\right)} \left(\frac{x}{2}\right)^{2k}\right. \\
&\qquad \left. - \frac{x}{2} \sum_{k=0}^{\infty} \frac{1}{\Gamma\left(k+\frac{3}{2}\right)\Gamma\left(k+\frac{5}{4}\right)} \left(\frac{x}{2}\right)^{2k}\right]
\end{aligned} \tag{3.7}$$

Then we recall the identities eqs. (10.7.11) and (10.7.18) from [21]

$$\begin{aligned}
\sum_{k=0}^{\infty} \frac{1}{k!\Gamma(k+a)} x^{2k} &= x^{1-a} \mathrm{I}_{a-1}(2x) \\
\sum_{k=1}^{\infty} \frac{1}{\Gamma\left(k+\frac{1}{2}\right)\Gamma(k+a)} x^{2k} &= x^{\frac{3}{2}-a} \mathbf{L}_{a-\frac{1}{2}}(2x)
\end{aligned} \tag{3.8}$$

where $\mathbf{L}_\nu(x)$ is a modified Struve function. The asymptotic behaviour of this is given in eq. (12.2.6) of [22] for $x \to \infty$ and we finally obtain

$$\begin{aligned}
\Psi\left(\frac{1}{2}, x\right) &= \sqrt{\pi}\left(\frac{x}{2}\right)^{\frac{1}{4}}\left[\mathrm{I}_{-\frac{1}{4}}(x) - \mathbf{L}_{-\frac{1}{4}}(x)\right] \\
&\simeq \frac{2}{\Gamma\left(\frac{1}{4}\right)} \frac{1}{x}\left\{1 + \mathcal{O}\left(\frac{1}{x^2}\right)\right\}
\end{aligned} \tag{3.9}$$

3.3 $a = \frac{3}{4}$

For this particular case the series reduces to an exponential

$$\Psi\left(\frac{3}{4}, x\right) = \mathrm{e}^{-x} \tag{3.10}$$

3.4 $a = 1$

In this case the model is at its upper critical dimension. The calculation proceeds along the same lines as above and with the same relations as for the case $a = \frac{1}{2}$ we find

$$\begin{aligned}
\Psi(1, x) &= \sqrt{\pi}\left(\frac{1}{\sqrt{\pi}\Gamma\left(\frac{3}{4}\right)} + \left(\frac{x}{2}\right)^{\frac{3}{4}}\left[\mathbf{L}_{\frac{1}{4}}(x) - \mathrm{I}_{\frac{1}{4}}(x)\right]\right) \\
&\simeq -\frac{1}{2\Gamma\left(\frac{3}{4}\right)} \frac{1}{x^2}\left\{1 + \mathcal{O}\left(\frac{1}{x^2}\right)\right\}
\end{aligned} \tag{3.11}$$

where the analytic continuation $\Gamma(-1/4) = -4\Gamma(3/4)$ was used. We note that in this case the correlations show a predominantly antiferromagnetic behaviour. In particular, since $\Psi(1, 0) > 0$, this implies that there exists some x_0 such that $\Psi(1, x_0) = 0$, that is, the universal part of the correlation function vanishes. A numerical calculation yields $x_0 \simeq 2.80187\ldots$ and from figure 1, it can be seen that this is the only zero of $\Psi(1, x)$ for x positive.

4 Lifshitz points of arbitrary order

Going beyond the simplest case $L = 2$, we could attempt to repeat the approach of the last section. In fact, we may write for any L the scaling function in terms of the generalized hypergeometric function ${}_pF_q$. For $L = 3$, this leads to

$$\Xi(3,a;x) = \frac{\Gamma(a)}{\sqrt{\pi}\Gamma(5/6)}\,{}_1F_4\left(a;\frac{1}{3},\frac{1}{2},\frac{2}{3},\frac{5}{6};-\frac{x^3}{27}\right) \tag{4.1}$$
$$-3x\frac{\Gamma\left(a+\frac{1}{3}\right)}{2\pi}\,{}_1F_4\left(a+\frac{1}{3};\frac{2}{3},\frac{5}{6},\frac{7}{6},\frac{4}{3};-\frac{x^3}{27}\right)+6x^2\frac{\Gamma\left(a+\frac{2}{3}\right)}{\sqrt{\pi}\Gamma(1/6)}\,{}_1F_4\left(a+\frac{2}{3};\frac{7}{6},\frac{4}{3},\frac{3}{2},\frac{5}{3};-\frac{x^3}{27}\right)$$

A much simpler form can be obtained for the two special cases

$$\Xi\left(3,\frac{1}{2};x\right) = \sum_{k=0}^{\infty}\frac{(-x)^k}{k!\Gamma\left(\frac{k}{3}+\frac{5}{6}\right)} = \left(3888\pi^3\right)^{1/6}\mathrm{Ai}\left(-\sqrt[6]{12}x^{1/2}\right)\mathrm{Ai}\left(\sqrt[6]{12}x^{1/2}\right) \tag{4.2}$$

and

$$\Xi\left(3,\frac{5}{6};x\right) = \sum_{k=0}^{\infty}\frac{(-x)^k}{k!\Gamma\left(\frac{k}{3}+\frac{1}{2}\right)} \tag{4.3}$$
$$= -\pi^{3/2}\left[\mathrm{Ai}\left(-\sqrt[6]{12}x^{1/2}\right)\mathrm{Ai}'\left(\sqrt[6]{12}x^{1/2}\right)+\mathrm{Ai}'\left(-\sqrt[6]{12}x^{1/2}\right)\mathrm{Ai}\left(\sqrt[6]{12}x^{1/2}\right)\right]$$

where $\mathrm{Ai}(x)$ is the Airy function and the prime denotes the derivative. For the proof of these see appendix B. We shall see below that these two cases are rather distinctive in their asymptotic behaviour for large values of x. We display the scaling functions for $L = 3$ in figure 2. As we have already noted for the case $L = 2$ above, we may have either long-range ferromagnetic or antiferromagnetic behaviour. In distinction to the examples seen so far, in these two cases it is known from the properties of $\mathrm{Ai}(x)$ that there are infinitely many zeroes of the scaling functions in the cases $a = 1/2$ and $a = 5/6$.

While these examples use some peculiarities for a given value of L or a, we now examine the asymptotic behaviour as $x \to \infty$. This follows from a general theorem due to Wright [23] on the asymptotic behaviour of an extension of the generalized hypergeometric function. We summarize those of his results relevant for us in appendix A. As it has been shown for the Lifshitz point of first order in the previous section, the form of the asymptotic behaviour of $\Xi(L,a;x)$ depends quite sensitively on the value of a.

4.1 Algebraic asymptotic behaviour

We first consider the case of generic values of $0 < a \leq 1$. Then from theorem 1 of appendix A [23], we see that the asymptotic behaviour of the function $\Xi(L,a;x)$ is given by the poles of the coefficients of its series expansion eq. (2.20). For generic values of a, the Γ-function in the numerator will not cancel with one of those in the denominator. Working out the residues at those points where $\Gamma(k/L+a)$ has a pole, we find the asymptotic behaviour of $\Xi(L,a;x)$ for $x \to \infty$ to be algebraic and given by

$$\Xi\left(L,\frac{d-d_-}{2};x\right) \simeq \sum_{l\geq 0} P_l\, x^{-L\left(l+\frac{d-d_-}{2}\right)} \tag{4.4}$$

with

$$P_l = \frac{(-1)^l}{\Gamma(l+1)}\frac{L\Gamma\left(L\left(l+\frac{d-d_-}{2}\right)\right)}{\prod_{n=1}^{L-1}\Gamma\left(-l-\frac{d-d_-}{2}+\frac{n}{L}+\frac{1}{2L}\right)} \tag{4.5}$$

The reader may compare this general form with the specific results for $L = 2$ found in section 3.

Since the leading term is given by $l = 0$, let us consider P_0. We first note that there will be a cancellation of factors in eq. (2.20) if

$$a = a_n := \frac{2n+1}{2L} \quad ; \quad n = 1,\ldots,L-1 \tag{4.6}$$

and we take the convention that $a_0 := 0$. In fact, if $a = a_n$, we have to reconsider the asymptotic behaviour and we shall do so below. However, we note that P_0 changes sign if we pass from $a < a_n$ to $a > a_n$ for some $n = 1,\ldots L-1$. We therefore see that

$$\begin{aligned} P_0 &> 0 \quad ; \text{ if } a_n < a < a_{n+1} \text{ with } n \text{ even} \\ P_0 &< 0 \quad ; \text{ if } a_n < a < a_{n+1} \text{ with } n \text{ odd} \end{aligned} \tag{4.7}$$

From table 1, it is now easy to see for which values of d and m the leading long-range behaviour of the spin-spin correlation function will be ferromagnetic or antiferromagnetic, respectively. In fact, it is quite surprising to see that already at the Lifshitz point the anisotropies of the model can become so strong that the ferromagnetic behaviour may be changed into an effective antiferromagnetic behaviour. In those cases, where the system is antiferromagnetic for large values of x, it follows that the universal scaling function vanishes for some finite x_0, since $\Xi(L,a;0) > 0$. If $x = x_0$, one phenomenologically observes effective exponents different from those quoted in the introduction. In fact, the mechanism of modifying scaling relations of apparent exponents by the vanishing of the scaling function (in distinction to the presence of dangerous irrelevant variables) well below the upper critical dimension appears to occur quite generally, see [24]. However, as can be seen from figure 2, even in cases where $P_0 > 0$, this does not necessarily imply that the correlations are ferromagnetic for all values of x. An example is provided by $\Xi(3,1;x)$ which has two zeroes at $x_0^{(1)} \simeq 1.231\ldots$ and $x_0^{(2)} \simeq 5.116\ldots$, see figure 2. The correlations are antiferromagnetic for $x_0^{(1)} < x < x_0^{(2)}$. Ferromagnetic correlations appear to be kept for all x, however, if $a < a_1$.

4.2 Exponential-like asymptotic behaviour

The case where $a = a_n$ with a_n given by eq. (4.6) marks just the borderline between regions of long-range ferromagnetic and antiferromagnetic behaviour. If $1 \leq n \leq L-1$,

two Γ-functions in eq. (2.20) with the same argument cancel and we get

$$\Xi(L, a_n; x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^k}{k! \prod_{\ell=1}^{n-1} \Gamma\left(\frac{k}{L} + \frac{1}{2L} + \frac{\ell}{L}\right) \prod_{\ell=n+1}^{L-1} \Gamma\left(\frac{k}{L} + \frac{1}{2L} + \frac{\ell}{L}\right)} \tag{4.8}$$

The asymptotic behaviour of this series as $x \to \infty$ is exponential-like as found from theorem 2 of appendix A [23]

$$\begin{aligned}\Xi(L, a_n; x) \simeq\ & \alpha (2\pi)^{1-\frac{L}{2}} \left(\frac{2L}{L-1}\right)^{\frac{1}{2}} \left(\frac{x}{L}\right)^{\frac{L}{2(L-1)}\left(\frac{1}{2L}-1+a_n\right)} \\ & \times \exp\left[2\frac{L-1}{L} L^{\frac{L}{2(L-1)}-1} x^{\frac{L}{2(L-1)}} \cos\left(\frac{\pi}{2}\frac{L}{L-1}\right)\right] \\ & \times \cos\left[2\frac{L-1}{L} L^{\frac{L}{2(L-1)}-1} x^{\frac{L}{2(L-1)}} \sin\left(\frac{\pi}{2}\frac{L}{L-1}\right) + \frac{\pi}{2}\frac{L}{L-1}\left(\frac{1}{2L} - 1 + a_n\right)\right] \\ & \times \left\{1 + \mathcal{O}\left(x^{-\frac{L}{2(L-1)}}\right)\right\}\end{aligned} \tag{4.9}$$

where $\alpha = 1/2$ if $L = 2$ and $\alpha = 1$ if $L > 2$. Note that for $L \geq 2$ the argument of the exponential is always negative. We also see the presence of an oscillating term, which is absent only for $L = 2$. This indicates the presence of an infinite set of values of x for which the scaling function will vanish.

5 Discussion

We have found exactly the spin-spin correlation function for the anisotropic Lifshitz points of arbitrary order L realized in the spherical model with competing interactions extending beyond the nearest neighbors. The calculation was performed using the scaling limit where $r_{\|}, r_{\perp} \to \infty$ simultaneously but such that the ratio $r_{\|}/r_{\perp}^{\theta}$ is kept fixed, where $\theta = 1/L$. The result can be generally written in the form

$$C(r_{\|}, r_{\perp}; T_c) = B_1 r_{\perp}^{-(d-d_-)} \Phi\left(B_2 \frac{r_{\|}}{r_{\perp}^{\theta}}\right) \tag{5.1}$$

The explicit expressions for the scaling function Φ and the non-universal metric factors B_1 and B_2 are given in eq. (2.18). We have described in sections 3 and 4 the explicit representation of Φ in terms of well-known transcendental functions. Our results are as follows.

1. The general form eq. (5.1) is in agreement with the expected anisotropic scale invariance. The scaling function Φ only depends on the number of dimensions d, the number m of dimensions with competing interactions present and the order L of the Lifshitz point. It is independent, for example, of the values κ_i and we confirm the expected universality. Properties of the model dependent on further details of the lattice only enter into the metric factors $B_{1,2}$. We also note that the dependence on m of Φ only enters via the lower critical dimension d_-.

2. In general, the leading asymptotic behaviour of Φ for large values of its argument is given by a remarkably simple structure

$$\Phi\left(\frac{r_{\|}}{r_{\perp}^{\theta}}\right) \simeq \mathcal{A}\left(\frac{r_{\|}}{r_{\perp}^{\theta}}\right)^{-(d-d_-)/\theta} \tag{5.2}$$

where $\mathcal{A}$ is a known constant, see eq. (4.4). This is consistent with the known critical exponents. If we had known beforehand that the leading behaviour of $\Phi(x)$ for x large would be a power law, we could have predicted eq. (5.2) from matching the correlation function scaling forms eq. (1.2).

3. The scaling amplitude $\mathcal{A}$ may be either positive or negative, corresponding to long-range ferromagnetic or antiferromagnetic behaviour, respectively. It is surprising to see that already at a Lifshitz point, the effect of the competing interactions may become so strong as to be capable to create effective antiferromagnetic correlations. Which of the two possible situations is realized only depends on the quantity $d - d_-$ as given in eq. (4.7), since $\mathcal{A}$ is proportional to P_0.

4. In those cases where the long-range behaviour is antiferromagnetic, there is always a particular choice $x = x_0$ of the scaling variable such that the universal scaling function $\Phi(x_0) = 0$. The long-range correlation is ferromagnetic for $x < x_0$ and antiferromagnetic for $x > x_0$. Antiferromagnetic correlations will be present for at least some values of x if $d - d_- > 3/L$.

5. The borderline between the long-range ferromagnetic and antiferromagnetic behaviour occurs when $d - d_- = \theta(2n+1)$ with n being a positive integer and is characterized by an exponential-like behaviour

$$\begin{aligned}\Phi\left(\frac{r_{\|}}{r_{\perp}^{\theta}}\right) \simeq\ & \alpha \left(\frac{r_{\|}}{r_{\perp}^{\theta}}\right)^{\frac{1}{1-\theta}\left(\frac{\theta}{2}-1+\frac{d-d_-}{2}\right)} \exp\left(\beta \left(\frac{r_{\|}}{r_{\perp}^{\theta}}\right)^{\frac{1}{1-\theta}}\right) \\ & \times \cos\left(\gamma + \delta \left(\frac{r_{\|}}{r_{\perp}^{\theta}}\right)^{\frac{1}{1-\theta}}\right)\end{aligned} \tag{5.3}$$

where α, β, γ and δ are known constants, see eq. (4.9). We have seen that in this case there may occur infinitely many changes between long-range ferromagnetic and antiferromagnetic behaviour as the scaling variable is varied. For a Lifshitz point of first order, no zeroes occur.

6. Our results may be considered as an analogy to the calculation of time-delayed correlation functions in dynamical problems. The analogy with these works for the cases $m = 1$, where the role of time is played by $r_{\|}$ and where the analogue of a dynamical exponent $z = 1/L$, and for $m = d - 1$, where $z = L$ and the role of time is played by $r_{\perp}$.

7. Considering the case of a dynamical exponent $z = 2$, we see that indeed for $L = 2$, $d = 6$ and $m = 5$, the prediction eq. (1.12) following from the hypothesis of

Schrödinger invariance at an strongly anisotropic critical point is reproduced. For all other $d < d_+$, the upper critical dimension, the scaling function has a different form. Since local scale invariance is central to this hypothesis, a consideration of the cases where $d > d_+$ does not appear to be of much interest in this context.

8. Considering the analogy with a dynamical exponent $z \neq 2$, a simple pattern emerges for the cases where the scaling function has a leading exponential-like behaviour of the form (where μ is a non-universal constant)

$$\Phi\left(\frac{r^z}{t}\right) \sim \exp\left(-\mu\left(\frac{r^z}{t}\right)^{1/(z-1)}\right) \tag{5.4}$$

for large values of its argument, and where we suppressed the oscillating and power-like prefactors. If $z = L$, this case is realized for the dimensions $d = L + 2(n+1)$, $n = 1, \ldots, L-1$, while for L even and $z = 1/L$, this case appears only for $d = 4$. We note that the form of this result contains the number of dimensions only implicitly through the value of the dynamical exponent z. The form eq. (5.4) is quite distinct from the conformal invariance prediction [16] of the z-independence of the time-dependent correlation function.

Summarising, we have seen that already such a simple strongly anisotropic model like the spherical model with competing beyond nearest neighbor interactions such as to display Lifshitz points has quite a complicated behaviour of its spin-spin correlation function. The results are in agreement with scale invariance and allow for the first time to ask questions about the form of the scaling function itself. While in a few cases, the results can be understood in terms of local scale invariance, it remains a challenge to develop a better conceptual understanding of these fascinating phenomena.

Acknowledgement: It is a pleasure to thank M. Droz for useful comments.

Appendix A

We recall a few results on the asymptotics of the following extension of the generalized hypergeometric function

$${}_p\mathcal{F}_q(x) = \sum_{k=0}^{\infty} \frac{f(k)}{k!}(-x)^k \tag{A.1}$$

where

$$f(t) = \left(\prod_{r=1}^{p} \Gamma(\alpha_r t + \beta_r)\right)\left(\prod_{r=1}^{q} \Gamma(\rho_r t + \mu_r)\right)^{-1} \tag{A.2}$$

If $\alpha_r = 1$ and $\rho_r = 1$ for all values r occurring, we recover the generalized hypergeometric function as defined e.g. in [25]. The numbers α_r and ρ_r are all real and positive and

$$\kappa = 1 + \sum_{r=1}^{q} \rho_r - \sum_{r=1}^{p} \alpha_r > 0 \tag{A.3}$$

For our purposes, where

$$\kappa = 2 - \frac{2}{L} \tag{A.4}$$

we need the asymptotic behaviour along the positive real axis for values of $1 \leq \kappa < 2$. For the convenience of the reader we restate the relevant theorems obtained by Wright [23]. The full asymptotic expansion for any complex argument and for any $\kappa > 0$ can be found in [23].

Theorem 1: *If $0 < \kappa < 2$, then the asymptotic expansion of ${}_p\mathcal{F}_q(x)$ for $x \to \infty$ is*

$${}_p\mathcal{F}_q = J(x) \tag{A.5}$$

where $J(x)$ is defined below.

Theorem 2: *If $f(t)$ has only a finite number of poles or none, then $\kappa \geq 1$ and the asymptotic expansion of ${}_p\mathcal{F}_q(x)$ for $x \to \infty$ is given by*

$$\begin{aligned} {}_p\mathcal{F}_q(x) &= I(Z) + I(\bar{Z}) + H(x) \quad (1 < \kappa < 2), \\ {}_p\mathcal{F}_q(x) &= I(Z) + H(x) \quad (\kappa = 1) \end{aligned} \tag{A.6}$$

where $I(X), H(x)$ and Z are defined below. If $f(t)$ has no poles, then $H(x) = 0$.

The following notation is used. Let

$$h = \left(\prod_{r=1}^{p} \alpha_r^{\alpha_r}\right)\left(\prod_{r=1}^{q} \rho_r^{-\rho_r}\right), \quad \vartheta = \sum_{r=1}^{p} \beta_r - \sum_{r=1}^{q} \mu_r + \frac{1}{2}(q-p) \tag{A.7}$$

The variable Z is defined as

$$Z = \kappa(hx)^{1/\kappa} e^{i\pi/\kappa} \tag{A.8}$$

and we write $I(X)$ for the exponential asymptotic expansion

$$I(X) = A_0 X^{\vartheta} e^X \left[1 + \mathcal{O}(X^{-1})\right] \tag{A.9}$$

where

$$A_0 = (2\pi)^{\frac{1}{2}(p-q)} \kappa^{-\frac{1}{2}-\vartheta} \prod_{r=1}^{p} \alpha_r^{\beta_r - \frac{1}{2}} \prod_{r=1}^{q} \rho_r^{\frac{1}{2}-\mu_r}, \tag{A.10}$$

and $J(y)$ for the algebraic expansion

$$J(y) = \sum_{r=1}^{p} \sum_{l \geq 0} P_{r,l} y^{-(l+\beta_r)/\alpha_r} \tag{A.11}$$

where $P_{r,l}$ are defined from the poles of $f(t)$ in the following way. The poles of $f(t)$ are among those of $\prod_{r=1}^{p} \Gamma(\alpha_r t + \beta_r)$ at the points

$$t = -\frac{l + \beta_r}{\alpha_r}. \tag{A.12}$$

If $f(t)$ has a pole of degree s at this point, we write for the residue

$$s P_{r,l} y^{-(l+\beta_r)/\alpha_r} = \mathrm{Res}\left(\Gamma(-t) f(t) y^t\right) \tag{A.13}$$

If $f(t)$ has only a finite number l_r of poles, then $P_{r,l} = 0$ when $l > l_r$ and $H(y)$ is the finite sum

$$H(y) = \sum_{r=1}^{p} \sum_{l=0}^{l_r} P_{r,l} y^{-\frac{l+\beta_r}{\alpha_r}} \tag{A.14}$$

Appendix B

We analyse a few sums arising in the calculation of some of the scaling functions discussed in the text. We begin with

$$\psi(x) := \Xi\left(3, \frac{1}{2}; x\right) = \sum_{k=0}^{\infty} \frac{(-x)^k}{k!\Gamma\left(\frac{k}{3}+\frac{5}{6}\right)} \tag{B.1}$$

where x is real and positive. To bring this into a more tractable form, we use the identity

$$(3k)! = (2\pi)^{-1} 3^{3k+\frac{1}{2}} k! \Gamma\left(k+\frac{1}{3}\right)\Gamma\left(k+\frac{2}{3}\right) \tag{B.2}$$

and get $\psi = \psi_1 + \psi_2 + \psi_3$, where each term is treated separately. The first one is

$$\begin{aligned}\psi_1 &= \sum_{k=0}^{\infty} \frac{(-x)^{3k}}{(3k)!\Gamma(k+5/6)} \\ &= \frac{2^{7/6}\pi}{\sqrt{6\pi}} \sum_{k=0}^{\infty} \frac{(-1)^k \left(\frac{4}{27}x^3\right)^k}{k!\Gamma(k+2/3)\Gamma(2k+2/3)}\end{aligned} \tag{B.3}$$

where we reused eq. (2.17). Next, we recall the identity eq. (10.40.2) from [21]

$$\sum_{k=0}^{\infty} \frac{(-1)^k x^{4k}}{k!\Gamma(k+a)\Gamma(2k+a)} = x^{2-2a} \mathrm{I}_{a-1}(2x)\mathrm{J}_{a-1}(2x) \tag{B.4}$$

where $\mathrm{I}_\nu(x)$ and $\mathrm{J}_\nu(x)$ are Bessel functions, and get

$$\psi_1 = \frac{2}{3}\sqrt{\pi x}\,\mathrm{I}_{-\frac{1}{3}}(X)\mathrm{J}_{-\frac{1}{3}}(X) \tag{B.5}$$

with the abbreviation

$$X := \left(\frac{64}{27}\right)^{1/4} x^{3/4} \tag{B.6}$$

The second term is treated in an analogous fashion and we have

$$\begin{aligned}\psi_2 &= \sum_{k=0}^{\infty} \frac{(-x)^{3k+1}}{(3k+1)!\Gamma(k+7/6)} \\ &= -\frac{2^{11/6}\pi}{\sqrt{6\pi}}\frac{x}{3} \sum_{k=0}^{\infty} \frac{(-1)^k \left(\frac{4}{27}x^3\right)^k}{k!\Gamma(k+4/3)\Gamma(2k+4/3)} \\ &= -\frac{2}{3}\sqrt{\pi x}\,\mathrm{I}_{\frac{1}{3}}(X)\mathrm{J}_{\frac{1}{3}}(X)\end{aligned} \tag{B.7}$$

The third term finally is

$$\begin{aligned}\psi_3 &= \sum_{k=0}^{\infty} \frac{(-x)^{3k+2}}{(3k+2)!\Gamma(k+3/2)} \\ &= \frac{2\pi}{\sqrt{3}}\left(\frac{x}{3}\right)^2 \sum_{k=0}^{\infty} \frac{(-x/3)^{3k}}{k!\Gamma(k+3/2)\Gamma(k+4/3)\Gamma(k+5/3)} \\ &= \frac{2}{3}\sqrt{\pi x} \sum_{k=0}^{\infty} \frac{(-1)^k \left[\left(\frac{4}{27}\right)^{1/4} x^{3/4}\right]^{4k+2}}{(2k+1)!\Gamma(k+4/3)\Gamma(k+5/3)}\end{aligned} \tag{B.8}$$

We now use the identity eq. (10.40.8) from [21]

$$\begin{aligned}&\sum_{k=0}^{\infty} \frac{(-1)^k x^{4k+2}}{(2k+1)!\Gamma(k+3/2-a)\Gamma(k+3/2+a)} \\ &= \frac{1}{2\sin\pi a}\left[\mathrm{J}_{2a}(2x)\mathrm{I}_{-2a}(2x) - \mathrm{J}_{-2a}(2x)\mathrm{I}_{2a}(2x)\right]\end{aligned} \tag{B.9}$$

and find

$$\psi_3 = \frac{2}{3}\sqrt{\pi x}\left[\mathrm{J}_{\frac{1}{3}}(X)\mathrm{I}_{-\frac{1}{3}}(X) - \mathrm{J}_{-\frac{1}{3}}(X)\mathrm{I}_{\frac{1}{3}}(X)\right] \tag{B.10}$$

We collect the results and get

$$\begin{aligned}\psi(x) &= \frac{2}{3}\sqrt{\pi x}\left[\mathrm{J}_{-\frac{1}{3}}(X)\mathrm{I}_{-\frac{1}{3}}(X) - \mathrm{J}_{\frac{1}{3}}(X)\mathrm{I}_{\frac{1}{3}}(X) + \mathrm{J}_{\frac{1}{3}}(X)\mathrm{I}_{-\frac{1}{3}}(X) - \mathrm{J}_{-\frac{1}{3}}(X)\mathrm{I}_{\frac{1}{3}}(X)\right] \\ &= \frac{2}{3}\sqrt{\pi x}\left[\mathrm{J}_{\frac{1}{3}}(X) + \mathrm{J}_{-\frac{1}{3}}(X)\right]\cdot\left[\mathrm{I}_{-\frac{1}{3}}(X) - \mathrm{I}_{\frac{1}{3}}(X)\right] \\ &= \left(3888\pi^3\right)^{1/6} \mathrm{Ai}\left(-\sqrt[6]{12}x^{1/2}\right)\mathrm{Ai}\left(\sqrt[6]{12}x^{1/2}\right)\end{aligned} \tag{B.11}$$

where $\mathrm{Ai}(x)$ is the Airy function. This is the result quoted in the text.

We next consider the function

$$\varphi(x) := \Xi\left(3, \frac{5}{6}; x\right) = \sum_{k=0}^{\infty} \frac{(-x)^k}{k!\Gamma\left(\frac{k}{3}+\frac{1}{2}\right)} \tag{B.12}$$

We use again eq. (B.2) to get the decomposition $\varphi = \varphi_1 + \varphi_2 + \varphi_3$ and turn to study these terms separately. The first one is

$$\begin{aligned}\varphi_1 &= \sum_{k=0}^{\infty} \frac{(-x)^{3k}}{(3k)!\Gamma(k+1/2)} \\ &= 2\sqrt{\frac{\pi}{3}} \sum_{k=0}^{\infty} \frac{(-1)^k \left(\frac{4}{27}x^3\right)^k}{(2k)!\Gamma(k+1/3)\Gamma(k+2/3)}\end{aligned} \tag{B.13}$$

This is rewritten via the identity

$$\begin{aligned}&\sum_{k=0}^{\infty} \frac{(-1)^k x^{4k}}{(2k)!\Gamma(k+1/2-a)\Gamma(k+1/2+a)} \\ &= -\frac{x}{4\sin a\pi}\{\mathrm{J}_{2a}(2x)\mathrm{I}_{-2a-1}(2x) - \mathrm{J}_{-2a}(2x)\mathrm{I}_{2a-1}(2x) \\ &\quad -\mathrm{J}_{-2a+1}(2x)\mathrm{I}_{2a}(2x) + \mathrm{J}_{2a+1}(2x)\mathrm{I}_{-2a}(2x)\}\end{aligned} \tag{B.14}$$

We postpone the proof of this and proceed with the calculation. We get

$$\varphi_1 = -\sqrt{\frac{\pi}{12}}X\left\{\mathrm{J}_{\frac{1}{3}}(X)\mathrm{I}_{-\frac{4}{3}}(X) - \mathrm{J}_{-\frac{1}{3}}(X)\mathrm{I}_{-\frac{2}{3}}(X) - \mathrm{J}_{\frac{2}{3}}(X)\mathrm{I}_{-\frac{1}{3}}(X) + \mathrm{J}_{\frac{4}{3}}(X)\mathrm{I}_{-\frac{1}{3}}(X)\right\} \tag{B.15}$$

For the second term we have

$$\begin{aligned}\varphi_2 &= \sum_{k=0}^{\infty}\frac{(-x)^{3k+1}}{(3k+1)!\Gamma(k+5/6)}\\ &= -2^{5/3}\sqrt{\frac{\pi}{3}}\frac{x}{3}\sum_{k=0}^{\infty}\frac{(-1)^k\left(\frac{4}{27}x^3\right)^k}{k!\Gamma(k+2/3)\Gamma(2k+5/3)}\end{aligned} \tag{B.16}$$

This is evaluated by using the identity

$$\sum_{k=0}^{\infty}\frac{(-1)^k x^{4k}}{k!\Gamma(k+a)\Gamma(2k+a+1)} = \frac{x^{1-2a}}{2}\left[\mathrm{I}_{a-1}(2x)\mathrm{J}_a(2x)+\mathrm{I}_a(2x)\mathrm{J}_{a-1}(2x)\right] \tag{B.17}$$

which we prove below. We find

$$\varphi_2 = -\sqrt{\frac{\pi}{12}}X\left[\mathrm{I}_{-\frac{1}{3}}(X)\mathrm{J}_{\frac{2}{3}}(X)+\mathrm{I}_{\frac{2}{3}}(X)\mathrm{J}_{-\frac{1}{3}}(X)\right] \tag{B.18}$$

and for the last term we get, again using eq. (B.17)

$$\begin{aligned}\varphi_3 &= \sum_{k=0}^{\infty}\frac{(-x)^{3k+2}}{(3k+2)!\Gamma(k+7/6)}\\ &= 2^{7/3}\sqrt{\frac{\pi}{3}}\left(\frac{x}{3}\right)^2\sum_{k=0}^{\infty}\frac{(-1)^k\left(\frac{4}{27}x^3\right)^k}{k!\Gamma(k+4/3)\Gamma(2k+7/3)}\\ &= \sqrt{\frac{\pi}{12}}X\left[\mathrm{I}_{\frac{1}{3}}(X)\mathrm{J}_{\frac{4}{3}}(X)+\mathrm{I}_{\frac{4}{3}}(X)\mathrm{J}_{\frac{1}{3}}(X)\right]\end{aligned} \tag{B.19}$$

Combining these three terms, we find

$$\begin{aligned}\varphi(x) &= \sqrt{\frac{\pi}{12}}X\left\{\mathrm{J}_{\frac{1}{3}}(X)\left[\mathrm{I}_{\frac{4}{3}}(X)-\mathrm{I}_{-\frac{4}{3}}(X)\right]+\mathrm{J}_{-\frac{1}{3}}(X)\left[\mathrm{I}_{-\frac{2}{3}}(X)-\mathrm{I}_{\frac{2}{3}}(X)\right]\right.\\ &\qquad\left.+\mathrm{J}_{\frac{2}{3}}(X)\left[\mathrm{I}_{\frac{1}{3}}(X)-\mathrm{I}_{-\frac{1}{3}}(X)\right]+\mathrm{J}_{\frac{4}{3}}(X)\left[\mathrm{I}_{\frac{1}{3}}(X)-\mathrm{I}_{\frac{1}{3}}(X)\right]\right\}\\ &= \sqrt{\frac{\pi}{12}}X\left\{\left[\mathrm{J}_{\frac{1}{3}}(X)+\mathrm{J}_{-\frac{1}{3}}(X)\right]\mathrm{K}_{\frac{2}{3}}(X)+\left[\mathrm{J}_{-\frac{2}{3}}(X)-\mathrm{J}_{\frac{2}{3}}(X)\right]\mathrm{K}_{\frac{1}{3}}(X)\right\}\end{aligned} \tag{B.20}$$

recalling the familiar relationship between the modified Bessel functions $\mathrm{I}_\nu(X)$ and $\mathrm{K}_\nu(X)$ and using the recursions eq. (B.26) below and

$$\mathrm{K}_{\nu+1}(x) = \mathrm{K}_{\nu-1}(x)+\frac{2\nu}{x}\mathrm{K}_\nu(x) \tag{B.21}$$

Using the relationship with the Airy function $\mathrm{Ai}(x)$ and its derivative the final result is

$$\varphi(x) = -\pi^{3/2}\left[\mathrm{Ai}\left(-\sqrt[6]{12}x^{1/2}\right)\mathrm{Ai}'\left(\sqrt[6]{12}x^{1/2}\right)+\mathrm{Ai}'\left(-\sqrt[6]{12}x^{1/2}\right)\mathrm{Ai}\left(\sqrt[6]{12}x^{1/2}\right)\right] \tag{B.22}$$

which is the form stated in the text. In the same way, the representation of $\Xi(3,a;x)$ in terms of the generalized hypergeometric function ${}_1F_4$ can be obtained.

We now prove the identities needed in the calculation. For the proof of eq. (B.17), let

$$\begin{aligned}T &:= \sum_{k=0}^{\infty}\frac{(-1)^k x^{4k}}{k!\Gamma(k+a)\Gamma(2k+a+1)}\\ &= \sum_{k=0}^{\infty}\frac{(k+a)(-1)^k x^{4k}}{k!\Gamma(k+a+1)\Gamma(2k+a+1)}\\ &= \left(a+\frac{x}{4}\frac{\partial}{\partial x}\right)\left(x^{-2a}\mathrm{I}_a(2x)\mathrm{J}_a(2x)\right)\end{aligned} \tag{B.23}$$

where eq. (B.4) was used. We then use

$$\frac{d}{dx}\left(x^{-a}\mathrm{J}_a(x)\right) = -x^{-a}\mathrm{J}_{a+1}(x)\ ,\quad \frac{d}{dx}\left(x^{-a}\mathrm{I}_a(x)\right) = x^{-a}\mathrm{I}_{a+1}(x) \tag{B.24}$$

and find

$$T = \frac{x^{1-2a}}{2}\left[\left(\frac{a}{x}\mathrm{I}_a(2x)+\mathrm{I}_{a+1}(2x)\right)\mathrm{J}_a(2x)+\left(\frac{a}{x}\mathrm{J}_a(2x)-\mathrm{J}_{a+1}(2x)\right)\mathrm{I}_a(2x)\right] \tag{B.25}$$

Then eq. (B.17) follows from the recursion relations of the Bessel functions

$$\mathrm{I}_{\nu-1}(x) = \frac{2\nu}{x}\mathrm{I}_\nu(x)+\mathrm{I}_{\nu+1}(x)\ ,\quad \mathrm{J}_{\nu-1}(x) = \frac{2\nu}{x}\mathrm{J}_\nu(x)-\mathrm{J}_{\nu+1}(x) \tag{B.26}$$

For the proof of eq. (B.14), let

$$S := \sum_{k=0}^{\infty}\frac{(-1)^k x^{4k}}{(2k)!\Gamma(k+1/2-a)\Gamma(k+1/2+a)} \tag{B.27}$$

We separate off the term with $k=0$. For the remaining sum, we make a shift in the summation index and have the decomposition $S = S_0+S_1$ where

$$S_0 = \frac{\cos a\pi}{\pi} \tag{B.28}$$

$$\begin{aligned}S_1 &= -\frac{1}{2}\sum_{k=0}^{\infty}\frac{1}{k+1}\frac{(-1)^k x^{4k+4}}{(2k+1)!\Gamma(k+3/2-a)\Gamma(k+3/2+a)}\\ &= -2\int_0^x dy\sum_{k=0}^{\infty}\frac{(-1)^k y^{4k+3}}{(2k+1)!\Gamma(k+3/2-a)\Gamma(k+3/2+a)}\\ &= -\frac{1}{4\sin a\pi}\int_0^{2x}du\,u\left[\mathrm{J}_{2a}(u)\mathrm{I}_{-2a}(u)-\mathrm{J}_{-2a}(u)\mathrm{I}_{2a}(u)\right]\end{aligned} \tag{B.29}$$

where the identity eq. (B.9) was used. To evaluate this, we use the following, see eq. (11.3.29) in [22]

$$\begin{aligned}\int_0^x du\,u\mathrm{J}_\nu(ku)\mathrm{J}_{-\nu}(lu) &= \frac{x}{2}\left[k\mathrm{J}_{\nu+1}(kx)\mathrm{J}_{-\nu}(lx)-l\mathrm{J}_\nu(kx)\mathrm{J}_{-\nu+1}(lx)\right]\\ &\quad -\nu\mathrm{J}_\nu(kx)\mathrm{J}_{-\nu}(lx)+\lim_{\epsilon\to 0}\nu\mathrm{J}_\nu(k\epsilon)\mathrm{J}_{-\nu}(l\epsilon)\end{aligned} \tag{B.30}$$

We now use the relation with the modified Bessel function

$$\mathrm{I}_\nu(x) = \exp\left(\frac{-\nu\pi i}{2}\right)\mathrm{J}_\nu(ix) \tag{B.31}$$

and take $k = 1$ and $l = i$. With the leading behaviour of the Bessel functions for small values of their arguments [22] and some algebra, we obtain the identity

$$\begin{aligned}\int_0^x du\, u\mathrm{J}_\nu(u)\mathrm{I}_{-\nu}(u) &= \frac{x}{2}\left[\mathrm{J}_{\nu+1}(x)\mathrm{I}_{-\nu}(x) + \mathrm{J}_\nu(x)\mathrm{I}_{-\nu+1}(x)\right] \\ &\quad -\nu\mathrm{J}_\nu(x)\mathrm{I}_{-\nu}(x) + \frac{\sin\pi\nu}{\pi}\end{aligned} \tag{B.32}$$

Insertion into eq. (B.29) then yields the following

$$\begin{aligned}S = -\frac{x}{4\sin\pi a}\Big[&-\frac{2a}{x}\left(\mathrm{J}_{2a}(2x)\mathrm{I}_{-2a}(2x) + \mathrm{J}_{-2a}(2x)\mathrm{I}_{2a}(2x)\right) \\ &+ \mathrm{J}_{2a+1}(2x)\mathrm{I}_{-2a}(2x) + \mathrm{J}_{2a}(2x)\mathrm{I}_{-2a+1}(2x) - \mathrm{J}_{-2a+1}(2x)\mathrm{I}_{2a}(2x) - \mathrm{J}_{-2a}(2x)\mathrm{I}_{2a+1}(2x)\Big]\end{aligned} \tag{B.33}$$

and eq. (B.14) is obtained with the help of the recursion relation eq. (B.26). This completes the proof.

References

[1] R.M. Hornreich, M. Luban and S. Shtrikman, Phys. Rev. Lett. **35**, 1678 (1975); Phys. Lett. **55A**, 269 (1975)

[2] C.C. Becerra, Y. Shapira, N.F. Oliveira Jr. and T.S. Chang, Phys. Rev. Lett. **44**, 1692 (1980)

[3] W. Selke, in C. Domb and J. Lebowitz (Eds), *Phase Transitions and Critical Phenomena*, Vol. 15, Academic Press (New York 1992), p. 1

[4] R.J. Elliot, Phys. Rev **124**, 346 (1961); M.E. Fisher and W. Selke, Phys. Rev. Lett. **44**, 1502; (1980) P. Bak and J.v. Boehm, Phys. Rev. **B21**, 5297 (1980)

[5] W. Selke, Z. Phys. **B27**, 81 (1977); Phys. Lett. **61A**, 443 (1977)

[6] J.F. Nicoll, G.F. Tuthill, T.S. Chang and H.E. Stanley, Phys. Lett. **58A**, 1 (1976)

[7] S. Redner and H.E. Stanley, J.Phys. **C10**, 4765 (1977)

[8] M.N. Barber and M.E. Fisher, Ann. of Phys. **77**, 1 (1973)

[9] G.S. Joyce in C. Domb and M.S. Green (Eds), *Phase Transitions and Critical Phenomena*, Vol. 2, Academic Press (New York 1972), p. 375

[10] R. J. Baxter, *Exactly Solved Models in Statistical Mechanics*, Academic Press (New York 1982)

[11] S. Singh and R.K. Pathria, Phys. Rev. **B31**, 4483 (1985); **B36**, 3769 (1987)

[12] J.G. Brankov and N.S. Tonchev, Physica **189A**, 583 (1992)

[13] M. Zanetti and G.F. Mazenko, Phys. Rev. **B35**, 5043 (1987); A. Coniglio and M. Zanetti, Europhys. Lett. **10**, 575 (1989); T.J. Newman and A.J. Bray, J. Phys. **A23**, 4491 (1990)

[14] A.M. Khorunzhy, B.A. Khoruzhenko, L.A. Pastur and M.V. Shcherbina in C. Domb and J. Lebowitz (Eds), *Phase Transitions and Critical Phenomena*, Vol. 15, Academic Press (New York 1992), p. 73

[15] J.L. Cardy in E. Brézin and J. Zinn-Justin (Eds), *Fields, Strings and Critical Phenomena*, Les Houches XLIX, North Holland (Amsterdam 1990), p. 169; P. Christe and M. Henkel, *Introduction to Conformal Invariance and its Applications to Critical Phenomena*, Springer Lecture Notes in Physics, to appear

[16] J.L. Cardy, J. Phys. **A18**, 2771 (1985)

[17] M. Henkel, Geneva preprint UGVA/DPT 1992/09-783, to appear in Int. J. Mod. Phys. **C5**

[18] R.J. Glauber, J. Math. Phys. **4**, 294 (1963)

[19] R.M. Hornreich, M. Luban and S. Shtrikman, Physica **86A**, 465 (1976); W. Selke J. Magn. Mag. Mat. **9**, 7 (1978)

[20] E.W. Montroll and G.H. Weiss, J. Math. Phys. **6**, 167 (1965)

[21] E.R. Hansen, *A Table of Series and Products*, Prentice Hall (Englewood Cliffs 1975)

[22] M. Abramowitz and I.A. Stegun (Eds), *Handbook of Mathematical Functions*, Dover (New York 1965)

[23] E.M. Wright, Journal London Math. Soc. (2) **10**, 287 (1935); Proc. London Math. Soc. (2) **46**, 389 (1940); Journal London Math. Soc. (2) **27**, 256 (1952) (erratum)

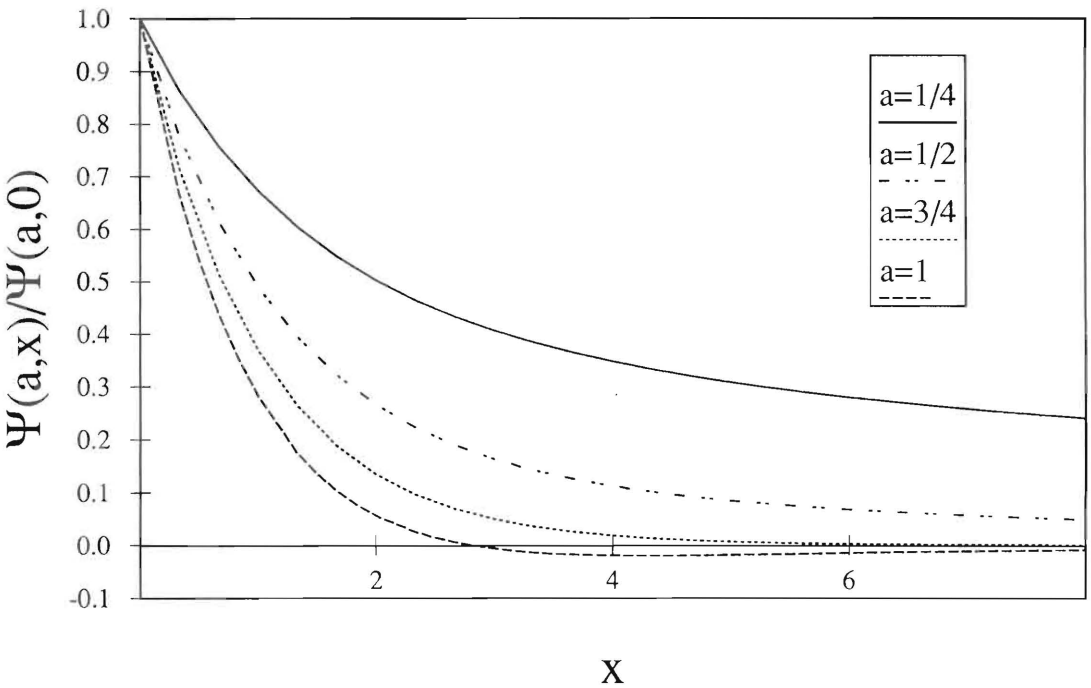
[24] M. Henkel and V. Privman, Phys. Rev. Lett. **65**, 1777 (1990)

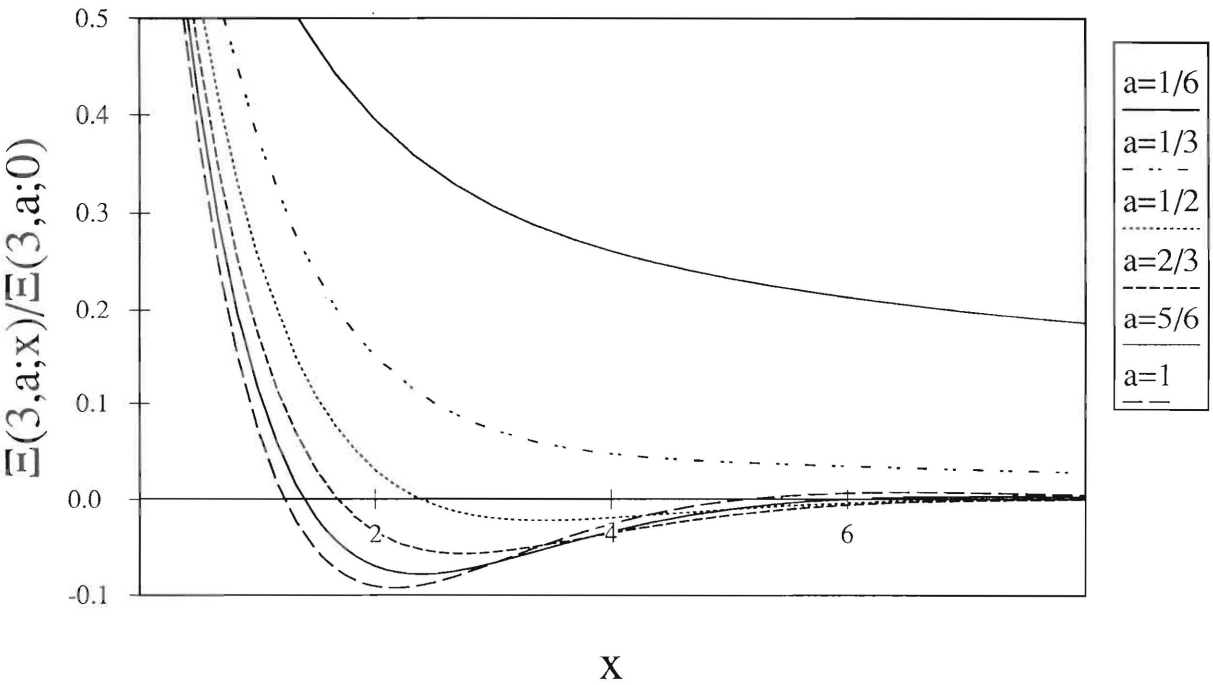
[25] A. Erdélyi, W. Magnus, F. Oberhettinger and F.G. Tricomi, *Higher Transcendental Functions*, Vol. 1, McGraw-Hill (New York 1955)

Figure captions

Figure 1: Normalized scaling function $\Psi(a,x)/\Psi(a,0)$ for the values $a = 1/4, 1/2, 3/4$ and $a = 1$ as a function of x.

Figure 2: Normalized scaling functions $\Xi(3,a;x)/\Xi(3,a;0)$ for $a = 1/6, 1/3, 1/2, 2/3, 5/6$ and $a = 1$ as a function of x.


1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0.0
-0.1
2
4
6
Ψ(a,x)/Ψ(a,0)
a=1/4
a=1/2
a=3/4
a=1
x


0.5
0.4
0.3
0.2
0.1
0.0
-0.1
2
4
6
Ξ(3,a;x)/Ξ(3,a;0)
a=1/6
a=1/3
a=1/2
a=2/3
a=5/6
a=1
x