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UG-2/92
March 1992

Recent Developments in W-Gravity

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ABSTRACT

We discuss recent developments in the gauging of W-symmetries, leading to the construction of so-called W-gravity theories.

[1]Talk given at the 5th Regional Conference on Mathematical Physics, 15-22 December 1991, Edirne, Turkey.

1. Introduction

Two-dimensional conformal or Virasoro-symmetries play an important role in the study of both conformal field theory, the basic building block of string theory, and critical phenomena in statistical mechanical models. W-symmetries are extensions of the Virasoro-symmetries. There are several reasons to consider such extensions. First of all, they can be used to obtain a better understanding of conformal field theory itself. They explain the existence of off-diagonal modular invariants in conformal field theory and can be used to investigate the representation theory of the Virasoro algebra for values c of the central charge at $c \geq 1$. Secondly, W-symmetries are "natural" symmetries. They occur as symmetries of a wide range of models such as Toda field theories, nonlinear differential equations, matrix models of two-dimensional gravity and discrete states of non-critical two-dimensional strings. Finally, a more ambitious approach is to consider W-symmetries as "fundamental" symmetries. In the same way as the Virasoro-symmetry leads to the construction of two-dimensional gravity and string theories, W-symmetries lead to so-called W-gravity and W-string theories.

There exist different kinds of W-symmetries. In particular, one can divide them into two groups: (1) W_N-symmetries are extensions of the Virasoro algebra with a finite number of (bosonic, integer) higher-spin generators. The simplest example is the W_3-algebra where a single higher-spin $s = 3$ generator is added to the spin-2 Virasoro generator [1]. (2) W_∞-symmetries are extensions with an infinite number of higher-spin generators, each spin occurring once. Symmetries of this kind were first considered in [2, 3].

It is natural to consider the supersymmetric generalization of W-symmetries. One expects such symmetries to be useful for the study of superconformal field theories. They also occur as a "natural" symmetry of supersymmetric nonlinear differential equations. Finally, they might lead to the construction of W-superstrings in the same way as the super-Virasoro algebra is related to the construction of "ordinary" superstring theory. In the latter approach one expects the additional supersymmetry to be helpful in removing the tachyonic states present in the bosonic W-strings. Like in the bosonic case, one can divide the supersymmetric W-symmetries into two groups. The ones with a finite number of higher-spin generators are generically denoted as super-W_N symmetries. On the other hand, W-supersymmetries with an infinite number of generators are indicated as super-W_∞ symmetries.

We will mainly concentrate here on the gauging of the super-W_∞ symmetries and the construction of the corresponding W-supergravity theory. For a more general discussion of W-gravity theories and an extensive list of references we refer to the following articles which have appeared recently in the literature [4, 5, 6, 7, 8]. The organization of this contribution is as follows. In the next section we will first discuss some general features of the underlying super-W algebra. In the next section we

will describe the construction of the corresponding W-supergravity theory. Finally, our conclusions will be given in section 4.

2. W-Supersymmetries

W-supersymmetries are extensions of the super-Virasoro algebra. Our starting point is the so-called $N = 1$ super-Virasoro algebra. The generators of this algebra are given by an $N = 1$ superfield $W^{(3/2)}$ of spin $s = 3/2$. In terms of the singular parts of Operator Product Expansions (OPE) the algebra without central charge is given by (we use the notation and conventions of [9])

$$W^{(3/2)}(1)W^{(3/2)}(2) \sim \{\frac{3}{2}\frac{\theta_{12}W^{(3/2)}}{z_{12}^2} - \frac{1}{2}\frac{D_2W^{(3/2)}}{z_{12}} + \frac{\theta_{12}\partial_2 W^{(3/2)}}{z_{12}}\} \tag{1}$$

In order to describe extensions of the $N = 1$ super-Virasoro algebra it is convenient to first extend its finite-dimensional $osp(1,2)$ subalgebra. This subalgebra is obtained by restricting the parameters $k_{(3/2)}$ multiplying the generators $W^{(3/2)}$ by the following differential equation:

$$D^5 k_{(3/2)} = 0 \tag{2}$$

where $D = \partial/\partial\theta - \theta\partial$ is the supercovariant derivative. This leaves us with five independent generators generating the $osp(1,2)$ algebra. An explicit realization of these generators in components is provided by the following differential operators:

$$\begin{aligned}
L_1 &= \partial \\
L_0 &= z\partial + \lambda + \frac{1}{2}\theta\frac{\partial}{\partial\theta} \\
L_{-1} &= z^2\partial + 2\lambda z + z\theta\frac{\partial}{\partial\theta} \\
G_{1/2} &= \frac{\partial}{\partial\theta} + \theta\partial \\
G_{-1/2} &= z\frac{\partial}{\partial\theta} + \theta[z\partial + 2\lambda]
\end{aligned} \tag{3}$$

The parameter λ is related to the $osp(1,2)$ second-order Casimir operator in the following way:

$$\begin{aligned}
C_2(osp(1,2)) &= L_0^2 - \frac{1}{2}\{L_1, L_{-1}\} + \frac{1}{4}[G_{1/2}, G_{-1/2}] \\
&= \lambda(\lambda - \frac{1}{2})
\end{aligned} \tag{4}$$

The above generators can be viewed as lying at the bottom of a so-called super-wedge algebra [10, 11] which is a supersymmetric generalization of the wedge algebras of [2, 3]. We denote this super-wedge algebra by super-$\mathcal{B}_\lambda$. The complete super-$\mathcal{B}_\lambda$ algebra provides a higher-spin extension of the $osp(1,2)$ algebra. For $\lambda = 1/4$ it corresponds to one of the higher-spin superalgebras of Fradkin and Vasiliev [12].

The super-$\mathcal{B}_\lambda$ algebra can be obtained as the enveloping algebra of $osp(1,2)$ factored over the ideal generated by $C_2(osp(1,2) - \lambda(\lambda - 1/2))$ in the following way [11]. Since the $sl(2)$ generators $L_{\pm 1}$ and L_0 can be expressed in terms of anticommutators of the supercharges $G_{\pm 1/2}$ we may restrict ourselves to polynomials of the latter. It is convenient to use a two-component notation for these supercharges where G_α with $\alpha = 1, 2$ denotes $(G_{1/2}, G_{-1/2})$. Using this notation, the commutator of two supercharges is given by

$$[G_\alpha, G_\beta] = \{2(\lambda - \frac{1}{4})K + \frac{1}{2}\}\epsilon_{\alpha\beta} \tag{5}$$

where K is given by

$$K = 1 - 2\theta\frac{\partial}{\partial\theta} \tag{6}$$

The operator K satisfies

$$K^2 = 1 \qquad\qquad \{K, G_\alpha\} = 0 \tag{7}$$

The presence of this operator indicates that we have two unit elements which are given by (linear combinations of) 1 and $\theta\partial/\partial\theta$. This leads to a doubling of the generators occurring in the enveloping algebra whose elements can be written as [13]

$$F(G_\alpha, K) = \sum_{k=0}^{\infty}\{F_0^{\alpha_1\ldots\alpha_k} + KF_1^{\alpha_1\ldots\alpha_k}\}G_{\alpha_1}\ldots G_{\alpha_k} \tag{8}$$

where the coefficients are symmetric in the $sl(2)$ indices $\alpha_1\ldots\alpha_k$. In particular, there is a doubling of the supersymmetry charges, which are given by G_α and $G_\alpha K$. Together with the $sl(2)$ generators $L_{\pm 1}$ and L_0 and the $s = 1$ generator

$$L = -2\lambda + \theta\frac{\partial}{\partial\theta} \tag{9}$$

they generate an $osp(2,2)$ algebra. Therefore the super-$\mathcal{B}_\lambda$ algebra has a hidden $N = 2$ supersymmetry. Each element of the algebra can be represented by a specific differential operator in the superspace coordinates z and θ.

We are now in a position to discuss the extension of the complete $N = 1$ super-Virasoro algebra. To show how this is done, it is instructive to first consider the

extension of the $osp(1,2)$ algebra to the full $N=1$ super-Virasoro algebra. Making use of the super-covariant derivative D we first rewrite the 5 generators of the $osp(1,2)$ algebra given in eq. 3 in terms of superfields as follows:

$$L_\lambda^{(3/2)}(k_{(3/2)}) = -k_{(3/2)}D^2 - \frac{1}{2}(Dk_{(3/2)})D - \lambda(D^2 k_{(3/2)}) \tag{10}$$

with the parameters $k_{(3/2)}$ satisfying the constraint eq. 2. The full $N=1$ super-Virasoro algebra is now obtained by discarding this constraint. The super-$\mathcal{B}_\lambda$ algebra can be extended to a higher-spin extension of the $N=1$ super-Virasoro algebra in a similar way. We first rewrite the generators of the super-$\mathcal{B}_\lambda$ algebra in terms of super-differential operators as follows ($s = 1/2, 1, 3/2, \ldots$):

$$L_\lambda^{(s)}(k_{(s)}) = \sum_{i=0}^{2s-1} A^i(s,\lambda)(D^{2s-i-1}k_{(s)})D^i \tag{11}$$

where the parameters $k_{(s)}$ satisfy the differential equations

$$D^{4s-1}k_{(s)} = 0 \tag{12}$$

The coefficients $A^i(s,\lambda)$ are fixed by requiring that the generators transform as non-decomposable representations of $osp(1,2)$ and are given by:

$$\begin{aligned} A^i(s,\lambda) &= (-)^{[s+1/2]+(2s+1)(i+1)+1}\frac{[s-1/2]!}{[i/2]![s-i/2-1/2]!} \\ &\times \frac{([i/2+1/2]+2\lambda)_{[s]-[i/2+1/2]}}{([s+i/2+1/2])_{2s-[s+i/2+1/2]}} \end{aligned} \tag{13}$$

Next we discard the constraint eq. 12. We thus obtain a higher-spin extension of the $N=1$ super-Virasoro algebra. To indicate the hidden $N=2$ supersymmetry we denote this algebra by $N=2$ super-$W_\infty(\lambda)$. It turns out that different choices of the parameter λ lead to the same algebra but in a different basis. The algebra was first given in the $\lambda=0$ basis in [10]. The generators of the algebra are given by $N=1$ superfields $W_\lambda^{(s)}$ for $s = 1/2, 1, 3/2, \ldots$. Due to the hidden $N=2$-supersymmetry all $N=1$ superfields combine into $N=2$ superfields.

The structure constants of the $N=2$ super-$W_\infty(\lambda)$ algebra have been calculated. Since the expressions are rather involved we will not give them here. The generic structure of the algebra is that the commutator of two generators of given spin s and t leads to a set of generators with highest spin $s+t-1/2$ or $s+t-3/2$ depending on whether s and t are integer or half-integer. The algebra simplifies considerably if a certain contraction is performed. In this contraction only the highest-spin generator in the commutator of two generators survive. One thus obtains the so-called $N=2$ super-w_∞ algebra [14] whose generators we denote by

$w^{(s)}$ $(s = 1/2, 1, 3/2, \ldots)$. Its explicit form in terms of the singular parts of OPE's is given by:

$$w^{(s)}(1)w^{(t)}(2) \sim -2\frac{\theta_{12}w^{(s+t-1/2)}}{z_{12}} \tag{14}$$

if s, t are integer. In all other cases with $s, t \neq 1/2$ the OPE expansions are given by

$$\begin{aligned} w^{(s)}(1)w^{(t)}(2) \sim\ & (-)^{|2s+1|_2}\{(s+t-\frac{3}{2})\frac{\theta_{12}w^{(s+t-3/2)}}{z_{12}^2} \\ & -\frac{1}{2}\frac{D_2 w^{(s+t-3/2)}}{z_{12}} + (s-\frac{1}{2})\frac{\theta_{12}\partial_2 w^{(s+t-3/2)}}{z_{12}}\} \end{aligned} \tag{15}$$

where $|s|_2$ is equal to zero for s even and 1 for s odd. The OPE's where s and/or t equals $1/2$ can be found in [9].

3. Gauging

It turns out that in order to realize the $N = 2$ super-w_∞ algebra as a Poisson bracket algebra one needs to introduce at least two $N = 1$ free scalar superfields ϕ and $\bar{\phi}$. This is due to the hidden $N = 2$ supersymmetry whose realization requires at least two $N = 1$ superfields. The explicit form of the generators in terms of these superfields is given by:

$$\begin{aligned} w^{(s)} &= (\partial\phi)^{s-1}D\phi D\bar{\phi} && (s \text{ integer}) \\ w^{(s)} &= (\partial\phi)^{s-1/2}D\bar{\phi} + \frac{1}{2}D\{D\phi(\partial\phi)^{s-3/2}D\bar{\phi}\} && (s \text{ half} - \text{integer}) \end{aligned} \tag{16}$$

Like for any Poisson bracket algebra in two dimensions, the gauging is achieved by adding to the kinetic terms of the superfields gauge field times current terms. We thus obtain an action $S = 1/\pi \int d^2Z L$, where L is given by

$$L = D\phi\bar{D}\bar{\phi} + \sum_{s=1/2}^{\infty} A_{(s)}w^{(s)} \tag{17}$$

Note that the kinetic term by itself is invariant under supersymmetries with a holomorphic parameter, i.e. $k_{(s)} = k_{(s)}(Z)$. Gauging means that we require the action to be invariant under supersymmetries with an arbitrary parameter $k_{(s)} = k_{(s)}(Z, \bar{Z})$, i.e. $\bar{D}k_{(s)} \neq 0$. One finds that the Noether procedure stops after one term and the final action is linear in the gauge fields $A_{(s)}$. The transformation rules for the matter fields follow from the general formula

$$\delta(k_{(s)})\phi(2) = \hbar^{-1} \sum_{s\geq 1/2} \oint \frac{dZ_1}{2\pi i} k_{(s)}(1) w^{(s)}(1)\phi(2) \tag{18}$$

and are given in [9]. The transformation rules of the gauge fields $A_{(s)}$ are basically given by the structure constants of the $N = 2$ super-w_∞ algebra. Their explicit form is given by[2]

$$\begin{aligned} \delta A_{(s)} &= \bar{D}k_{(s)} - 2 \sum_{t=1,2,\ldots}^{s-1/2} A_{(t)} k_{(s-t+1/2)} \\ &+ \sum_{t=\frac{1}{2},\frac{3}{2},\ldots}^{s-1} \{-(t-\frac{1}{2})A_{(t)}\partial k_{(s-t+3/2)} \\ &- \frac{1}{2} DA_{(t)} Dk_{(s-t+3/2)} + (s-t+1)\partial A_{(t)} k_{(s-t+3/2)}\} \end{aligned} \tag{19}$$

for half-integer s and

$$\begin{aligned} \delta A_{(s)} &= \bar{D}k_{(s)} + \sum_{t=\frac{1}{2},1,\ldots}^{s+t-3/2} \{-(t-\frac{1}{2})A_{(t)}\partial k_{(s-t+3/2)} \\ &+ \frac{1}{2}(-)^{|2t|_2} DA_{(t)} Dk_{(s-t+3/2)} + (s-t+1)\partial A_{(t)} k_{(s-t+3/2)}\} \end{aligned} \tag{20}$$

for integer s. It is understood that $k_{(s)} \equiv 0$ if $s \leq 1/2$. The transformations of the gauge fields under the $s = 1/2$ transformations are given in [9]. This finishes our discussion of the chiral $N = 2$ w_∞-supergravity theory.

Using the above field-theoretic realization, one finds an interesting relation between the full $N = 2$ super-$W_\infty(\lambda)$ algebra described in the previous section and the contracted $N = 2$ super-w_∞ algebra. A similar relation was derived for the bosonic case in [15]. It turns out that the contraction parameter relating the $N = 2$ super-$W_\infty(\lambda)$ algebra to the $N = 2$ super-w_∞ algebra can be identified with Planck's constant $\hbar$. The full $N = 2$ super-$W_\infty(\lambda)$ algebra can now be seen as a quantum deformation of the $N = 2$ super-w_∞ algebra. In the field-theoretic realization given above this quantum deformation arises when one removes potential anomalies that arise in the quantisation of the chiral $N = 2$ w_∞-supergravity theory. This leads effectively to a renormalization of the currents given in eq. 16. For instance, in the $\lambda = 0$ basis one finds that the renormalizations of the first few currents is given by

[2]We thank E. Sezgin for pointing out a simplification of the following formulae as compared to the ones given in [9] (see also [7]).

$$
\begin{aligned}
W^{(1/2)} &= D\bar{\phi} \\
W^{(1)} &= D\phi D\bar{\phi} + \sqrt{\hbar}\partial\phi \\
W^{(3/2)} &= \frac{1}{2}\partial\phi D\bar{\phi} + \frac{1}{2}D\phi\partial\bar{\phi} + \frac{1}{2}\sqrt{\hbar}\partial D\phi \\
W^{(2)} &= \partial\phi D\phi D\bar{\phi} + \sqrt{\hbar}\{\frac{1}{3}\partial D\phi D\bar{\phi} + \frac{1}{2}(\partial\phi)^2 \\
&+ \frac{1}{3}\partial\phi\partial\bar{\phi} - \frac{1}{3}D\phi\partial D\bar{\phi}\} + \frac{\hbar}{6}\partial^2\phi \qquad (21) \\
W^{(5/2)} &= \frac{1}{2}(\partial\phi)^2 D\bar{\phi} - \frac{1}{2}D\phi\partial D\phi D\bar{\phi} + \frac{1}{2}D\phi\partial\phi\partial\bar{\phi} \\
&+ \sqrt{\hbar}\{\frac{1}{6}\partial^2\phi D\bar{\phi} + \frac{1}{3}\partial D\phi\partial\bar{\phi} + \frac{1}{2}\partial D\phi\partial\phi \\
&- \frac{1}{3}\partial\phi\partial D\bar{\phi} - \frac{1}{6}D\phi\partial^2\bar{\phi}\} + \frac{\hbar}{12}\partial^2 D\phi
\end{aligned}
$$

The classical currents $w^{(s)}$ and the quantum currents $W^{(s)}$ are related by

$$
w^{(s)} = \lim_{\hbar\to 0} W^{(s)} \qquad (22)
$$

Taking multiple contractions between the quantum currents one recovers the $N = 2$ super-$W_\infty(\lambda)$ algebra in the $\lambda = 0$ basis. For more details we refer to [9].

One thus finds that there exists a consistent quantum theory of $N = 2$ w_∞ supergravity. The underlying symmetry algebra of the quantum theory is given by the $N = 2$ super-$W_\infty(\lambda)$ algebra.

4. Conclusions

We have shown how W_∞-supersymmetries can be gauged and quantised. The procedure is rather similar to the bosonic case [4, 5, 6, 7, 8] and can be applied to a quite general class of W-symmetries. A natural question to ask is whether these theories can be used for the construction of W (super)-string theories providing a generalization of the usual string theory which is based on the gauging and quantisation of the Virasoro-symmetries. Recently, the spectra of such W-strings have been discussed (see, for instance, [16, 17, 18].

One difference between usual strings and W-strings is that the first have a natural interpretation in terms of a Riemannian geometry whereas for the latter a similar geometric interpretation has not yet been found. There have been interesting developments indicating a geometric interpretation of W-symmetries [19, 20, 21]. It would be interesting to see whether this geometric approach can be applied to the W (super)-string.

Concerning the supersymmetry aspects, discussed in this contribution, we note that w_∞-supergravity differs in the following way from the bosonic case [9]. Starting from the classical currents given in eq. 16, one cannot write a current $w^{(s)}$ of given spin s as the product of two lower-spin currents $w^{(t)}w^{(u)}$ with $s = t + u$. Therefore, the telescoping procedure of [22] does not apply in this case. Our results therefore cannot be used to obtain a two-scalar superfield realization of a classical w_N-supergravity theory. We therefore expect that, using a two-scalar superfield realization, a Poisson bracket representation of a super-w_N algebra necessarily involves background charges. A "quantum" telescoping procedure does exist but the nonlinear terms in the quantum algebra are multiplied by inverse powers of $\hbar$ such that one cannot take the limit $\hbar \to 0$.

Acknowledgements

I would like to thank the participants of the conference for useful discussions and the organizers for their kind invitation. Section 3 is based upon the work of [11] and section 4 on that of [9]. This work has been made possible by a fellowship of the Royal Netherlands Academy of Arts and Sciences (KNAW).

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