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Contractions of Lie Groups Applied to Differential Equations

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ABSTRACT. The purpose of this paper is to initiate the study of the relationship between systems of differential equations whose respective symmetry groups are related by a contraction. As an example, we have constructed a "basis" of $so(2,1)$-invariant scalar equations of order two that contract to $p(1,1)$-invariant equations. A step toward a systematical study is presented.

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1. Introduction

The application of continuous (Lie) groups to systems of differential equations was started by Sophus Lie around the middle of the nineteenth century. He essentially unified all the known integration techniques. Since then, important developments were done (See, for instance, the Reference [1]). On the other hand, the theory of contractions of Lie groups was introduced in the fifties by E. Inönü and E. P. Wigner [2]. It stemed from the need to relate mathematically the symmetries underlying different physical phenomena. The purpose of this paper is to combine the two subjects. This article constitutes, to our knowledge, the first attempt to relate systematically different systems of differential equations, whose respective symmetry Lie groups are related through a contraction.

Many fundamental equations of physics (gravitation, particle physics, physics of fluids, etc.) are complicated and in general, nonlinear. Their symmetry Lie group can help to simplify them (through the symmetry reduction method), to obtain new solutions from known ones, to classify the solutions, etc. Like numerical integration and reduction to linear or integrable nonlinear equations, the group theoretical methods help to solve nonlinear differential equations. The theory of contractions was motivated by the study of physical phenomena. Given two physical theories or physical systems, from which one is a limit (or an approximation) of the other, the contraction procedure allows one to obtain the Lie group/algebra (and its representations) of the approximate theory from that of the exact one. The specific example considered in the original paper by Inönü and Wigner [2] was the contraction from the Poincaré group to the Galilei group, which are the symmetry groups of the Einsteinian and Galilean Relativity, respectively. Along with the contractions of the Lie algebras, methods to contract their representations were devised [3–9].

Hence, it is natural to apply the contraction methods to the Lie algebras underlying systems of differential equations. By suitably identifying some contraction parameters into an *initial* system of differential equations, such that for some limit values, these parameters lead to a different system of equations, one would possibly find that its solutions *contract* to the solutions of the *contracted* system of equations. (Such an approach offers new possibilities in relating different special functions, as done in Reference [2]). Physical example is an electrical circuit, with an associated equation or system of equations, and some symmetry group. Then if some component of the circuit burns out, the equations may be modified continuously in such a way that their group is a contraction of that associated with the original system. Another example occurs in quantum field theory, where one can take some

constants (*e.g.* masses, coupling constants, etc.) in the Lagrangian going to zero.

For example, Inönü and Wigner [2] pointed out that the asymptotic behavior of the associated Legendre function $\bar{P}_l^m$ into the Bessel function J_m,

$$J_m(\rho) = \lim_{l\to\infty} \bar{P}_l^m\left(\cos(\rho/l)\right),$$

is related to the contraction of the group of rotations $SO(3)$ into the Euclidean group in the plane $E(2)$. This relationship induces a limit process linking the respective differential equations. In fact, the associated Legendre function satisfies the modified Legendre equation,

$$(1-x^2)y_{xx} - 2xy_x + \left[l(l+1) - \frac{m^2}{1-x^2}\right] y = 0,$$

whereas the Bessel function satisfies the Bessel equation,

$$\rho^2 y_{\rho\rho} + \rho y_\rho + (\rho^2 - m^2)y = 0.$$

Both equations arise in various domains of physics. Changing x into $\cos(\rho/l)$ (where l is a constant), the Legendre equation becomes (after multiplication by $\sin^2(\rho/l)$)

$$l^2\sin^2(\rho/l)y_{\rho\rho} + l\cos(\rho/l)\sin(\rho/l)y_\rho + \left[\sin^2(\rho/l)l(l+1) - m^2\right] y = 0.$$

In the limit $l \to \infty$ (such thatsin$(\rho/l) \sim \rho/l$, $\cos(\rho/l) \sim 1$), it becomes the Bessel equation. However, the structure of this relationship is different from that considered in this paper.

The study of the contractions of systems of differential equations (and their solutions) can be done in two ways. The first and the most natural one is to identify contraction parameters (*e.g.* physical constants) into the equations and see if these parameters pass into the expression of their symmetry Lie algebra, and if, in the singularity limit, they lead to a contracted Lie algebra that corresponds to the contracted system. The second way to proceed is rather to introduce the contraction parameters into the structure of a Lie algebra right from the beginning. Then we construct a "basis" of differential equations invariant under this group (such as done, for instance, in Reference [10]) and look for what happens in the contraction limit. Here we haved used the latter way to proceed for it is more systematical. Within this point of view, the interest lies essentially in the Lie algebras considered, while for the former point of view, the interest rather lies in the equations themselves.

In the next section, we outline the mathematics needed. As a first working example, we find in Section 3 the scalar equations of order 2, in two dimensional spacetime, with the symmetry groups $SO(2,1)$ (de Sitter) and $P(1,1)$ (Poincaré). In the Section 4, we present a first step toward a systematical study, based on the method of graded contractions [11–13]. Concluding remarks are in Section 5.

2. Mathematical preliminaries

We first sketch the method to obtain the symmetry Lie group of a system of differential equations. After, we describe the general theory of contractions of Lie algebras, and particularly the theory of graded contractions.

Hereafter, we consider Lie point symmetries of differential equations. Thus, when we talk of "the symmetry group of a system of differential equations", we mean the largest local Lie group of point transformations that acts on the independent and dependent variables of this system and leaves invariant its solution set. In other words, it transforms solutions into each other. More details and examples about this are in Reference [1].

Consider a system of m differential equations of order k,

$$\Delta^a(x, u^{(k)}) = 0, \tag{2.1}$$

where $a = 1, \ldots, m$, and where we have p independent real variables,

$$x = (x_1, \ldots, x_p) \in \mathbb{R}^p, \tag{2.2}$$

and q dependent real variables,

$$u = (u_1, \ldots, u_q) \in \mathbb{R}^q. \tag{2.3}$$

In (2.1), $u^{(k)} = \{u, u_{(1)}, \ldots, u_{(k)}\}$ is the set of all derivatives (up to order k) for every u.

The symmetry group G of the system (2.1) is defined as the set of diffeomorphisms $\mathcal{X}_g(x,u)$ and $\mathcal{U}_g(x,u)$,

$$(x,u) \to (\mathcal{X}_g(x,u), \mathcal{U}_g(x,u)), \tag{2.4}$$

that leave the system (2.1) invariant.

In order to get a geometrical picture of the problem, note that we consider functions f of x,

$$u_\alpha = f_\alpha(x), \qquad (\alpha = 1, \ldots, q) \tag{2.5}$$

and their k-th *prolongations*,

$$\mathrm{pr}^{(k)} f(x) \equiv \{f, f_x, \ldots, f_{x^k}\}, \tag{2.6}$$

where f_{x^k} denotes k-th derivatives. This induces an extended space, called *jet space*, which consists in the set of independent variables, of dependent variables and their derivatives. The action (2.4) induces a "prolonged" action of G over the jet space. That the prolonged action leaves invariant the system of equations is the fundamental criterion for G to be the *symmetry group* of that system.

As S. Lie did, it is much more manageable to work out the Lie algebra L (of G), realized in terms of the vector fields

$$v = \sum_{i=1}^{p} \xi^i(x,u)\frac{\partial}{\partial x_i} + \sum_{\alpha=1}^{q} \phi^\alpha(x,u)\frac{\partial}{\partial u_\alpha}. \tag{2.7}$$

The coefficients ξ^i and ϕ^α determine L. They must satisfy the condition that the k-th prolongation $\mathrm{pr}^{(k)}v$ annihilates (2.1) on its solution surface,

$$\mathrm{pr}^{(k)}v \cdot \Delta^a(x,u^{(k)}) \mid_{\Delta^b=0} = 0, \tag{2.8}$$

with $a, b = 1, \ldots, m$.

Let us introduce the notation

$$u_{\alpha,J} = \frac{\partial^{|J|}u_\alpha}{\partial x_1^{j_1}\cdots\partial x_p^{j_p}}, \tag{2.9}$$

where $J = (j_1, \ldots j_p)$ is a set of nonnegative integers and $\mid J \mid \equiv j_1 + \cdots + j_p$. We also need the total derivative operator,

$$D_i = \frac{\partial}{\partial x_i} + \sum_{\alpha=1}^{q}\sum_{J} u_{\alpha,J+J_i}\frac{\partial}{\partial u_{\alpha,J}}, \quad 0 \leq \mid J \mid \leq k, \tag{2.10}$$

for $i = 1, \ldots, p$, and where J_i denotes a p-tuple with 1 on the i-th position and zeros elsewhere.

The prolongation formula is

$$\mathrm{pr}^{(k)}v = v + \sum_{\alpha=1}^{q}\sum_{J} \phi^{\alpha,J}(x,u^{(k)})\frac{\partial}{\partial u_{\alpha,J}}, \tag{2.11}$$

where $1 \leq \mid J \mid \leq k$. The coefficients $\phi^{\alpha,J}$ are obtained from the recursive formula

$$\phi^{\alpha,J+J_i} = D_i\phi^{\alpha,J} - \sum_{j=1}^{p}\left(D_i\xi^j\right)u_{\alpha,J+J_j}, \quad \mid J \mid \geq 1, \tag{2.12}$$

starting from the first order coefficients

$$\phi^{\alpha,J_i} = D_i\phi^\alpha - \sum_{j=1}^{p} \left(D_i\xi^j\right) u_{\alpha,J_j}, \tag{2.13}$$

which is nothing but (2.12) with $J = (0)$.

The prolongation formula (2.11) is a differential operator acting on the jet space and, when substituted into (2.8), provides a system of auxiliary equations to be solved for the ξ^i and ϕ^α of (2.7). These in turn determine the Lie algebra L. Many concrete examples are in the Section 2.4 of Reference [1].

As noted in the Introduction, here we reverse the method by starting from a given Lie algebra, forming a realization (2.7) and looking for a "basis" of differential equations which satisfy (2.8). Such a construction is presented in Reference [10].

We now turn to a brief description of the general theory of contractions of Lie algebras [2–9]. In general, a contraction is a limit process which allows to obtain, from a given Lie algebra, another Lie algebra (in general, nonisomorphic) named *contracted algebra*, or *contraction*. Clearly, both have same dimension. The traditional contraction methods consist in taking a Lie algebra L with commutation relations $[,]$, forming a family of Lie algebras L^ε, which depend on a parameter (or set of parameters) ε, by modifying the commutators as

$$[a,b]_\varepsilon = \sigma_\varepsilon^{-1}\left([\sigma_\varepsilon(a), \sigma_\varepsilon(b)]\right). \tag{2.14}$$

The automorphism σ_ε is nonsingular for every ε except the singularity points ε'. However, at these points, the commutator

$$[a,b]' = \lim_{\varepsilon\to\varepsilon'} [a,b]_\varepsilon, \tag{2.15}$$

may be well-defined and then, the commutator $[,]'$ determines a contracted algebra L'. It is often convenient to take L associated to all $\varepsilon = 1$, and L', to some $\varepsilon = 0$. In this formulation, the ε-dependent automorphism is the key object. For example, in the original article [2], it was defined to be

$$\sigma_\varepsilon = Id_A \dotplus \varepsilon\, Id_B, \tag{2.16}$$

where A, B are subspaces of L (A being a Lie subalgebra) such that

$$L = A \dotplus B. \tag{2.17}$$

The singularity point is $\varepsilon' = 0$. Many generalizations and applications of the concept to the representations of the Lie algebras were introduced [3–9].

In Section 4, we shall use the method of *graded contractions* [11–13]. It first consists in grading (*i.e.* vector decomposing) the Lie algebra L as follows,

$$L = \bigoplus_{\mu \in \Gamma} L_\mu, \tag{2.18}$$

with

$$0 \neq [L_\mu, L_\nu] \subseteq L_{\mu+\nu}, \tag{2.19}$$

for all $\mu, \nu, \mu + \nu \in \Gamma$. Systematical study of the Lie gradings was initiated in Reference [14]. For our purpose, Γ is an abelian finite group. It thus consists in a tensor product of cyclic groups. The grading subspaces are then eigenspaces of L under some set of commuting automorphisms of finite order. Everywhere in the paper, we use the notation $+$ for the product in Γ.

Rather than introducing an ε-dependent family of automorphisms singular at some point, we modify the commutator directly,

$$[L_\mu, L_\nu]_\varepsilon \equiv \varepsilon_{\mu,\nu}[L_\mu, L_\nu]. \tag{2.20}$$

It clearly preserves the Γ-grading. By enforcing the Jacobi identities to the new commutator, we find that the ε-parameters must satisfy the 2-cocycle relations

$$\varepsilon_{\mu,\nu}\varepsilon_{\mu+\nu,\rho} = \varepsilon_{\nu,\rho}\varepsilon_{\nu+\rho,\mu}, \tag{2.21}$$

for all $\mu, \nu, \rho \in \Gamma$. The ε-parameters are symmetric, (*i.e.* $\varepsilon_{\mu,\nu} = \varepsilon_{\nu,\mu}$). A solution of (2.21) provides us with a contraction of the Lie algebra L.

The contraction of representations is treated similarly. Along with decomposition (2.18), we decompose the (to be contracted) L-module V as

$$V = \bigoplus_{\mu \in \Gamma} V_\mu, \tag{2.22}$$

such that

$$0 \neq L_\mu \cdot V_\nu \subseteq V_{\mu+\nu}. \tag{2.23}$$

The action of L over V is modified by introducing parameters ψ,

$$L_\mu \overset{\psi}{\cdot} V_\nu \equiv \psi_{\mu,\nu} L_\mu \cdot V_\nu. \tag{2.24}$$

For a given contraction ε, the ψ-parameters must satisfy (from defining representation relations)

$$\varepsilon_{\mu,\nu}\psi_{\mu+\nu,\rho} = \psi_{\nu,\rho}\psi_{\mu,\nu+\rho} = \psi_{\mu,\rho}\psi_{\nu,\mu+\rho}, \tag{2.25}$$

for all $\mu, \nu, \rho \in \Gamma$. A solution ψ provides a contraction of the representation. The method to contract tensor product of representations is introduced in Reference [12]. We shall not use it hereafter.

The basic idea of the present paper is to introduce contraction parameters into a realization (2.7) of a given Lie algebra L, to form the prolongation operator (2.11), look for the invariant differential equations, and see what happens when the parameters go to their limit values. The way we introduce the parameters is dictated by the specific contraction method employed.

Having explained how to find the symmetry group of a system of differential equations, and the general contraction theory, we end this section by stating more precisely the question studied in this paper. Consider a real Lie algebra L and a realization V_L in terms of vector fields defined over the space ($\sim \mathbb{R}^{p+q}$) of variables x and u. Let $\Delta^{(k)}_{V_L}$ be a set of elementary L-invariants of order k , obtained via the realization V_L. Now suppose we use some contraction procedure to construct a family of Lie algebras L^ε and realizations V_{L^ε}. Let $L' = \lim_{\varepsilon \to \varepsilon'} L^\varepsilon$ be a contraction of L for which we have the limit realization $V_{L'} = \lim_{\varepsilon \to \varepsilon'} V_{L^\varepsilon}$. Now suppose we can construct the invariant systems $\Delta^{(k)}_{V_{L^\varepsilon}}$ and $\Delta^{(k)}_{V_{L'}}$. The question is whether it is possible to construct a realization V_{L^ε} such hat

$$\Delta^{(k)}_{V_{L'}} = \lim_{\varepsilon \to \varepsilon'} \Delta^{(k)}_{V_{L^\varepsilon}}. \tag{2.26}$$

Hereafter, we do not attempt to treat this question in all its generality. Rather, we present in Section 3 a "plausibility example", and in Section 4, we make one step toward a systematical construction, by using the theory of graded contractions.

3. Application to $(1+1)$-de Sitter group

In the present section, we construct a realization for $L = so(2,1)$ (de Sitter group in $(1+1)$ dimensions) for scalar equations of order two, with two independent variables x and t, such that (2.26) is satisfied for $L' = p(1,1)$ (Poincaré in $(1+1)$ dimensions). Hence we display a basis of $so(2,1)$-invariant differential equations that suitably contract to $p(1,1)$-invariant equations. More information about the de Sitter groups and their contractions to the "kinematical groups" is found in References [15–17].

Half of the work is already done, for the $p(1,1)$-invariants are given in [10] . The Lie algebra $p(1,1) = \{P_0, P_1, K\}$ has commutation relations

$$[P_0, K] = P_1, \ [P_1, K] = P_0, \ [P_0, P_1] = 0, \tag{3.1}$$

and can be realized as

$$\begin{aligned} P_0 &= \partial_t, \\ P_1 &= \partial_x, \\ K &= x\partial_t + t\partial_x. \end{aligned} \tag{3.2}$$

Note that there is another class of realizations (see [10]), but we shall not consider it hereafter.

The invariance conditions (2.8) read

$$\begin{aligned} &\mathrm{pr}^{(2)} P_\mu \cdot \Delta = P_\mu \cdot \Delta = 0, \qquad (\mu = 0, 1), \\ &\mathrm{pr}^{(2)} K \cdot \Delta = [x\partial_t + t\partial_x - u_t\partial_{u_x} - u_x\partial_{u_t} - 2u_{xt}(\partial_{u_{xx}} + \partial_{u_{tt}}) \\ &- (u_{tt} + u_{xx})\partial_{u_{xt}}] \cdot \Delta = 0. \end{aligned} \tag{3.3}$$

We must solve the system of characteristics for this set of equations. It provides us with the following five elementary invariants

$$\begin{aligned} \Delta_1 &= u, \\ \Delta_2 &= u_t^2 - u_x^2, \\ \Delta_3 &= u_{tt} - u_{xx}, \\ \Delta_4 &= (u_t - u_x)^2(u_{xx} + 2u_{xt} + u_{tt}), \\ \Delta_5 &= (u_t + u_x)^2(u_{xx} - 2u_{xt} + u_{tt}). \end{aligned} \tag{3.4}$$

The more general $p(1,1)$-invariant equation has the form

$$\mathcal{F}(\Delta_1, \ldots, \Delta_5) = 0. \tag{3.5}$$

The details about this construction, and other examples, are given in Reference [10].

Now we turn to $so(2,1)$, and construct a realization that contracts to (3.2) and such that its invariants contract to (3.4). The commutation relations of $so(2,1) = \{P_0, P_1, K\}$ are

$$[P_0, K] = P_1, \ [P_1, K] = P_0, \ [P_1, P_0] = K. \tag{3.6}$$

It is isomorphic to $sl(2, \mathbb{R}) = \{L_0, L_+, L_-\}$,

$$[L_+, L_-] = 2L_0, \ [L_0, L_\pm] = \pm L_\pm. \tag{3.7}$$

The latter is realized in terms of the vector fields over a 2-dimensional manifold (with independent variables $x^+ = x + t$, $x^- = x - t$),

$$\begin{aligned} L_+ &= \partial_-, \\ L_0 &= -x^-\partial_- + x^+\partial_+, \\ L_- &= -(x^-)^2\partial_- + (2x^-x^+ - \alpha)\partial_+, \end{aligned} \tag{3.8}$$

where α is an arbitrary constant.

This realization may be *Inönü-Wigner*-contracted to the realization (3.2) of $p(1,1)$ by

(1) defining $L^\varepsilon_\pm = \varepsilon L_\pm$, $L^\varepsilon_0 = L_0$,
(2) changing the variable $x^- \to x^-/\varepsilon$,
(3) choosing $\alpha = -1/\varepsilon$,

before going to the limit $\varepsilon = 0$. Thus, before contraction, the realization is

$$\begin{aligned} L^\varepsilon_+ &= \partial_-, \\ L^\varepsilon_0 &= -x^-\partial_- + x^+\partial_+, \\ L^\varepsilon_- &= -\varepsilon^2(x^-)^2\partial_- + (2\varepsilon^2 x^- x^+ - \varepsilon\alpha)\partial_+. \end{aligned} \tag{3.9}$$

We see that in L^ε_-, only the last term does survive after the contraction.

The second-order prolongation system (analogous to (3.3)) is

$$\begin{aligned} &\mathrm{pr}^{(2)} L^\varepsilon_+ = \partial_-, \\ &\mathrm{pr}^{(2)} L^\varepsilon_0 = -x^-\partial_- + x^+\partial_+ + u_-\partial_{u_-} - u_+\partial_{u_+} + 2u_{--}\partial_{u_{--}} - 2u_{++}\partial_{u_{++}}, \\ &\mathrm{pr}^{(2)} L^\varepsilon_- = -\varepsilon^2(x^-)^2\partial_- + (2\varepsilon^2 x^- x^+ - \alpha\varepsilon)\partial_+ + 2\varepsilon^2(x^- u_- - x^+ u_+)\partial_{u_-} \\ &- 2\varepsilon^2 x^- u_+\partial_{u_+} + 2\varepsilon^2(u_- + 2x^- u_{--} - 2x^+ u_{-+})\partial_{u_{--}} \\ &- 2\varepsilon^2(x^+ u_{++} + u_+)\partial_{u_{-+}} - 4\varepsilon^2 x^- u_{++}\partial_{u_{++}}, \end{aligned} \tag{3.10}$$

where $u_+ = \frac{\partial u}{\partial x^+}$, etc. The associated system of characteristics leads to five elementary invariants,

$$\begin{aligned} \Delta^\varepsilon_1 &= u, \\ \Delta^\varepsilon_2 &= u_{++}/u_+^2, \\ \Delta^\varepsilon_3 &= 2\varepsilon^2 x^+ u_+ + \varepsilon\alpha\frac{u_- u_{++}}{u_+} - \varepsilon\alpha u_{-+}, \\ \Delta^\varepsilon_4 &= \varepsilon^2(x^+)^2 u_+^2 - \varepsilon\alpha u_- u_+, \\ \Delta^\varepsilon_5 &= \varepsilon^2\alpha^2 u_{-+}^2 - \varepsilon^2\alpha^2 u_{--}u_{++} - 4\alpha\varepsilon^3 x^+ u_+ u_{-+} \\ &+ 4\alpha\varepsilon^3 u_- u_+ - 2\alpha\varepsilon^3 x^+ u_- u_{++} + 4\varepsilon^4(x^+)^3 u_+ u_{++}. \end{aligned} \tag{3.11}$$

Going to the limit $\varepsilon = 0$, with $\alpha\varepsilon = -1$, they contract to

$$\begin{aligned} \Delta'_1 &= \lim_{\varepsilon\to 0}\Delta^\varepsilon_1 = u, \\ \Delta'_2 &= u_{++}/u_+^2, \\ \Delta'_3 &= -\frac{u_- u_{++}}{u_+} + u_{-+}, \\ \Delta'_4 &= u_- u_+, \\ \Delta'_5 &= u_{-+}^2 - u_{--}u_{++}. \end{aligned} \tag{3.12}$$

Substituting back the initial independent variables $x = (x^+ + x^-)/2$, $t = (x^+ - x^-)/2$, the contracted invariants read

$$\begin{aligned}
\Delta_1' &= u, \\
\Delta_2' &= \frac{u_{xx} + 2u_{xt} + u_{tt}}{(u_x + u_t)^2}, \\
\Delta_3' &= -\frac{1}{4}\frac{(u_x - u_t)(u_{xx} + 2u_{xt} + u_{tt})}{(u_x + u_t)} + \frac{1}{4}(u_{xx} - u_{tt}), \\
\Delta_4' &= \frac{1}{4}(u_x^2 - u_t^2), \\
\Delta_5' &= \frac{1}{16}\left[(u_{xx} - u_{tt})^2 - (u_{xx} + u_{tt})^2 + 4u_{xt}^2\right],
\end{aligned} \tag{3.13}$$

which all satisfy (3.3) and hence are all $p(1,1)$ invariants! Moreover, they constitute a *complete basis*, for all $p(1,1)$-invariants (3.4) can be expressed in terms of them,

$$\begin{aligned}
\Delta_1 &= \Delta_1', \\
\Delta_2 &= 4\Delta_4', \\
\Delta_3 &= -4(\Delta_3' + \Delta_2'\Delta_4'), \\
\Delta_4 &= 16\Delta_2'\Delta_4'^{\,2}, \\
\Delta_5 &= 16\frac{(\Delta_3' + \Delta_2'\Delta_4')^2 - \Delta_5'}{\Delta_2'}.
\end{aligned} \tag{3.14}$$

The Δs are in (3.4) and the Δ's, in (3.13). Note that the change to variables x, t may be performed in (3.11).

In summary, we have found a basis of $so(2,1)$-invariant scalar differential equations of order two, in two dimensions, such that

$$\mathcal{F}(\Delta_1^\varepsilon, \ldots, \Delta_5^\varepsilon) = 0 \tag{3.15}$$

is the more general $so(2,1)$ invariant equation. In the limit $\varepsilon \to 0$, they contract to a $p(1,1)$-invariant equation (3.5) and so should do their solutions. However, the contractions of the solutions (related to the contractions of special functions) is to be considered elsewhere.

4. Application of graded contractions

We now suggest a systematical approach to the problem, using the theory of *graded contractions*. It can be applied at once to all Lie algebras. The basic idea is to start from a graded representation, as shown in (2.22)-(2.23),

and to realize that representation in terms of vector fields on a space having the same dimension as the representation. If these vector fields induce prolonged vector fields (2.11) that do provide well-defined invariant equations (through (2.8)), then it is completely straightforward, using (2.24), to introduce contraction parameters into the vector fields and hence, into the invariant equations. Note that as a particular case, one can study the contractions of the ordinary invariants of Lie algebras (such as classified, for instance, in [18]). To do so, one just has to consider the adjoint representation and to construct the invariants from the vector fields themselves (and not from their prolongations).

We now explain this more precisely. Consider a general d-dimensional real Lie algebra L, having basis $\{a_1, \ldots, a_d\}$ and an n-dimensional representation $T(L)$,

$$T^i_{jk} \equiv [T(a_i)]_{jk}\,, \tag{4.1}$$

where $i = 1, \ldots, d; j, k = 1, \ldots, n$. For the adjoint representation, the Ts clearly correspond to the structure constants. From (2.23), we see that if $a_i \in L_\mu$ and $k \in V_\nu$ then $j \in \mu + \nu$. If we include the grading index, (4.1) reads

$$T^{\mu i}_{(\mu+\nu)j,\nu k} \equiv [T(a_{\mu i})]_{(\mu+\nu)j,\nu k}\,. \tag{4.2}$$

Representation (4.1) provides us with the following n-dimensional vector field realization of L,

$$v(a_i) = \sum_{j,k} x_j T^i_{jk} \partial_k, \tag{4.3}$$

or, in terms of (4.2),

$$v(a_{\mu i}) = \sum_{j,k} x_{(\mu+\nu)j} T^{\mu i}_{(\mu+\nu)j,\nu k} \partial_{\nu k}. \tag{4.4}$$

It is straightforward to build the prolonged operator (2.11) by using (2.12)-(2.13), and by identifying the variables x and u to the basis elements a_i. But it is not so easy to find the invariants, as done in Section 3. Whenever it is possible, we introduce contraction parameters, following (2.24), so that (4.4) becomes

$$v^\psi(a_{\mu i}) = \sum_{j,k} x_{(\mu+\nu)j} \psi_{\mu,\nu} T^{\mu i}_{(\mu+\nu)j,\nu k} \partial_{\nu k}. \tag{4.5}$$

The ψ parameters will clearly be contained within the expression of the invariant equations. It is also clear that, by taking the limit values of the ψs

in these equations, we shall obtain equations invariant under the contracted algebra. Hence, from the set of invariant equations of the initial Lie algebra alone, we can readily obtain invariant equations of all its contractions.

When we consider contractions of the invariants of the Lie algebra, we must start with the adjoint representation,

$$T^i_{jk} = c^i_{jk}, \tag{4.6}$$

where the cs are the structure constants of L. The ensuing contracted vector fields are, in analogy with (4.5),

$$v^\varepsilon(a_{\mu i}) = \sum_{j,k} x_{(\mu+\nu)j}\varepsilon_{\mu,\nu}c^{\mu i}_{(\mu+\nu)j,\nu k}\partial_{\nu k}. \tag{4.7}$$

As a simple example, we go back to the contraction from $so(2,1)$ to $p(1,1)$, and consider the ordinary invariants of the algebras (*not* the invariant differential equations). We denote the basis of $so(2,1)$ and $p(1,1)$ as

$$a_1 = K,\ a_2 = P_0,\ a_3 = P_1. \tag{4.8}$$

From (3.6), (4.3) and (4.6), we have that for $so(2,1)$,

$$\begin{aligned} v(a_1) &= -x_3\partial_2 - x_2\partial_3, \\ v(a_2) &= x_3\partial_1 - x_1\partial_3, \\ v(a_3) &= x_2\partial_1 + x_1\partial_2, \end{aligned} \tag{4.9}$$

from which we deduce the invariant

$$x_1^2 - x_2^2 + x_3^2 \to a_1^2 - a_2^2 + a_3^2 = K^2 - P_0^2 + P_1^2. \tag{4.10}$$

For $p(1,1)$, we have from (3.1),

$$\begin{aligned} v(a_1) &= -x_3\partial_2 - x_2\partial_3, \\ v(a_2) &= x_3\partial_1, \\ v(a_3) &= x_2\partial_1, \end{aligned} \tag{4.11}$$

and the ensuing invariant is

$$x_2^2 - x_3^2 \to a_2^2 - a_3^2 = P_0^2 - P_1^2. \tag{4.12}$$

Now let us endow $so(2,1)$ with the following $\Gamma = \mathbb{Z}_2$-grading structure,

$$\begin{aligned} so(2,1) &= L_0 \dotplus L_1, \\ L_0 &= \{a_1\},\ L_1 = \{a_2, a_3\}, \end{aligned} \tag{4.13}$$

so that the vector fields (4.9) become, following (4.7),

$$\begin{aligned} v^\varepsilon(a_1) &= \varepsilon_{0,1}(-x_3\partial_2 - x_2\partial_3), \\ v^\varepsilon(a_2) &= \varepsilon_{0,1}x_3\partial_1 - \varepsilon_{1,1}x_1\partial_3, \\ v^\varepsilon(a_3) &= \varepsilon_{0,1}x_2\partial_1 - \varepsilon_{1,1}x_1\partial_2. \end{aligned} \tag{4.14}$$

The Lie algebra $p(1,1)$ is obtained by taking $\varepsilon_{0,1} = 1$ and $\varepsilon_{1,1} = 0$. It thus "kills" $[a_2, a_3]$. The general invariant provided by (4.14) is

$$\varepsilon_{1,1}x_1^2 - \varepsilon_{0,1}x_2^2 + \varepsilon_{0,1}x_3^2 \to \varepsilon_{1,1}a_1^2 - \varepsilon_{0,1}a_2^2 + \varepsilon_{0,1}a_3^2. \tag{4.15}$$

In the limit leading to $p(1,1)$, it becomes (4.12).

5. Concluding remarks

We recall that the purpose of this paper is to initiate research in the application of contractions of Lie algebras/groups to the symmetries of systems of differential equations. As a particular example, we have shown how $so(2,1)$-invariant scalar equations of order two in (1+1) dimensions are contracted to equations invariant under $p(1,1)$. Generalizations can be made in many ways. Here we studied Lie point symmetries and a possible generalization is to consider also the "generalized symmetries", discussed in Reference [1]. In Section 4 we considered linear realizations of the algebra, that is, the coefficients of the ∂s were linear in the xs. It would be interesting to generalize this construction to nonlinear realizations, and see how to introduce contraction parameters into these realizations. In practice, the symmetry groups of differential equations are often infinite dimensional. The contraction procedure of Section 4 directly applies to infinite dimensional Lie algebras and representations. The actual application of the method to the present problem still remains to be done.

As we mentioned in the Introduction, the reverse study (*i.e.* starting from a given system of differential equations) is more natural. It should be interesting to look at concrete examples (from various domains of physics) where constants in the equations can be seen as actual contraction parameters. The passage from the solutions of the initial (or exact) system to the approximate ones should be made in terms of these constants. The question related to the contraction of solutions, and to the general study of contractions of special functions (using the material of References [19–20]) can naturally be regarded from the point of view of the present article. The whole subject certainly deserves further study.

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