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USING QUANTUM ALGEBRAS IN q-SPECIAL FUNCTION THEORY

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Abstract

We show how the representation theory of the quantum algebra $\mathcal{U}_q(sl(2))$ can be used to make advances in the study of the q-hypergeometric series ${}_2\phi_1(a,b;c;q,z)$.

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The well-established connection between quantum algebras and q-special functions provides a vantage stanpoint in the study of their properties.[1-12] As in standard Lie theory,[13] these functions arise as matrix elements of certain generating operators of the quantum algebras in specific representations, and also as basis vectors of the corresponding representation spaces. This algebraic setting naturally leads to generating relations, orthogonality properties and addition formulas involving the q-special functions.

In these letter we shall illustrate the power of this "group-theoretic" interpretation by examining the q-hypergeometric series ${}_2\phi_1(a,b;c;q,z)$ and its relation to the repersentation theory of the quantum algebra $\mathcal{U}_q(sl(2))$. We shall give a few examples of generating functions and summation formulas involving ${}_2\phi_1$, concentrating on the methodology, rather than exploiting all the possible formulas that can be obtained in this way. Our aim is in fact to show the usefulness and simplicity of the present approach, without claiming to be exaustive.

In the quantum algebra interpretation of the hypergeometric function ${}_2\phi_1$, an important role is played by the following two q-exponential functions[14]

$$e_q(z) = \sum_{n=0}^{\infty} \frac{1}{(q;q)_n}\, z^n = \frac{1}{(z;q)_\infty}\,, \qquad |z|<1\,, \tag{1a}$$

$$E_q(z) = \sum_{n=0}^{\infty} \frac{q^{\frac{1}{2}n(n-1)}}{(q;q)_n}\, z^n = (-z;q)_\infty\,, \tag{1b}$$

where, for a and α arbitrary complex numbers, $(a;q)_\alpha$ stands for the q-shifted factorial

$$(a;q)_\alpha = \frac{(a;q)_\infty}{(aq^\alpha;q)_\infty}\,, \tag{2}$$

with

$$(a;q)_\infty = \prod_{k=0}^{\infty}(1-aq^k)\,, \qquad |q|<1\,. \tag{3}$$

Note that $e_q(z)\,E_q(-z)=1$, and that $\lim_{q\to 1^-} e_q\big(z(1-q)\big) = \lim_{q\to 1^-} E_q\big(z(1-q)\big) = e^z$. We shall denote by T_z the q-dilatation operator which acts as

$$T_z\,\varphi(z) = \varphi(qz)\,, \tag{4}$$

on functions of the variable z; out of it, the q-difference operators

$$D_z^+ = z^{-1}(1-T_z)\,, \tag{5a}$$

$$D_z^- = z^{-1}(1-T_z^{-1})\,, \tag{5b}$$

are constructed. Observe that $\frac{1}{(1-q)}D_z^+ \to d/dz$ and $\frac{1}{(1-q^{-1})}D_z^- \to d/dz$ as $q\to 1$. The basic hypergeometric series ${}_r\phi_s$ is defined by[14]

$$\begin{aligned} &{}_r\phi_s(a_1,a_2,\ldots,a_r;b_1,\ldots,b_s;q;z) \\ &\quad = \sum_{n=0}^{\infty} \frac{(a_1;q)_n(a_2;q)_n\ldots(a_r;q)_n}{(q;q)_n(b_1;q)_n\ldots(b_s;q)_n}\left[(-1)^n q^{\frac{n(n-1)}{2}}\right]^{1+s-r} z^n\,, \end{aligned} \tag{6}$$

with $q \neq 0$ when $r > s+1$. Since $(q^{-m};q)_n = 0$, for $n = m+1, m+2, \ldots$, the series ${}_r\phi_s$ terminates if one of the numerator parameters $\{a_i\}$ is of the form q^{-m} with $m = 0,1,2\ldots$, and $q \neq 0$. In the following, we shall concentrate on the case

$$ {}_2\phi_1(a,b;c;q,z) = \sum_{n=0}^{\infty} \frac{(a;q)_n (b;q)_n}{(q;q)_n (c;q)_n} z^n , \tag{7} $$

in terms of which various matrix elements will be expressed. Notice that as $c \to q^{1-m}$, with m a positive integer, this function satisfies the following limit relation[6]

$$ \frac{1}{(q;q)_{-m}} \, {}_2\phi_1(a,b,q^{1-m};q;z) = z^m \, {}_2\phi_1(aq^m, bq^m, q^{m+1};q,z) \, \frac{(a;q)_m (b;q)_m}{(q;q)_m} . \tag{8} $$

Furthermore, in the following we shall always assume $|q| < 1$.

The quantum universal enveloping algebra $\mathcal{U}_q(sl(2))$ is the Hopf algebra generated by the elements k, k^{-1}, e and f satisfying the relations[15,16]

$$ k\,e\,k^{-1} = q^{1/2} e , \qquad k\,f\,k^{-1} = q^{-1/2} f , \qquad [e,f] = \frac{k^2 - k^{-2}}{q^{1/2} - q^{-1/2}} , \tag{9} $$

and $k\,k^{-1} = k^{-1}k = 1$. The coproduct $\Delta : \mathcal{U}_q(sl(2)) \to \mathcal{U}_q(sl(2)) \otimes \mathcal{U}_q(sl(2))$, antipode $S : \mathcal{U}_q(sl(2)) \to \mathcal{U}_q(sl(2))$ and counit $\varepsilon : \mathcal{U}_q(sl(2)) \to \mathbf{C}$ are defined by:

$$ \begin{aligned} &\Delta(k) = k \otimes k , && \Delta(e) = e \otimes k + k^{-1} \otimes e , && \Delta(f) = f \otimes k + k^{-1} \otimes f , \\ &S(k) = k^{-1} , && S(e) = -q^{1/2} e , && S(f) = -q^{-1/2} f , \\ &\varepsilon(k) = 1 , && \varepsilon(e) = 0 , && \varepsilon(f) = 0 . \end{aligned} \tag{10} $$

We now introduce a left $\mathcal{U}_q(sl(2))$-module $V^{(\lambda,m_0)} = \bigoplus_{j\in I} \mathbf{C}\xi_j$, where $I = \{i \mid i = m_0 + n,\ n \in \mathbf{Z}\}$, and λ, m_0 are complex numbers.[9,17] $V^{(\lambda,m_0)}$ is infinite dimensional, unless $m_0 + \lambda$ and $m_0 - \lambda$ are both integers. The corresponding representation is characterized by the following action of the generators on the basis vectors ξ_j, $j \in I$:

$$ \begin{aligned} k\,\xi_j &= q^{-j/2} \xi_j , \\ e\,\xi_j &= q^{(1-2\lambda)/4} \frac{1-q^{\lambda+j}}{1-q} \xi_{j-1} , \\ f\,\xi_j &= q^{(1-2\lambda)/4} \frac{1-q^{\lambda-j}}{1-q} \xi_{j+1} . \end{aligned} \tag{11} $$

Given any $a \in \mathcal{U}_q(sl(2))$, its matrix elements a_{ij} in this representation are defined by

$$ a\,\xi_j = \sum_{i\in\mathbf{Z}+m_0} \xi_i \, a_{ij} . \tag{12} $$

Analogy with ordinary Lie theory suggests to consider the following element of $\mathcal{U}_q(sl(2))$:

$$ U(\alpha,\beta) = E_q(\alpha\,e)\, e_q(\beta\,f) , \tag{13} $$

where α, β are complex parameters. Indeed, set $k = q^{-h/2}$ and $q = e^{-\eta}$. In the limit $q \to 1^-$, $\eta \to 0^+$, the defining relations (9) become those of $sl(2)$: $[h,e] = -e$, $[h,f] = f$, $[e,f] = -2h$, and $U\big((1-q)\alpha, (1-q)\beta\big)$ goes into the $SL(2)$ group element $e^{\alpha e} e^{\beta f}$.

The matrix elements $U_{ij}(\alpha,\beta)$ of $U(\alpha,\beta)$ on $V^{(\lambda,m_0)}$ turn out to be expressible in terms of the function ${}_2\phi_1(a,b;c;q,z)$. In fact, with the help of (1), using (11) and identities involving q-shifted factorials, it is straightforward to show that

$$ \begin{aligned} U_{ij}(\alpha,\beta) = &\left(\beta \frac{q^{(1-2\lambda)/4}}{1-q}\right)^{i-j} \frac{(q^{\lambda-i+1};q)_{i-j}}{(q;q)_{i-j}} \\ &\times {}_2\phi_1\left(q^{\lambda+i+1}, q^{i-\lambda}; q^{i-j+1}; q, -\alpha\beta \frac{q^{(1-2i)/2}}{(1-q)^2}\right) , \quad \text{if } i-j \geq 0 , \end{aligned} \tag{14a} $$

$$ \begin{aligned} U_{ij}(\alpha,\beta) = &\, q^{(j-i)(j-i-1)/2} \left(\alpha \frac{q^{(1-2\lambda)/4}}{1-q}\right)^{j-i} \frac{(q^{\lambda+i+1};q)_{j-i}}{(q;q)_{j-i}} \\ &\times {}_2\phi_1\left(q^{\lambda+j+1}, q^{j-\lambda}; q^{j-i+1}; q, -\alpha\beta \frac{q^{(1-2i)/2}}{(1-q)^2}\right) , \quad \text{if } i-j \leq 0 , \end{aligned} \tag{14b} $$

with $i,j \in \mathbf{Z} + m_0$. This establishes most simply the connection between the basic hypergeometric series ${}_2\phi_1$ and $\mathcal{U}_q(sl(2))$. Notice that by using the limiting formula (8), the above two expressions for $U_{ij}(\alpha,\beta)$ are valid irrespective from the sign of $i-j$.

Off course, other combinations of little and big exponentials can also be used. For instance, in the case of

$$ \widetilde{U}(\alpha,\beta) = E_q(\beta\,f)\, e_q(\alpha\,e) , \tag{15} $$

one similarly obtains ($i,j \in \mathbf{Z} + m_0$)

$$ \begin{aligned} \widetilde{U}_{ij}(\alpha,\beta) = &\, q^{(i-j)(i-j-1)/2} \left(\beta \frac{q^{(1-2\lambda)/4}}{1-q}\right)^{i-j} \frac{(q^{\lambda-i+1};q)_{i-j}}{(q;q)_{i-j}} \\ &\times {}_2\phi_1\left(q^{\lambda-j+1}, q^{-j-\lambda}; q^{i-j+1}; q, -\alpha\beta \frac{q^{(1+2i)/2}}{(1-q)^2}\right) , \quad \text{if } i-j \geq 0 , \end{aligned} \tag{16a} $$

$$ \begin{aligned} \widetilde{U}_{ij}(\alpha,\beta) = &\left(\alpha \frac{q^{(1-2\lambda)/4}}{1-q}\right)^{j-i} \frac{(q^{\lambda+i+1};q)_{j-i}}{(q;q)_{j-i}} \\ &\times {}_2\phi_1\left(q^{\lambda-i+1}, q^{-i-\lambda}; q^{j-i+1}; q, -\alpha\beta \frac{q^{(1+2i)/2}}{(1-q)^2}\right) , \quad \text{if } i-j \leq 0 . \end{aligned} \tag{16b} $$

The case of $e_q(\alpha\,e)\, E_q(\beta\,f)$ has been disscussed in Ref.[9], while the matrix elements of operators in $\mathcal{U}_q(sl(2))$ involving two little or two big q-exponentials can be expressed in terms of the q-hypergeometric series ${}_3\phi_1$ and ${}_2\phi_2$, and will not be discussed here.

To get generating relations for the ${}_2\phi_1$ using the results (14) and (16), one first notices that it is possible to give a one-variable model for the representation (11), where the generators are expressed as q-difference operators in the complex variable z acting on the space of all linear combinations of the functions z^n, $n \in \mathbb{Z}$. Indeed, by taking

$$\begin{aligned} k &= q^{-m_0/2}\, T_z^{-1/2} \,, \\ e &= q^{(1-2\lambda)/4} \left(\frac{1}{1-q} D_z^+ + \frac{1-q^{\lambda+m_0}}{1-q} \frac{1}{z} T_z \right) \,, \\ f &= q^{(1-2\lambda)/4} \left(\frac{z^2}{1-q} D_z^- + \frac{1-q^{\lambda-m_0}}{1-q} z\, T_z^{-1} \right) \,, \end{aligned} \tag{17}$$

and $\xi_j = z^n$, $j = m_0 + n$, $n \in \mathbb{Z}$, for the basis vectors, one can check that the relations (11) are satisfied. Let us now act directly with the operator $U(\alpha,\beta)$, with e and f expressed as in (17), on z^{j-m_0}. After using the definition (1) of the q-exponentials and suitable identities involving q-shifted factorials, one explicitly finds

$$\begin{aligned} U(\alpha,\beta)\, z^{j-m_0} = \; & z^{j-m_0} \sum_{m,l\in\mathbb{Z}} q^{-l(l-1)/2} \frac{(q^{j-\lambda};q)_l}{(q;q)_l} \left(-\beta z \frac{q^{-j+(1+2\lambda)/4}}{1-q} \right)^l \\ & \times \frac{(q^{-\lambda-j-l};q)_m}{(q;q)_m} \left(-\frac{\alpha}{z} \frac{q^{j+l+(1+2\lambda)/4}}{1-q} \right)^m \,. \end{aligned} \tag{18}$$

The sum over m can be explicitly performed thanks to Heine's q-binomial theorem:[14]

$$\sum_{n=0}^{\infty} \frac{(q^{-\alpha};q)_n}{(q;q)_n} z^n = \frac{(zq^{-\alpha};q)_\infty}{(z;q)_\infty} = (zq^{-\alpha};q)_\alpha \,, \qquad |z|<1, \quad |q|<1 \,. \tag{19}$$

The remaining sum over l gives a ${}_2\phi_0$ q-hypergeometric series, so that

$$\begin{aligned} U(\alpha,\beta)\, z^{j-m_0} = \; & z^{j-m_0} \left(-\frac{\alpha}{z} \frac{q^{(1-2\lambda)/4}}{1-q}; q \right)_{\lambda+j} \\ & \times {}_2\phi_0\left(-\frac{\alpha}{z} \frac{q^{j+(1+2\lambda)/4}}{1-q}, q^{j-\lambda}; q, \beta z \frac{q^{-j+(1+2\lambda)/4}}{1-q} \right) \,. \end{aligned} \tag{20}$$

Recall now the definition (12) for the matrix elements of $a \equiv U(\alpha,\beta)$, and insert (20) for the l.h.s., while in the r.h.s. substitute for $U_{ij}(\alpha,\beta)$ the result (14a). After using the following transformation rule for the ${}_2\phi_1$ series,[14]

$${}_2\phi_1(q^a, q^b; q^c; q, z) = (q^{a+b-c}z;q)_{c-a-b}\; {}_2\phi_1(q^{c-a}, q^{c-b}; q^c; q, q^{a+b-c}z) \,, \tag{21}$$

set $j = m_0$, $x = -\alpha\beta\, q^{m_0+1/2}/(1-q)^2$ and $y = -(1-q)q^{(2\lambda+3)/4}\, z/\alpha$ to get the following generating formula for the q-hypergeometric function ${}_2\phi_1$:

$$\begin{aligned} & (xq^{-2m_0};q)_{2m_0}\, (q/y;q)_{m_0+\lambda}\; {}_2\phi_0\left(q^{m_0+\lambda+1}/y, q^{m_0-\lambda}; q, xyq^{-2m_0-1}\right) \\ & \quad = \sum_{k\in\mathbb{Z}} \frac{(q^{m_0-\lambda};q)_k}{(q^{2m_0+1}/x;q)_k\, (q;q)_k}\, y^k\; {}_2\phi_1\left(q^{\lambda-m_0+1}, q^{-m_0-\lambda}; q^{k+1}; q, x\right) \,. \end{aligned} \tag{22}$$

Here again the summand in the r.h.s. is well defined for all integers k, thanks to the limiting relation (8). An equivalent generating formula can be similarly obtained starting with the operator $\widetilde{U}(\alpha,\beta)$.

An orthogonality relation involving two ${}_2\phi_1$ functions can be derived by combining the results (14) and (16). By recalling that $e_q(z)\, E_q(-z) = 1$, one sees that

$$U(\alpha,\beta)\, \widetilde{U}(-\alpha,-\beta) = \mathbf{1} \,; \tag{23}$$

acting on ξ_j, one then finds ($i,j \in \mathbb{Z} + m_0$)

$$\delta_{i-j,0} = \sum_{l\in\mathbb{Z}+m_0} U_{il}(\alpha,\beta)\, \widetilde{U}_{lj}(-\alpha,-\beta) \,. \tag{24}$$

Insert now for the matrix elements U_{il} and $\widetilde{U}_{lj}$ the expression (14b) and the one that it is obtained from (16a) with the use of the transformation formula (21). After some simplifications and the redefinition $z = -\alpha\beta\, q^{1/2}/(1-q)^2$, one finally arrives at the following relation

$$\begin{aligned} \delta_{i-j,0} = \; & \frac{q^{(i^2+j^2)/2}\, (q/z;q)_j}{(q^{\lambda+1};q)_i\, (q^{-\lambda};q)_j} \sum_{l\in\mathbb{Z}+m_0} q^{l(l-1)/2} \left(-zq^{-i-j}\right)^l \frac{(z;q)_l\, (q^{\lambda+1};q)_l\, (q^{-\lambda};q)_l}{(q;q)_{l-j}\, (q;q)_{l-i}} \\ & \times {}_2\phi_1\left(q^{\lambda+l+1}, q^{l-\lambda}; q^{l-i+1}; q, zq^{-i}\right)\; {}_2\phi_1\left(q^{\lambda+l+1}, q^{l-\lambda}; q^{l-j+1}; q, zq^{-j}\right) \,. \end{aligned} \tag{25}$$

The matrix elements $U_{ij}(\alpha,\beta)$ and $\widetilde{U}_{ij}(\alpha,\beta)$ themselves define models of the module $V^{(\lambda,m_0)}$ and directly provide a two-variable realization of $\mathcal{U}_q(sl(2))$. To see this, for each element $a \in \mathcal{U}_q(sl(2))$ define the operator $\pi(a)$ acting on the variables α, β such that $\pi(a)\, U(\alpha,\beta) = U(\alpha,\beta)\, a$. Upon operating on ξ_j, one obtains:

$$\pi(a)\, U_{ij}(\alpha,\beta) = \sum_{l\in\mathbb{Z}+m_0} U_{il}(\alpha,\beta)\, a_{lj} \,, \tag{26}$$

that is, the functions $U_{ij}(\alpha,\beta)$ transform like the vectors ξ_j. One can easily construct the operators π for k, e and f, and thus for any $a \in \mathcal{U}_q(sl(2))$ by composition. For instance, one finds that

$$\pi^{(l)}(k) = q^{-l/2}\, T_\alpha^{-1/2}\, T_\beta^{1/2} \,, \tag{27a}$$

$$\pi^{(l)}(e) = -qD_\alpha^- - \frac{q^{1/2}}{(1-q)^2}\, \beta \left(q^l\, T_\alpha T_\beta^{-1} - q^{-l} T_\alpha^{-1} \right) \,, \tag{27b}$$

$$\pi^{(l)}(f) = D_\beta^+ \,, \tag{27c}$$

acting on the basis vectors

$$\xi_j^{(l)}(\alpha,\beta) = U_{lj}(\alpha,\beta) \,, \qquad j,l \in \mathbb{Z} + m_0 \,, \tag{28}$$

obey the same commutation relations as k, e and f.

The operators $\pi^{(l)}(k)$ and $\pi^{(l)}(f)$ are obtained by using the first two relations in (9) and from the property: $D^+_{\beta}\, e_q(\beta f) = e_q(\beta f)\, f$, respectively. To derive $\pi^{(l)}(e)$, one first acts with $-q\, D^-_{\alpha}$ on $U(\alpha,\beta)$ using $-q\, D^-_{\alpha}\, E_q(\alpha e) = E_q(\alpha e)\, e$, and then shows that

$$E_q(-\beta f)\, e\, e_q(\beta f) = e - \frac{q^{1/2}}{(1-q)^2}\, \beta\, \left(k^{-2} - (1-q^{-1}\beta f)^{-1} k^2\right) , \tag{29}$$

with the help of the following formula[2]

$$E_q(\zeta X)\; Y\, e_q(-\zeta X) = \sum_{n=0}^{\infty} \frac{\zeta^n}{(q;q)_n}\, [X,Y]_n , \tag{30}$$

where

$$[X,Y]_0 = Y, \qquad [X,Y]_{n+1} = q^n\, X\, [X,Y]_n - [X,Y]_n\, X, \quad \text{for } n = 1,2,\ldots .$$

Similar results are obtained using $\widetilde{U}_{ij}(\alpha,\beta)$.

The observation that the matrix elements $U_{ij}(\alpha,\beta)$ are also basis vectors for representations of $\mathcal{U}_q(sl(2))$ can be used to obtain further properties of the basic hypergeometric function ${}_2\phi_1$. First of all one can derive the q-difference equation obeyed by ${}_2\phi_1(a,b;c;q,z)$. The Casimir element C, which belongs to the center of $\mathcal{U}_q(sl(2))$, is given by

$$C = \frac{q^{1/2}k^2 + q^{-1/2}k^{-2} - 2}{(q^{1/2} - q^{-1/2})^2} + fe , \tag{31}$$

and on $V^{(\lambda,m_0)}$ assumes the value

$$C^{(\lambda)} = \frac{q^{\lambda+1/2} + q^{-\lambda-1/2} - 2}{(q^{1/2} - q^{-1/2})^2} . \tag{32}$$

Then, using *e.g.* (14*a*), from

$$\pi^{(l)}(C)\; U_{lj}(\alpha,\beta) = C^{(\lambda)}\; U_{lj}(\alpha,\beta) , \tag{33}$$

one can work out the equation satisfied by ${}_2\phi_1\left(q^{\lambda+i+1}, q^{i-\lambda}; q^{i-j+1}; q, -\alpha\beta\, \frac{q^{(1-2i)/2}}{(1-q)^2}\right)$. With the obvious identifications for a, b, c and z, one finds

$$\begin{aligned} \Big\{ z(c - abqz)\left(D^+_z\right)^2 + \Big[(1-c) + \big((1-a)(1-b) - (1-c)\big) z\Big] D^+_z \\ -(1-a)(1-b) \Big\}\, {}_2\phi_1(a,b;c;q,z) = 0 . \end{aligned} \tag{34}$$

Since we have a two-variable model of the module $V^{(\lambda,m_0)}$, from the general definition of matrix elements (12) one can write, recalling (28),

$$\widetilde{U}(\alpha,\beta)\; \xi^{(l)}_j(x,y) = \sum_{i\in\mathbb{Z}+m_0} U_{li}(x,y)\, \widetilde{U}_{ij}(\alpha,\beta) , \tag{35}$$

where the model independent matrix elements $\widetilde{U}_{ij}(\alpha,\beta)$ are still given by (16). This allows deriving addition formulas for the q-hypergeometric series ${}_2\phi_1$: one just needs to evaluate explicitly the l.h.s. of (35), *i.e.* to compute directly the action of $\widetilde{U}(\alpha,\beta)$ on the basis functions $\xi^{(l)}_j(x,y)$, when e and f are realized as in (27), α and β being replaced by x and y.

We start by first taking $\alpha = 0$, and compute the action of $\widetilde{U}(0,-\beta) \equiv E_q(-\beta f)$ on $\xi^{(l)}_j(x,y)$. Inserting the series expansion (7) for ${}_2\phi_1$ in (16*a*) and using the following summation formula,[2]

$$E_q(-\beta\, D^+_y)\, y^n = y^n\, (\beta/y;q)_n , \tag{36}$$

one easily finds $(l - j \geq 0)$

$$\begin{aligned} E_q(-\beta f)\, \xi^{(l)}_j(x,y) = \left(y\frac{q^{(1-2\lambda)/4}}{1-q}\right)^{l-j} \frac{(\beta/y;q)_{l-j}\, (q^{\lambda-l+1};q)_{l-j}}{(q;q)_{l-j}} \\ \times\, {}_3\phi_2\left(q^{\lambda+l+1}, q^{l-\lambda}, q^{l-j}\beta/y; q^{l-j+1}, 0; q, -xy\frac{q^{(1-2l)/2}}{(1-q)^2}\right) . \end{aligned} \tag{37}$$

Insert now this result and the explicit expressions (14*a*) and (16*a*) for the matrix elements $U_{li}(x,y)$ and $\widetilde{U}_{ij}(0,-\beta)$ in (35); setting $z = -xyq^{(1-2l)/2}/(1-q)^2$, $w = q^{\lambda-j}\beta/y$, one finally obtains $(j,l \in \mathbb{Z} + m_0)$

$$\begin{aligned} \frac{(wq^{j-\lambda};q)_{l-j}\, (q^{\lambda-l+1};q)_{l-j}}{(q;q)_{l-j}}\; {}_3\phi_2\left(q^{\lambda+l+1}, q^{l-\lambda}, wq^{l-\lambda}; q^{l-j+1}, 0; q, z\right) \\ = \sum_{k=0}^{\infty} \frac{(q^{\lambda-l+1};q)_{l-j-k}\, (q^{j-\lambda};q)_k}{(q;q)_{l-j-k}\, (q;q)_k}\; w^k\; {}_2\phi_1\left(q^{\lambda+l+1}, q^{l-\lambda}; q^{l-j-k+1}; q, z\right) . \end{aligned} \tag{38}$$

This gives another generating relation for the ${}_2\phi_1$ q-hypergeometric series.

In order to compute the action of $\widetilde{U}(\alpha,\beta)$ on $\xi^{(l)}_j(x,y)$ for $\alpha \neq 0$, one can now proceed in steps. One writes

$$\widetilde{U}(\alpha,\beta)\, \xi^{(l)}_j(x,y) = \sum_{i\in\mathbb{Z}+m_0} \left(E_q(\beta f)\, \xi^{(l)}_i(x,y)\right) \widetilde{U}_{ij}(\alpha,0) , \tag{39}$$

and then uses the result (37) together with (16*b*). Explicitly, $(l - j \geq 0)$

$$\begin{aligned} \widetilde{U}(\alpha,\beta)\, \xi^{(l)}_j(x,y) = \left(y\frac{q^{(1-2\lambda)/4}}{1-q}\right)^{l-j} \frac{(-\beta/y;q)_{l-j}\, (q^{\lambda-l+1};q)_{l-j}}{(q;q)_{l-j}} \\ \times \sum_{k=0}^{\infty} q^{-k(k-1)/2} \left(-\alpha y\frac{q^{(1+2j)/2}}{(1-q)^2}\right)^k \frac{(q^{\lambda-j+1};q)_k\, (q^{-\lambda-j};q)_k\, (-q^{l-j}\beta/y;q)_k}{(q;q)_k\, (q^{l-j+1};q)_k} \\ \times\, {}_3\phi_2\left(q^{\lambda+l+1}, q^{l-\lambda}, -q^{l+k-j}\beta/y; q^{l+k-j+1}, 0; q, -xy\frac{q^{(1-2l)/2}}{(1-q)^2}\right) . \end{aligned} \tag{40}$$

Insert now in (35) the expressions (14a) and (16a) for the matrix elements $U_{li}(x,y)$ and $\bar{U}_{ij}(\alpha,\beta)$. After some manipulations and the redefinitions $z = -xy\,q^{(1-2l)/2}/(1-q)^2$, $w = -\alpha\beta\,q^{(1+2j)/2}/(1-q)^2$, $t = \alpha y\,q^{(1+2j)/2}/(1-q)^2$, one gets the following formula:

$$\frac{(w/t;q)_{l-j}}{(q;q)_{l-j}} \sum_{n=0}^{\infty} q^{-n(n-1)/2}(-t)^n \frac{(q^{\lambda-j+1};q)_n\,(q^{-\lambda-j};q)_n\,(q^{l-j}w/t;q)_n}{(q;q)_n\,(q^{l-j+1};q)_n}$$
$$\times\ {}_3\phi_2\Big(q^{\lambda+l+1}, q^{l-\lambda}, q^{l+n-j}w/t; q^{l+n-j+1}, 0; q, z\Big) = \sum_{k\in\mathbb{Z}} \frac{q^{k(k-1)/2}}{(q;q)_k\,(q;q)_{l-j-k}} \Big(-\frac{w}{t}\Big)^k \tag{41}$$
$$\times\ {}_2\phi_1\big(q^{\lambda+l+1}, q^{l-\lambda}; q^{l-j-k+1}; q, z\big)\ {}_2\phi_1\big(q^{\lambda-j+1}, q^{-j-\lambda}; q^{k+1}; q, wq^k\big)\ .$$

This relation simplifies in the limit $t \to w/q$. In this case the l.h.s. factorizes into the product of a ${}_2\phi_1$ and a ${}_2\phi_0$ q-hypergeometric functions and one finally obtains the following addition formula:

$${}_2\phi_1\big(q^{\lambda+l+1}, q^{l-\lambda}; 0; q, z\big)\ {}_2\phi_0\big(q^{\lambda-j+1}, q^{-\lambda-j}; q, w/q\big) = \sum_{k\in\mathbb{Z}} \frac{(-1)^k q^{k(k+1)/2}}{(q;q)_{l-j-k}\,(q;q)_k}$$
$$\times\ {}_2\phi_1\big(q^{\lambda+l+1}, q^{l-\lambda}; q^{l-j-k+1}; q, z\big)\ {}_2\phi_1\big(q^{\lambda-j+1}, q^{-j-\lambda}; q^{k+1}; q, wq^k\big)\ . \tag{42}$$

This result can also be checked starting directly with the r.h.s of (41). Substitute for the two ${}_2\phi_1$ functions their series expansion (7). After some manipulations of the various q-shifted factorials, the bilateral sum over k can be explicitly performed with the help of the following Ramanujan's identity[14]

$$\sum_{k\in\mathbb{Z}} \frac{(a;q)_k}{(b;q)_k}\, z^k = \frac{(q;q)_\infty\,(b/a;q)_\infty\,(az;q)_\infty\,(q/az;q)_\infty}{(b;q)_\infty\,(q/a;q)_\infty\,(z;q)_\infty\,(b/az;q)_\infty}\ , \qquad |b/a| < |z| < 1\ . \tag{43}$$

Then, using the definition (2), in the limit $t \to w/q$ one recovers the l.h.s. of (42).

The addition formula (42) simplifies when $m_0 = 0$ and λ is a positive integer. In this case the module $V^{(m_0,\lambda)}$ is finite-dimensional (see (11)) and the various q-hypergeometric series in (42) reduce to polynomials. Indeed, recalling the definition of the little q-Jacobi polynomials,[14]

$$p_n(z;a,b;q) = {}_2\phi_1(q^{-n}, abq^{n+1}; aq; q, qz)\ , \tag{44}$$

and setting for simplicity $l = j$, $m = \lambda + j$ and $n = \lambda - j$, from (42) one gets

$$c_n\big(q^{m+1}; -q^n/z; q\big)\ \mathcal{P}_m\big(wq^{-m}; q^{n+1}; q\big)$$
$$= \sum_{-m}^{n} q^{k(k-n)} \begin{bmatrix} n \\ k \end{bmatrix} \frac{(q^{m+1};q)_k}{(q;q)_k}\, z^k\ p_{n-k}\big(z; q^k, q^{n-m}; q\big)\ p_m\big(wq^k; q^k, q^{n-m-k}; q\big)\ . \tag{45}$$

The c_n are q-Charlier polynomials,[14]

$$c_n(x;a;q) = {}_2\phi_1(q^{-n}, x; 0; q, -q^{n+1}/a)\ , \tag{46}$$

while the polynomials $\mathcal{P}_n$ are defined by

$$\mathcal{P}_n(z;a;q) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} (a;q)_k\, z^k\ , \tag{47}$$

with

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{(q;q)_n}{(q;q)_k\,(q;q)_{n-k}}\ .$$

They satisfy the following three-term recursion relation

$$\mathcal{P}_{n+1}(z;a;q) = \big[1 + z(1-aq^n)\big]\,\mathcal{P}_n(z;a;q) - z\,(1-q^n)\,\mathcal{P}_{n-1}(z;a;q)\ , \tag{48}$$

and reduce to the Rogers-Szegö polynomials[18-20] in the limit $a \to 0$.

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A different model for the algebra $\mathcal{U}_q(\mathcal{G}_{r,s})$ can be obtained by choosing the matrix elements $\widetilde{U}_{\mathbf{m}'\mathbf{n}',\mathbf{mn}}(\alpha,\beta,\gamma)$ as basis vectors. Explicitly, one finds

$$\begin{aligned} &R_i = -qD^-_{\alpha_i}\,, && S_i = -\gamma\,\partial_\gamma - \alpha_i\,\partial_{\alpha_i} + \omega_i\,, && i = 1,\dots,r\,,\\ &L_j = D^+_{\beta_j}\,, && T_j = -\gamma\,\partial_\gamma + \beta_j\,\partial_{\beta_j} + \sigma_j\,, && j = 1,\dots,s\,,\\ &V = D^+_\gamma\,, \end{aligned} \tag{3.8}$$

with

$$f_{\mathbf{m}-\mathbf{m}',\mathbf{n}-\mathbf{n}'} = \widetilde{U}_{\mathbf{m}'\mathbf{n}',\mathbf{mn}}(\alpha,\beta,\gamma)\,. \tag{3.9}$$

Identities for the q-hypergeometric series ${}_r\varphi_s$ similar to the ones in (3.6) can now be derived from this model. Starting again from the relations (2.5), one gets

$$\begin{aligned} &(1-q^{m_i}T_z)\;{}_r\varphi_s\left[\begin{matrix} q^{m_1},\dots,q^{m_r}\\ q^{n_1},\dots,q^{n_s}\end{matrix};q,z\right]\\ &\qquad = (1-q^{m_i})\;{}_r\varphi_s\left[\begin{matrix} q^{m_1},\dots,q^{m_{i-1}},q^{m_i+1},q^{m_{i+1}},\dots,q^{m_r}\\ q^{n_1},\dots,q^{n_s}\end{matrix};q,z\right]\,, \end{aligned} \tag{3.10a}$$

$$\begin{aligned} &(1-q^{n_j-1}T_z)\;{}_r\varphi_s\left[\begin{matrix} q^{m_1},\dots,q^{m_r}\\ q^{n_1},\dots,q^{n_s}\end{matrix};q,z\right]\\ &\qquad = (1-q^{n_j-1})\;{}_r\varphi_s\left[\begin{matrix} q^{m_1},\dots,q^{m_r}\\ q^{n_1},\dots,q^{n_{j-1}},q^{n_j-1},q^{n_{j+1}},\dots,q^{n_s}\end{matrix};q,z\right]\,, \end{aligned} \tag{3.10b}$$

$$\begin{aligned} &D^+_z\;{}_r\varphi_s\left[\begin{matrix} q^{m_1},\dots,q^{m_r}\\ q^{n_1},\dots,q^{n_s}\end{matrix};q,z\right]\\ &\qquad = \prod_{i=1}^{r}\prod_{j=1}^{s}\left(\frac{1-q^{m_i}}{1-q^{n_j}}\right)\;{}_r\varphi_s\left[\begin{matrix} q^{m_1+1},\dots,q^{m_r+1}\\ q^{n_1+1},\dots,q^{n_s+1}\end{matrix};q,z\right]\,. \end{aligned} \tag{3.10c}$$

From (2.7) one now sees that ${}_r\varphi_s$ obeys the following q-difference equation:

$$\begin{aligned} &\Big[(1-q^{m_1}T_z)\cdots(1-q^{m_r}T_z) - D^+_z\\ &\qquad\times(1-q^{n_1-1}T_z)\cdots(1-q^{n_s-1}T_z)\Big]\;{}_r\varphi_s\left[\begin{matrix} q^{m_1},\dots,q^{m_r}\\ q^{n_1},\dots,q^{n_s}\end{matrix};q,z\right] = 0\,. \end{aligned} \tag{3.11}$$

Since we have models for the representation (2.5), from the general definition of the matrix elements (2.9), one can write, *e.g.* using (3.9):

$$U(\alpha,\beta,\gamma)\;f_{\mathbf{m}-\mathbf{m}',\mathbf{n}-\mathbf{n}'}(x,y,z) = \sum_{\overline{\mathbf{m}},\overline{\mathbf{n}}}\widetilde{U}_{\mathbf{m}'\mathbf{n}',\overline{\mathbf{m}}\,\overline{\mathbf{n}}}(x,y,z)\;U_{\overline{\mathbf{m}}\,\overline{\mathbf{n}},\mathbf{mn}}(\alpha,\beta,\gamma)\,, \tag{3.12}$$

where the model independent matrix elements $U_{\overline{\mathbf{m}}\,\overline{\mathbf{n}},\mathbf{mn}}(\alpha,\beta,\gamma)$ are still given by (2.10). This allows deriving various identities involving the q-hypergeometric series ${}_r\phi_s$ and ${}_r\varphi_s$. To this end, one needs to evaluate explicitly the l.h.s. of (3.12), *i.e.* to compute directly the

action of $U(\alpha,\beta,\gamma)$ on the basis functions (3.9), when the algebra generators R_i, L_j and V are realized as in (3.8), with α_i, β_j and γ being replaced by x_i, y_j and z, respectively.

We shall start by considering the simple case in which all parameters α_i, β_j and γ but one are set to zero. Take the following operator $U(\alpha_1) = e_q(\alpha_1 R_1)$, whose matrix elements are easily seen to be given by

$$U_{\mathbf{m}'\mathbf{n}',\mathbf{mn}}(\alpha_1) = \frac{\alpha_1^{m_1'-m_1}}{(q;q)_{m_1'-m_1}}\;\delta_{m_2,m_2'}\cdots\delta_{m_r,m_r'}\;\delta_{n_1,n_1'}\cdots\delta_{n_s,n_s'}\,. \tag{3.13}$$

The action of $U(\alpha_1)\equiv e_q(-q\alpha_1 D^-_{x_1})$ on the basis function $f_{\mathbf{m}-\mathbf{m}',\mathbf{n}-\mathbf{n}'}(x,y,z)$ of (3.9) can be computed with the help of the following summation formula ($|\alpha/x|<1$)

$$\begin{aligned} e_q(-q\alpha D^-_x)\;x^n &= x^n\,\frac{1}{(-q\alpha/x;q)_{-n}}\\ &= q^{-n(n-1)/2}\,(-x/\alpha;q)_n\;\alpha^n\,. \end{aligned} \tag{3.14}$$

One finds

$$\begin{aligned} &e_q(-q\alpha_1 D^-_{x_1})\;f_{\mathbf{m}-\mathbf{m}',\mathbf{n}-\mathbf{n}'}(x,y,z)\\ &= \frac{1}{(-q\alpha_1/x_1;q)_{m_1-m_1'}}\prod_{i=1}^{r}\prod_{j=1}^{s}\left(q^{(m_i'-m_i)(m_i'-m_i-1)/2}\,\frac{x_i^{m_i'-m_i}}{(q;q)_{m_i'-m_i}}\;\frac{y_j^{n_j-n_j'}}{(q;q)_{n_j-n_j'}}\right)\\ &\qquad\times{}_r\varphi_{s+1}\left[\begin{matrix} q^{m_1-m_1'},\dots,q^{m_r-m_r'}\\ q^{n_1-n_1'+1},\dots,q^{n_s-n_s'+1},-q^{m_1-m_1'+1}\alpha_1/x_1\end{matrix};q,(-q)^r\frac{y_1\cdots y_s z}{x_1\cdots x_r}\right]\,. \end{aligned} \tag{3.15}$$

Insert now this result and the explicit expressions (2.12) and (2.10) for the matrix elements $\widetilde{U}_{\mathbf{m}'\mathbf{n}',\overline{\mathbf{m}}\,\overline{\mathbf{n}}}(x,y,z)$ and $U_{\overline{\mathbf{m}}\,\overline{\mathbf{n}},\mathbf{mn}}(\alpha_1)$ in (3.12); setting $u = (-q)^r y_1\cdots y_s z/x_1\cdots x_r$ and $w = -q\alpha_1/x_1$, one finally obtains, after suitable simplifications:

$$\begin{aligned} &\frac{1}{(w;q)_{m_1}}\;{}_r\varphi_{s+1}\left[\begin{matrix} q^{m_1},\dots,q^{m_r}\\ q^{n_1},\dots,q^{n_s},q^{m_1}w\end{matrix};q,u\right]\\ &\qquad = \sum_{k=0}^{\infty} w^k\,\frac{(q^{m_1};q)_k}{(q;q)_k}\;{}_r\varphi_s\left[\begin{matrix} q^{m_1+k},q^{m_2},\dots,q^{m_r}\\ q^{n_1},\dots,q^{n_s}\end{matrix};q,u\right]\,,\quad |w|<1\,. \end{aligned} \tag{3.16}$$

This can be considered to be a generating relation for the series ${}_r\varphi_s$ in terms of a suitable ${}_r\varphi_{s+1}$ function.

Similar results are obtained by considering the operator $U(\beta_1) = E_q(\beta_1 L_1)$, with matrix elements

$$U_{\mathbf{m}'\mathbf{n}',\mathbf{mn}}(\beta_1) = q^{(n_1-n_1')(n_1-n_1'-1)/2}\,\frac{\beta_1^{n_1-n_1'}}{(q;q)_{n_1-n_1'}}\;\delta_{m_1,m_1'}\cdots\delta_{m_r,m_r'}\;\delta_{n_2,n_2'}\cdots\delta_{n_s,n_s'}\,. \tag{3.17}$$

The action of $E_q(\beta_1 D^+_{y_1})$ on the basis function $f_{m-m',n-n'}(x,y,z)$ can be computed by using in this case the following summation formula ($|\beta/y|<1$)

$$E_q(\beta D^+_y)\; y^n = y^n\; (-\beta/y;q)_n\;. \tag{3.18}$$

The details of the calculation are similar to the ones described before, we shall therefore only give the final formula:

$$\begin{aligned}(w;q)_{n_1}\;{}_{r+1}\varphi_s&\left[\begin{matrix}q^{m_1},\ldots,q^{m_r},w\,q^{n_1}\\ q^{n_1+1},\ldots,q^{n_s+1}\end{matrix};q,u\right]\\ &=\sum_{k=0}^{\infty} w^k\, q^{kn_1}\,\frac{(q^{-n_1};q)_k}{(q;q)_k}\;{}_r\varphi_s\left[\begin{matrix}q^{m_1},\ldots,q^{m_r}\\ q^{n_1-k+1},q^{n_2+1},\ldots,q^{n_s+1}\end{matrix};q,u\right]\;,\quad |w|<1\;.\end{aligned} \tag{3.19}$$

The formulas (3.16) and (3.19) can also be checked directly by substituting for the functions ${}_r\varphi_s$ appearing in the l.h.s. their series expansions and then performing the sum over k with the help of the Heine's q-binomial theorem:[5]

$$\sum_{k=0}^{\infty}\frac{(q^{-\alpha};q)_k}{(q;q)_k}z^k=\frac{(zq^{-\alpha};q)_\infty}{(z;q)_\infty}=(zq^{-\alpha};q)_\alpha\;,\qquad |z|<1,\;\;|q|<1\;. \tag{3.20}$$

To get a summation formulas from (3.12) one needs to consider the general case where all parameters α_i, β_j and γ are nonzero. Using again the summation rules (3.14) and (3.18) the action on the l.h.s. of (3.12) can be expressed in terms of a ${}_{(r+s+1)}\varphi_{(r+s)}$ function. With this result and recalling the explicit form of the matrix elements appearing in the r.h.s. of (3.12), after some straightforward simplifications one arrives at the following formula

$$\begin{aligned}{}_{(r+s+1)}\varphi_{(r+s)}&\left[\begin{matrix}q^{m_1-m'_1},\ldots,q^{m_r-m'_r},q^{n_1-n'_1}\sigma_1,\ldots,q^{n_s-n'_s}\sigma_s,\eta\\ q^{m_1-m'_1}\rho_1,\ldots,q^{m_r-m'_r}\rho_r,q^{n_1-n'_1+1},\ldots,q^{n_s-n'_s+1}\end{matrix};q,w\right]\\ &=\sum_{\overline{m}\,\overline{n}}\prod_{i=1}^{r}\prod_{j=1}^{s}\Bigg((q^{[(\overline{m}_i-m_i)(\overline{m}_i+m_i-1)+(n_j-\overline{n}_j)(n_j-\overline{n}_j-1)]/2}\frac{(\rho_i;q)_{m_i-m'_i}}{(\sigma_j;q)_{n_j-n'_j}}\\ &\qquad\times\begin{bmatrix}m'_i-m_i\\ \overline{m}_i-m_i\end{bmatrix}_q\begin{bmatrix}n_j-n'_j\\ \overline{n}_j-n'_j\end{bmatrix}_q(-\rho_i\,q^{-m'_i})^{\overline{m}_i-m_i}(-\sigma_j)^{n_j-\overline{n}_j}\Bigg)\\ &\times{}_r\varphi_s\left[\begin{matrix}q^{\overline{m}_1-m'_1},\ldots,q^{\overline{m}_r-m'_r}\\ q^{\overline{n}_1-n'_1+1},\ldots,q^{\overline{n}_s-n'_s+1}\end{matrix};q,w\right]\;{}_r\phi_s\left[\begin{matrix}q^{m_1-\overline{m}_1},\ldots,q^{m_r-\overline{m}_r}\\ q^{n_1-\overline{n}_1+1},\ldots,q^{n_s-\overline{n}_s+1}\end{matrix};q,\frac{\sigma_1\cdots\sigma_s}{\rho_1\cdots\rho_s}\eta\,w q^{\overline{M}-\overline{N}}\right]\end{aligned} \tag{3.21}$$

where we have used the redefinitions $w=y_1\cdots y_s z/x_1\cdots x_r$, $\rho_i=-q\alpha_i/x_i$, $\sigma_j=-\beta_j/y_j$, $\eta=-\gamma/z$, $\overline{M}=\sum_{i=0}^{r}(\overline{m}_i-m_i)$ and $\overline{N}=\sum_{j=0}^{s}(\overline{n}_j-n_j)$, and

$$\begin{bmatrix}m\\ n\end{bmatrix}_q=\frac{(q;q)_m}{(q;q)_{m-n}\,(q;q)_n}\;, \tag{3.22}$$

is the q-binomial coefficient.

Identities involving the q-hypergeometric series ${}_r\phi_s$ can be similarly obtained by using the model (3.1) for the algebra $\mathcal{U}_q(\mathcal{G}_{r,s})$. The basic formula to start with is again (3.12), with U and $\widetilde{U}$ exchanged. For simplicity, we shall consider the cases for which only one of the parameters α_i, β_j and γ is nonzero. Since the computations are analogous to the ones described before, only the final formulas will be given. In the case $\alpha_1\neq 0$, one obtains

$$\begin{aligned}\frac{1}{(w;q)_{m_1}}\;{}_r\phi_{s+1}&\left[\begin{matrix}q^{m_1},\ldots,q^{m_r}\\ q^{n_1},\ldots,q^{n_s},q/w\end{matrix};q,q\,u/w\right]\\ &=\sum_{k=0}^{\infty}w^k\,\frac{(q^{m_1};q)_k}{(q;q)_k}\;{}_r\phi_s\left[\begin{matrix}q^{m_1+k},q^{m_2},\ldots,q^{m_r}\\ q^{n_1},\ldots,q^{n_s}\end{matrix};q,uq^{-k}\right]\;,\quad |w|<1\;,\end{aligned} \tag{3.23}$$

while for $\beta_1\neq 0$, one gets

$$\begin{aligned}(w;q)_{n_1}\;{}_{r+1}\phi_s&\left[\begin{matrix}q^{m_1},\ldots,q^{m_r},q/w\\ q^{n_1+1},\ldots,q^{n_s+1}\end{matrix};q,uw/q\right]\\ &=\sum_{k=0}^{\infty}w^k\,q^{kn_1}\,\frac{(q^{-n_1};q)_k}{(q;q)_k}\;{}_r\phi_s\left[\begin{matrix}q^{m_1},\ldots,q^{m_r}\\ q^{n_1-k+1},q^{n_2+1},\ldots,q^{n_s+1}\end{matrix};q,uq^{-k}\right]\;,\quad |w|<1\;.\end{aligned} \tag{3.24}$$

Again these formulas can be checked directly using (3.20).

The relations we have derived are just a few examples of the many identities that can be obtained from the quantum algebra interpretation of the q-hypergeometric functions ${}_r\phi_s$ and ${}_r\varphi_s$ that we have provided. Our aim was to show the usefulness and simplicity of this approach, without pretending to be exhaustive. As a final remark, notice that the two q-hypergeometric series ${}_1\phi_1$ and ${}_2\phi_1$ also appear in the representation theory of the lower dimensional quantum algebras $\mathcal{U}_q(\mathcal{E}(2))$ and $\mathcal{U}_q(sl(2))$,[11-18] $\mathcal{E}(2)$ being the Euclidean algebra in two dimensions.

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