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GENERALIZED q-BESSEL FUNCTIONS

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Abstract

An algebraic setting for a natural generalization of q-Bessel functions is provided. Many properties and formulas involving these functions are obtained and discussed.

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1. INTRODUCTION

Most special functions of mathematical physics naturally arise in Lie algebra theory. Many of them have q-analogs and their interpretation in terms of Lie theory is now systematically investigated. In strict analogy with the standard approach,[1] these basic or q-special functions[2] arise as matrix elements of certain algebra generators in specific representations of lower-dimensional universal enveloping Lie algebras, and also as basis vectors of the representation spaces.[3-11]

Basic analogs of the Bessel functions have been identified in realizations of the two-dimensional Euclidean algebra $\mathcal{E}(2)$ in the space of complex functions.[9,11] In this way, various generating relations and addition formulas for these q-special functions have been derived. Here we shall consider an extension $\mathcal{G}_{r,s}$ of the algebra $\mathcal{E}(2)$,[1] and similarly study realizations of it on functions of one and two complex variables. Natural generalizations of q-Bessel functions will be identified and various properties for these functions will be obtained within this Lie algebraic setting.

For convenience, we begin by listing some results and formulas of q-analysis.[2] For a and α arbitrary complex numbers, we denote by $(a;q)_\alpha$ the q-shifted factorial:

$$(a;q)_\alpha = \frac{(a;q)_\infty}{(aq^\alpha;q)_\infty} , \tag{1.1}$$

where

$$(a;q)_\infty = \prod_{k=0}^{\infty}(1-aq^k) , \qquad |q|<1 . \tag{1.2}$$

Note that for α a positive integer n,

$$(a;q)_n = (1-a)(1-aq)\dots(1-aq^{n-1}) . \tag{1.3}$$

These products satisfy various identities and we shall recall them whenever they will be required. Note also that $(q;q)_n/(1-q)^n \to n!$ as $q \to 1^-$. Of fundamental importance is Heine's q-binomial theorem which states that

$$\sum_{n=0}^{\infty} \frac{(a;q)_n}{(q;q)_n} z^n = \frac{(az;q)_\infty}{(z;q)_\infty} , \qquad |z|<1, \ \ |q|<1 . \tag{1.4}$$

Two q-exponential functions can be obtained from this formula:

$$e_q(z) = \sum_{n=0}^{\infty} \frac{1}{(q;q)_n} z^n = \frac{1}{(z;q)_\infty} , \qquad |z|<1 , \tag{1.5a}$$

$$E_q(z) = \sum_{n=0}^{\infty} \frac{q^{\frac{n(n-1)}{2}}}{(q;q)_n} z^n = (-z;q)_\infty . \tag{1.5b}$$

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Note that $e_q(z)\,E_q(-z) = 1$ and that

$$\lim_{q\to 1^-} e_q\big(z(1-q)\big) = \lim_{q\to 1^-} E_q\big(z(1-q)\big) = e^z \ . \tag{1.6}$$

Let T_z be the q-dilatation operator acting as follows on functions of the complex variable z

$$T_z\, f(z) = f(qz) \ . \tag{1.7}$$

Observe that, for n integer, $(T_z)^n\, f(z) \equiv T_z^n\, f(z) = f(q^n z)$; for α an arbitrary complex number, we define: $T_z^\alpha\, f(z) = f(q^\alpha z)$. The q-difference operators D_z^+ is given by

$$D_z^+ = z^{-1}(1 - T_z) \ , \tag{1.8}$$

and $\frac{1}{(1-q)} D_z^+ \to d/dz$ as $q \to 1$.

The basic hypergeometric series ${}_r\phi_s$ is defined by

$$\begin{aligned} &{}_r\phi_s(a_1, a_2, \ldots, a_r; b_1, \ldots, b_s; q; z) \\ &\quad = \sum_{n=0}^{\infty} \frac{(a_1;q)_n (a_2;q)_n \ldots (a_r;q)_n}{(q;q)_n (b_1;q)_n \ldots (b_s;q)_n} \left[(-1)^n q^{\frac{n(n-1)}{2}}\right]^{1+s-r} z^n \ , \end{aligned} \tag{1.9}$$

with $q \neq 0$ when $r > s+1$. Since $(q^{-m};q)_n = 0$, for $n = m+1, m+2, \ldots$, the series ${}_r\phi_s$ terminates if one of the numerator parameters $\{a_i\}$ is of the form q^{-m} with $m = 0, 1, 2 \ldots$, and $q \neq 0$. Unless explicitly stated, in the following we shall always assume $0 < q < 1$.

At least three different q-analogs of the Bessel functions of the first kind $J_\nu(z)$ have been studied in the literature. The first two has been introduced by Jackson, and further studied by Hahn (see [12,13] and references therein)

$$J_\nu^{(1)}(z;q) = \frac{1}{(q;q)_\nu} \left(\frac{z}{2}\right)^\nu {}_2\phi_1\Big(0, 0; q^{\nu+1}; q, -\frac{z^2}{4}\Big) \ , \tag{1.10a}$$

$$J_\nu^{(2)}(z;q) = \frac{1}{(q;q)_\nu} \left(\frac{z}{2}\right)^\nu {}_0\phi_1\Big(q^{\nu+1}; q, -\frac{z^2\, q^{\nu+1}}{4}\Big) \ . \tag{1.10b}$$

They are usually referred to the Jackson q-Bessel functions. The function $J_\nu^{(2)}(z;q)$ is an entire analytic function of z and it is connected to $J_\nu^{(1)}(z;q)$ through the formula:

$$J_\nu^{(2)}(z;q) = (-z^2/4;q)_\infty\, J_\nu^{(1)}(z;q) \ . \tag{1.11}$$

The third generalization of $J_\nu(z)$ was first introduced by Hahn, and then later studied by Exton (see [14,15] and references therein),

$$J_\nu(z;q) = \frac{1}{(q;q)_\nu}\, z^\nu\, {}_1\phi_1\Big(0, q^{\nu+1}; q, q\, z^2\Big) \ . \tag{1.12}$$

It is an entire analytic function of z. One can check that in the limit $q \to 1^-$ all three functions (1.10), (1.12) reduce to the ordinary Bessel function $J_\nu(z)$:

$$\begin{aligned} &\lim_{q\to 1^-} J_\nu^{(k)}\big((1-q)z;q\big) = J_\nu(z) \ , \qquad k = 1, 2 \ , \\ &\lim_{q\to 1^-} J_\nu\big((1-q)z/2;q\big) = J_\nu(z) \ . \end{aligned} \tag{1.13}$$

The basic Bessel functions satisfy the following recursion relation:

$$\frac{(1-q^\nu)}{z}\, J_\nu^{(k)}(z;q) = \frac{1}{2}\Big(J_{\nu-1}^{(k)}(z;q) + q^\nu\, J_{\nu+1}^{(k)}(z;q)\Big) \ , \qquad k = 1, 2 \ , \tag{1.14a}$$

$$(1 - q^\nu + z^2) J_\nu(z;q) = z\Big(J_{\nu-1}(z;q) + J_{\nu+1}(z;q)\Big) \ . \tag{1.14b}$$

Moreover, acting with the difference operator (1.8) on $J_\nu^{(1)}(z;q)$ one can check that

$$\Big(D_z^+ - \frac{z}{4}\Big)\, J_\nu^{(1)}(z;q) = \frac{1}{2}\Big(J_{\nu-1}^{(1)}(z;q) - J_{\nu+1}^{(1)}(z;q)\Big) \ . \tag{1.15}$$

From this, using (1.11), one can get an analogous formula for $J_\nu^{(2)}(z;q)$:

$$\Big[D_z^+ - \frac{z}{4} - \frac{z}{4}\Big(1 + q\big(1 + \frac{z^2}{4}\big)\Big) T_z\Big]\, J_\nu^{(2)}(z;q) = \frac{1}{2}\Big(J_{\nu-1}^{(2)}(z;q) - J_{\nu+1}^{(2)}(z;q)\Big) \ . \tag{1.16}$$

Modified q-Bessel functions can also be defined,[13]

$$I_\nu^{(k)}(z;q) = e^{-i\pi\nu/2}\, J_\nu^{(k)}(e^{i\pi/2} z;q) \qquad k = 1, 2 \ , \tag{1.17a}$$

$$I_\nu(z;q) = e^{-i\pi\nu/2}\, J_\nu(e^{i\pi/2} z;q) \ . \tag{1.17b}$$

These definition are analogues to the case $q = 1$. All the properties given above for $J_\nu^{(k)}(z;q)$, $k = 1, 2$, and $J_\nu(z;q)$ can be refrased in terms of the functions (1.17). For instance, the recursion relations (1.14) become,

$$(1 - q^\nu)\left(\frac{2}{z}\right) I_\nu^{(k)}(z;q) = I_{\nu-1}^{(k)}(z;q) - q^\nu\, I_{\nu+1}^{(k)}(z;q) \qquad k = 1, 2 \ , \tag{1.18a}$$

$$\big(1 - q^\nu - z^2\big)\, I_\nu(z;q) = z\big(I_{\nu-1}(z;q) - I_{\nu+1}(z;q)\big) \ , \tag{1.18b}$$

while (1.15) gives:

$$\Big(D_z^+ + \frac{1}{4}\Big)\, I_\nu^{(1)}(z;q) = \frac{1}{2}\Big(I_{\nu-1}^{(1)}(z;q) + I_{\nu+1}^{(1)}(z;q)\Big) \ . \tag{1.19}$$

Further properties of the q-Bessel functions are discussed in Refs.[9-19].

2. GENERALIZED q-BESSEL FUNCTIONS AND THE ALGEBRA $\mathcal{G}_{r,s}$

Given two positive integers (r,s), the three-dimensional algebra $\mathcal{G}_{r,s}$ is defined by the following commutation relations of its three generators:[1]

$$[J,P_+] = rP_+ \ , \qquad [J,P_-] = -sP_- \ , \qquad [P_+,P_-] = 0 \ . \tag{2.1}$$

Clearly, $\mathcal{G}_{1,1}$ is isomorphic to the two-dimensional Euclidean algebra $\mathcal{E}(2)$. We shall further assume that the two numbers r and s satisfy the following three requirements: 1) r and s are relatively prime; 2) r is odd; 3) if s is odd, then $r > s$, unless $r = s = 1$. Then, two Lie algebras $\mathcal{G}_{r,s}$ and $\mathcal{G}_{r',s'}$, with subscripts satisfying these properties, are isomorphic if and only if $r = r'$ and $s = s'$.

Each irreducible representation $Q_{r,s}(\omega, m_0)$ of $\mathcal{G}_{r,s}$ is characterized by two complex numbers ω and m_0, with $\omega \neq 0$ and $0 \leq \Re e\, m_0 < 1$.[1] In this representation the spectrum of J is the set $S = \{m_0 + n : \ n \in \mathbf{Z}\}$, and the representation space has basis vectors f_m, $m \in S$, such that

$$P_+ f_m = \omega f_{m+r} \ , \tag{2.2a}$$
$$P_- f_m = \omega f_{m-s} \ , \tag{2.2b}$$
$$J f_m = m f_m \ , \tag{2.2c}$$
$$(P_+)^r (P_-)^s f_m = \omega^{r+s} f_m \ , \tag{2.2d}$$

where $C^{r,s} \equiv (P_+)^r (P_-)^s$ is the Casimir operator.

An explicit realization of the representation $Q_{r,s}(\omega, m_0)$ can be easily constructed by taking[1]

$$J = m_0 + z\frac{d}{dz} \ , \qquad P_+ = \omega z^r \ , \qquad P_- = \omega z^{-s} \ . \tag{2.3}$$

These operators act on the space of all linear combinations of the functions z^n, $z \in \mathbf{C}$, $n \in \mathbf{Z}$, with basis vectors $f_m(z) = z^n$, where $m = m_0 + n$, n an integer.

The connection of the realization (2.3) with q-special functions is obtained by considering various combinations of q-exponentials of the algebra generators P_+ and P_- in the universal enveloping algebra of $\mathcal{G}_{r,s}$. We shall first examine the operator

$$U^{(1)}(\alpha,\beta) = e_{q^r}(\alpha P_+)\, e_{q^s}(\beta P_-) \ , \tag{2.4}$$

which in the realization (2.3) becomes

$$U^{(1)}(\alpha,\beta) = e_{q^r}(\alpha\omega z^r)\, e_{q^s}(\beta\omega z^{-s}) \ . \tag{2.5}$$

Its matrix elements $U^{(1)}_{kn}(\alpha,\beta)$, defined through

$$U^{(1)}(\alpha,\beta)\, f_{m_0+n} = \sum_{k=-\infty}^{\infty} U^{(1)}_{kn}(\alpha,\beta)\, f_{m_0+k} \ , \tag{2.6}$$

can be explicitly computed. By using the definition (1.5a) of the q-exponential function, one has

$$U^{(1)}(\alpha,\beta)\, z^n = \sum_{m,l=0}^{\infty} \frac{\omega^{m+l}}{(q^r;q^r)_l\,(q^s;q^s)_m}\, \alpha^l \beta^m\, z^{n+lr-ms} \ . \tag{2.7}$$

It is convenient to introduce two new summation indices k and j through the relations: $k = n + lr - ms$, $l = \mu_{k-n} + sj$ and $m = \nu_{k-n} + rj$. For a given integer n, the nonnegative integers μ_n and ν_n are uniquely determined by the following two properties:[1] $n = \mu_n r - \nu_n s$; if $n = \mu'_n r - \nu'_n s$, where μ'_n, ν'_n are nonnegative integers, then $\mu'_n + \nu'_n \geq \mu_n + \nu_n$. Since r and s are relativly prime, the integers μ_n and ν_n exist for all n. For instance, in the particular case $r = s = 1$, one has $\mu_n = n$, $\nu_n = 0$ for $n \geq 0$, and $\mu_n = 0$, $\nu_n = n$ for $n < 0$. Also notice that $\mu_{n+m} = \mu_n + \mu_m \bmod s$, $\nu_{n+m} = \nu_n + \nu_m \bmod r$.

One can then rewrite (2.7) as

$$\begin{aligned} U^{(1)}(\alpha,\beta)\, z^n = & \sum_{k=-\infty}^{\infty} z^k\, \omega^{\mu_{k-n}+\nu_{k-n}}\, \alpha^{\mu_{k-n}}\, \beta^{\nu_{k-n}} \\ & \times \sum_{j=0}^{\infty} \frac{1}{(q^r;q^r)_{\mu_{k-n}+sj}\,(q^s;q^s)_{\nu_{k-n}+rj}} \left(\omega^{(r+s)}\alpha^s\beta^r\right)^j \ , \end{aligned} \tag{2.8}$$

and extract from this expression the matrix elements $U^{(1)}_{kn}(\alpha,\beta)$. In terms of the definition

$$I^{(1)\,r,s}_n(z;q) = \left(\frac{z}{r+s}\right)^{\mu_n+\nu_n} \sum_{k=0}^{\infty} \frac{1}{(q^r;q^r)_{\mu_n+sk}\,(q^s;q^s)_{\nu_n+rk}} \left(\frac{z}{r+s}\right)^{k(r+s)} \ , \tag{2.9}$$

one explicitly has

$$U^{(1)}_{kn}(\alpha,\beta) = \left(\frac{\alpha}{\beta}\right)^{\frac{k-n}{r+s}} I^{(1)\,r,s}_{k-n}\left(\omega(r+s)\left(\alpha^s\beta^r\right)^{1/(r+s)};q\right) \ . \tag{2.10}$$

For $r = s = 1$ the function in (2.9) coincides with $I^{(1)}_n(z;q)$, so that $I^{(1)\,r,s}_n(z;q)$ can be considered a group theoretical generalization of the first modified q-Bessel function in (18.a), with integer index ν. From the definition (2.9) one can also check that

$$\lim_{q\to 1^-} I^{(1)\,r,s}_n\left((1-q^r)^{s/(r+s)}(1-q^s)^{r/(r+s)}\, z;q\right) = \left(\frac{s}{r}\right)^{\frac{n}{r+s}} I^{r,s}_n(z) \ , \tag{2.11}$$

where $I^{r,s}_n(z)$ is the ordinary generalized Bessel function introduced in Ref.[1]. It is defined by the series expansion

$$I^{r,s}_n(z) = \left(\frac{z}{r+s}\right)^{\mu_n+\nu_n} \sum_{k=0}^{\infty} \frac{1}{(\mu_n+sk)!\,(\nu_n+rk)!} \left(\frac{z}{r+s}\right)^{k(r+s)} \ , \tag{2.12}$$

and arises in the representation theory of the group associated with $\mathcal{G}_{r,s}$.

The q-Bessel function $I_n^{(1)\,r,s}(z;q)$ satisfies the following recursion relation

$$(1-q^n)\left(\frac{r+s}{z}\right) I_n^{(1)\,r,s}(z;q) = I_{n-r}^{(1)\,r,s}(z;q) - q^n\, I_{n+s}^{(1)\,r,s}(z;q)\ , \tag{2.13}$$

as can be proven again using (2.9). For $r=s=1$, one recovers (1.18a), while as $q\to 1^-$ one gets the recursion relation satisfied by the ordinary function $I_n^{r,s}(z)$:

$$n\left(\frac{r+s}{z}\right) I_n^{r,s}(z) = r\, I_{n-r}^{r,s}(z) - s\, I_{n+s}^{r,s}(z)\ . \tag{2.14}$$

Using the explicit representation (2.12), one can also check that:

$$\left(s\,\frac{d}{dz}+\frac{n}{z}\right) I_n^{r,s}(z) = I_{n-r}^{r,s}(z)\ , \tag{2.15a}$$

$$\left(r\,\frac{d}{dz}-\frac{n}{z}\right) I_n^{r,s}(z) = I_{n+s}^{r,s}(z)\ , \tag{2.15b}$$

from which (2.14) can be deduced. Similar relations hold also for $I_n^{(1)\,r,s}(z;q)$:

$$\left(\frac{r+s}{z}\right)\left(1-q^{\frac{nr}{r+s}}\,T_z^{\frac{rs}{r+s}}\right) I_n^{(1)\,r,s}(z;q) = I_{n-r}^{(1)\,r,s}(z;q)\ , \tag{2.16a}$$

$$\left(\frac{r+s}{z}\right)\left(1-q^{-\frac{ns}{r+s}}\,T_z^{\frac{rs}{r+s}}\right) I_n^{(1)\,r,s}(z;q) = I_{n+s}^{(1)\,r,s}(z;q)\ . \tag{2.16b}$$

The recursion relation (2.13) clearly is a consequence of these. After a suitable rescaling of z (see (2.11)), one can check that (2.16) reduce to (2.15) as $q\to 1^-$. Applying in succession $(r+s)$-times the operators in (2.16) one sees that the function $I_m^{(1)\,r,s}(z;q)$ is a solution of the following q-difference equation:

$$\begin{aligned}\left(\frac{r+s}{z}\right)\left(1-q^{\frac{r(n+r)}{r+s}}\,T_z^{\frac{rs}{r+s}}\right)\left(\frac{r+s}{z}\right)\left(1-q^{\frac{r(n+2r)}{r+s}}\,T_z^{\frac{rs}{r+s}}\right)\cdots\left(\frac{r+s}{z}\right)\left(1-q^{\frac{r(n+rs)}{r+s}}\,T_z^{\frac{rs}{r+s}}\right)\\ \times\left(\frac{r+s}{z}\right)\left(1-q^{\frac{-s(n+rs-s)}{r+s}}\,T_z^{\frac{rs}{r+s}}\right)\left(\frac{r+s}{z}\right)\left(1-q^{\frac{-s(n+rs-2s)}{r+s}}\,T_z^{\frac{rs}{r+s}}\right)\cdots\\ \times\left(\frac{r+s}{z}\right)\left(1-q^{\frac{-ns}{r+s}}\,T_z^{\frac{rs}{r+s}}\right) I_n^{(1)\,r,s}(z;q) = I_n^{(1)\,r,s}(z;q)\ .\end{aligned} \tag{2.17}$$

Further, notice that $I_n^{(1)\,r,s}(z;q)$ also satisfies the following four-terms relation:

$$\begin{aligned}\left(\frac{r+s}{z}\right)\left(1-q^{n\frac{r-s}{r+s}}\,T_z^{\frac{2rs}{r+s}}\right) I_n^{(1)\,r,s}(z;q) + \left(\frac{z}{r+s}\right) I_{n-r+s}^{(1)\,r,s}(z;q)\\ = I_{n-r}^{(1)\,r,s}(z;q) + I_{n+s}^{(1)\,r,s}(z;q)\ ;\end{aligned} \tag{2.18}$$

this is the genralization of (1.19), to which it reduces for $r=s=1$.

A generating formula for $I_n^{(1)\,r,s}(z;q)$ can be easily derived. Inserting the result (2.10) for the matrix elements $U_{kn}^{(1)}(\alpha,\beta)$ in (2.6) and letting $\alpha=\beta=1$, $n=0$, and $y=\omega(r+s)$ one gets

$$e_{q^r}\left(\frac{y}{r+s}\,z^r\right)\, e_{q^s}\left(\frac{y}{r+s}\,z^{-s}\right) = \sum_{k=-\infty}^{\infty} z^k\, I_k^{(1)\,r,s}(y;q)\ . \tag{2.19}$$

Let us now replace z by $e^{i\theta}$ in the previous formula. Recalling (1.5a), one can write

$$\frac{1}{\left(\frac{y}{r+s}e^{ir\theta};q^r\right)_\infty\left(\frac{y}{r+s}e^{-is\theta};q^s\right)_\infty} = \sum_{k=-\infty}^{\infty} e^{ik\theta}\, I_k^{(1)\,r,s}(y;q)\ . \tag{2.20}$$

Multplying both sides of this relation by $e^{-in\theta}/2\pi$ and integrating in $d\theta$ from 0 to 2π, one obtains the following integral representation ($|y|<r+s$)

$$I_n^{(1)\,r,s}(y;q) = \frac{1}{2\pi}\int_0^{2\pi}\frac{e^{-in\theta}}{\left(\frac{y}{r+s}e^{ir\theta};q^r\right)_\infty\left(\frac{y}{r+s}e^{-is\theta};q^s\right)_\infty}\,d\theta\ . \tag{2.21}$$

A generalization of the second modified q-Bessel function $I_n^{(2)\,r,s}(z;q)$, similar to that of Eq.(2.9) for $I_n^{(1)}(z;q)$, can also be considered

$$I_n^{(2)\,r,s}(z;q) = \left(\frac{z}{r+s}\right)^{\mu_n+\nu_n}\sum_{k=0}^{\infty}\frac{q^{[r(\mu_n+sk)^2+s(\nu_n+rk)^2]/2}}{(q^r;q^r)_{\mu_n+sk}\,(q^s;q^s)_{\nu_n+rk}}\left(\frac{z}{r+s}\right)^{k(r+s)}\ . \tag{2.22}$$

It arises in the study of the matrix elements of the following operator in the universal enveloping algebra of $\mathcal{G}_{r,s}$,

$$U^{(2)}(\alpha,\beta) = E_{q^r}(\alpha P_+)\; E_{q^s}(\beta P_-)\ . \tag{2.23}$$

Using (1.5b) and the explict realization (2.3), steps similar to the ones involved in deriving (2.8) give

$$\begin{aligned}U^{(2)}(\alpha,\beta)\,z^n = \sum_{k=-\infty}^{\infty} z^k\, q^{[r\mu_{k-n}(\mu_{k-n}-1)+s\nu_{k-n}(\nu_{k-n}-1)]/2}\,\omega^{\mu_{k-n}+\nu_{k-n}}\alpha^{\mu_{k-n}}\beta^{\nu_{k-n}}\\ \times\sum_{m=0}^{\infty}\frac{q^{rsm(m(r+s)/2+\mu_{k-n}+\nu_{k-n})}}{(q^r;q^r)_{\mu_{k-n}+sm}\,(q^s;q^s)_{\nu_{k-n}+rm}}\left(\omega^{r+s}\alpha^s\beta^r q^{-rs}\right)^m\ .\end{aligned} \tag{2.24}$$

The matrix elements of $U^{(2)}(\alpha,\beta)$ are still defined as in (2.6); thus, from (2.24) one finds

$$U_{kn}^{(2)}(\alpha,\beta) = \left(\frac{\alpha}{\beta}\,q^{-(r-s)/2}\right)^{\frac{k-n}{r+s}} I_{k-n}^{(2)\,r,s}\left(\omega(r+s)\left(\alpha^s\beta^r q^{-rs}\right)^{1/(r+s)};q\right)\ . \tag{2.25}$$

The function $I_n^{(2)\,r,s}(z;q)$ is an entire analytic function of z. When $r = s = 1$, it reduces to the second q-Bessel function in (1.17a): $I_n^{(2)\,1,1}(z;q) = q^{n^2/2}\, I_n^{(2)}(z;q)$, while in the limit $q \to 1^-$ one gets back the ordinary generalized Bessel function (2.12):

$$\lim_{q\to 1^-} I_n^{(2)\,r,s}\left((1-q^r)^{s/(r+s)}(1-q^s)^{r/(r+s)}\, z;q\right) = \left(\frac{s}{r}\right)^{\frac{n}{r+s}} I_n^{r,s}(z)\ . \tag{2.26}$$

Using the definition (2.21), one can chek the following recursion relation

$$(q^{-n/2}-q^{n/2})\left(\frac{r+s}{z}\right) I_n^{(2)\,r,s}(z;q) = q^{(n-r)/2}\, I_{n-r}^{(2)\,r,s}(z;q) - q^{-(n+s)/2}\, I_{n+s}^{(2)\,r,s}(z;q)\ , \tag{2.27}$$

which in the limit $q \to 1^-$ gives back (2.14), and for $r = s = 1$ reduces to (1.18a).

By inserting the explicit form (2.25) of the matrix elements $U_{kn}^{(2)}(\alpha,\beta)$ in (2.6), one gets a generating formula for $I_n^{(2)\,r,s}(z;q)$. Setting $\alpha = q^{r/2}$, $\beta = q^{s/2}$, $n = 0$ and $y = \omega(r+s)$, one obtains

$$E_{q^r}\left(\frac{y}{r+s}(z\sqrt{q})^r\right)\, E_{q^s}\left(\frac{y}{r+s}(z/\sqrt{q})^{-s}\right) = \sum_{k=-\infty}^{\infty} z^k\, I_k^{(2)\,r,s}(y;q)\ . \tag{2.28}$$

As shown before in the case of $I_n^{(1)\,r,s}(z;q)$, from this formula an integral representation for $I_n^{(2)\,r,s}(z;q)$ can be derived:

$$I_n^{(2)\,r,s}(z;q) = \frac{1}{2\pi}\int_0^{2\pi} e^{-in\theta}\left(-\frac{z}{r+s}q^{r/2}e^{ir\theta};q^r\right)_\infty \left(-\frac{z}{r+s}q^{-s/2}e^{-is\theta};q^s\right)_\infty d\theta\ . \tag{2.29}$$

A biorthogonal relation involving the two functions $I_n^{(1)\,r,s}(z;q)$ and $I_n^{(2)\,r,s}(z;q)$ can be obtained by combining together the two generating relations (2.19) and (2.28). First let $z \to q^{(s-r)/2(r+s)}\, z$ and $y \to -q^{-rs/(r+s)}\, y$ in (2.28) and then multiply it by (2.19). Using $e_q(x)\, E_q(-x) = 1$ one gets

$$1 = \sum_{k,l=-\infty}^{\infty} z^{k+l} q^{-\frac{1}{2}\left(\frac{r-s}{r+s}\right)}\, I_k^{(1)\,r,s}(y;q)\, I_l^{(2)\,r,s}\left(-y\, q^{-rs/(r+s)};q\right)\ ; \tag{2.30}$$

this double sum is absolutely convergent for $\left(\frac{|y|}{r+s}\right)^{1/s} < |z| < \left(\frac{r+s}{|y|}\right)^{1/r}$. Equating the coefficients of equal powers of z on both sides of (2.30), one finally finds, for $|y| < (r+s)^{rs/(r+s)}$,

$$\sum_{k=-\infty}^{\infty} q^{-\frac{1}{4}(m+n-2k)\left(\frac{r-s}{r+s}\right)}\, I_{k-m}^{(1)\,r,s}(y;q)\, I_{n-k}^{(2)\,r,s}\left(-y\, q^{-rs/(r+s)};q\right) = \delta_{m,n}\ . \tag{2.31}$$

Natural generalizations of the Hahn-Exton modified q-Bessel function (1.17b) are obtained by considering operators that involve both the q-exponential functions of (1.5). Consider first the following operator

$$U^{(3)}(\alpha,\beta) = e_{q^r}(\alpha P_+)\, E_{q^s}(\beta P_-)\ . \tag{2.32}$$

Its matrix elements in the representation (2.3), still defined as in (2.6), can be written as

$$U_{kn}^{(3)}(\alpha,\beta) = \left(q^s\,\frac{\alpha}{\beta}\right)^{\frac{k-n}{r+s}} I_{k-n}^{(3)\,r,s}\left(\omega(r+s)\left(\alpha^s\beta^r\, q^{-rs}\right)^{1/(r+s)};q\right)\ , \tag{2.33}$$

where

$$I_n^{(3)\,r,s}(z;q) = z^{\mu_n+\nu_n}\sum_{k=0}^{\infty}\frac{q^{s(\nu_n+rk)(\nu_n+rk+1)/2}}{(q^r;q^r)_{\mu_n+sk}\,(q^s;q^s)_{\nu_n+rk}}\, z^{k(r+s)}\ . \tag{2.34}$$

The function $I_n^{(3)\,r,s}(z;q)$ is an entire analytic function of z. It reduces to the Hahn-Exton q-Bessel function (1.17b) in the particular case $r = s = 1$; also, one checks that $\lim_{q\to 1^-} I_n^{(3)\,r,s}\left((1-q^r)^{s/(r+s)}(1-q^s)^{r/(r+s)}\, z;q\right) = \left(\frac{s}{r}\right)^{\frac{n}{r+s}} I_n^{r,s}((r+s)z)$. It obeys the following four-terms recursion relation:

$$(1-q^n)\, I_n^{(3)\,r,s}(z;q) - z^2\, I_{n+s-r}^{(3)\,r,s}(z;q) = z\left(I_{n-r}^{(3)\,r,s}(z;q) - I_{n+s}^{(3)\,r,s}(z;q)\right)\ , \tag{2.35}$$

which coincides with (1.18b) for $r = s = 1$, and reduces to (2.14) as $q \to 1^-$.

The generating formula that follow from (2.6) takes now the form, for $0 < |z| < |y|^{1/r}$,

$$e_{q^r}(yz^r)\, E_{q^s}\left(y\,(z/q)^{-s}\right) = \sum_{k=-\infty}^{\infty} z^k\, I_k^{(3)\,r,s}(y;q)\ , \tag{2.36}$$

from which the following integral repersentation can be deduced ($|z| < 1$)

$$I_n^{(3)\,r,s}(z;q) = \frac{1}{2\pi}\int_0^{2\pi} e^{-in\theta}\,\frac{\left(-zq^s e^{-is\theta};q^s\right)_\infty}{\left(ze^{ir\theta};q^r\right)_\infty}\, d\theta\ . \tag{2.37}$$

The matrix elements of the operator

$$U^{(4)}(\alpha,\beta) = E_{q^r}(\alpha P_+)\, e_{q^s}(\beta P_-)\ , \tag{2.36}$$

in the representation (2.3), can be expressed in terms of the following different generalization of the q-Bessel function (1.17b)

$$I_n^{(4)\,r,s}(z;q) = z^{\mu_n+\nu_n}\sum_{k=0}^{\infty}\frac{q^{r(\mu_n+sk)(\mu_n+sk+1)/2}}{(q^r;q^r)_{\mu_n+sk}\,(q^s;q^s)_{\nu_n+rk}}\, z^{k(r+s)}\ . \tag{2.39}$$

Explicitly, one gets

$$U^{(4)}_{kn}(\alpha,\beta) = \left(q^{-r}\,\frac{\alpha}{\beta}\right)^{\frac{k-n}{r+s}} I^{(4)\,r,s}_{k-n}\left(\omega(r+s)\left(\alpha^s\beta^r\,q^{-rs}\right)^{1/(r+s)};q\right) . \tag{2.40}$$

The function $I^{(4)\,r,s}_n(z;q)$ is an entire analytic function of z, which for $r = s = 1$ reduces to $q^{n/2}I^{r,s}_n(q^{n/2}z;q)$. It obeys the recursion relation:

$$(1-q^{-n})\,I^{(4)\,r,s}_n(z;q) - z^2\,I^{(4)\,r,s}_{n+s-r}(z;q) = z\left(I^{(4)\,r,s}_{n-r}(z;q) - I^{(4)\,r,s}_{n+s}(z;q)\right) . \tag{2.41}$$

A generating formula is again easily obtained from (2.6), for $|y|^{1/s} < |z|$,

$$E_{q^r}(y\,(qz)^r)\;e_{q^s}(y\,z^{-s}) = \sum_{k=-\infty}^{\infty} z^k\,I^{(4)\,r,s}_k(y;q) , \tag{2.42}$$

from which the following integral repersentation follows ($|z| < 1$)

$$I^{(4)\,r,s}_n(z;q) = \frac{1}{2\pi}\int_0^{2\pi} e^{-in\theta}\,\frac{\left(-zq^re^{ir\theta};q^r\right)_\infty}{\left(ze^{-is\theta};q^s\right)_\infty}\,d\theta . \tag{2.43}$$

To get a biorthogonality relation involving $I^{(3)\,r,s}_n(z;q)$ and $I^{(4)\,r,s}_n(z;q)$, replace y by $-y$ and z by z/q in (2.42), and then multiply it by (2.36). Using the properties of the q-exponential functions, one gets

$$1 = \sum_{k=-\infty}^{\infty} z^{k+l}q^{-l}\,I^{(3)\,r,s}_k(y;q)\,I^{(4)\,r,s}_l(-y;q) ; \tag{2.44}$$

this double sum is absolutly convergent for $q|y|^{1/s} < |z| < |y|^{-1/r}$. Equating the coefficients of equal powers of z on both sides of (2.44), one finds, for $|y| < q^{-sr/(r+s)}$,

$$\sum_{k=-\infty}^{\infty} q^kq^{-(n+m)/2}\,I^{(3)\,r,s}_{m-k}(y;q)\,I^{(4)\,r,s}_{k-n}(-y;q) = \delta_{n,m} . \tag{2.45}$$

3. REALIZATIONS OF $\mathcal{G}_{r,s}$ IN TWO VARIABLES

In the previous section we have seen that generalizations of q-Bessel functions arise in the study of realizations of $\mathcal{G}_{r,s}$ on the space of functions of one complex variable. These q-special functions $I^{(k)\,r,s}_n(z;q)$, $k = 1,2,3,4$, were defined through a series expansion

only for integer values of the index n. Following a standard procedure in the theory of complex functions[20] we shall use analytic continuation to define them also for n an arbitrary complex number. We shall carry out the detayls only in the case of $I^{(1)\,r,s}_n(z;q)$, but similar procedures can be adopted also for the remaining three generalized q-Bessel functions.

We start by applying the Cauchy residue theorem to the generating formula (2.19). For n integer, and $\left(\frac{|z|}{r+s}\right)^{1/s} < |z| < \left(\frac{r+s}{|z|}\right)$

$$I^{(1)\,r,s}_n(z;q) = \frac{1}{2\pi i}\int_C \frac{x^{-n-1}}{\left(\frac{z}{r+s}x^r;q^r\right)_\infty\left(\frac{z}{r+s}x^{-s};q^s\right)_\infty}\,dx , \tag{3.1}$$

where C is a simple closed contour encircling the origin in the x complex plane. Without changing the integral, the contour C can be deformed into a loop which consists of the lower edge of the cut along the negative real x-axis from $-\infty$ to $-\varepsilon$, a circle of radius ε and the upper edge of the cut from $-\varepsilon$ to $-\infty$. We shall use the standard notation $\int_{-\infty}^{(0+)}$ to indicate the integral along this path. Since r is odd, the integrand in (3.1) is bounded along the path for $\mathcal{R}e\,z > 0$; moreover, it is single-valued even for n complex. We can thus use

$$I^{(1)\,r,s}_n(z;q) = \frac{1}{2\pi i}\int_{-\infty}^{(0+)} \frac{x^{-n-1}}{\left(\frac{z}{r+s}x^r;q^r\right)_\infty\left(\frac{z}{r+s}x^{-s};q^s\right)_\infty}\,dx , \tag{3.2}$$

to define $I^{(1)\,r,s}_n(z;q)$ for arbitrary n and $\mathcal{R}e\,z > 0$. To obtain a series expansion, introduce a new integration variable $t = \left(\frac{z}{r+s}\right)^{1/r}x$:

$$I^{(1)\,r,s}_n(z;q) = \frac{1}{2\pi i}\left(\frac{z}{r+s}\right)^{n/r}\int_{-\infty}^{(0+)} \frac{t^{-n-1}}{\left(t^r;q^r\right)_\infty\left(\left(\frac{z}{r+s}\right)^{(r+s)/r}t^{-s};q^s\right)_\infty}\,dt . \tag{3.3}$$

This expression, which is not single-valued, can be used to define $I^{(1)\,r,s}_n(z;q)$ as an analytic function for all $z \neq 0$. In particular, $z^{-n/r}\,I^{(1)\,r,s}_n(z;q)$ is an entire function of $z^{(r+s)/r}$. Recalling the definition (1.5a), one can now expand in series the second q-exponential appearing in the integrand of (3.3), and integrate term by term:

$$\begin{aligned} I^{(1)\,r,s}_n(z;q) = \left(\frac{z}{r+s}\right)^{n/r}\sum_{k=0}^{\infty}\left(\frac{z}{r+s}\right)^{k(r+s)/r}\frac{1}{(q^s;q^s)_k} \\ \times\frac{1}{2\pi i}\int_{-\infty}^{(0+)}\frac{t^{-n-sk-1}}{\left(t^r;q^r\right)_\infty}\,dt . \end{aligned} \tag{3.4}$$

To proceed further, we use the following Ramanujan's formula[21]

$$\frac{(q^{1-z};q)_\infty\,(a;q)_\infty}{(q;q)_\infty\,(aq^{-z};q)_\infty} = \frac{\sin\pi z}{\pi}\int_0^\infty dt\,t^{z-1}\,\frac{(-at;q)_\infty}{(-t;q)_\infty} . \tag{3.5}$$

With the change of variable $t \to t^r$ and of base $q \to q^r$, and the redefinition $z \to z/r$, one gets, for r an odd positive integer,

$$\frac{(q^{r-z};q^r)_\infty\,(a;q^r)_\infty}{(q^r;q^r)_\infty\,(aq^{-z};q^r)_\infty} = r\,\frac{\sin\pi(z/r)}{\pi}\int_0^\infty dt\,t^{z-1}\,\frac{(-at^r;q^r)_\infty}{(-t^r;q^r)_\infty}\,. \tag{3.6}$$

Consider now the following contour integral

$$\mathcal{I} = \frac{1}{2\pi i}\int_{-\infty}^{(0+)} dt\,t^{z-1}\,\frac{(at^r;q^r)_\infty}{(t^r;q^r)_\infty}\,. \tag{3.7}$$

It can be evaluated in the following manner:

$$\begin{aligned}\mathcal{I} &= \frac{1}{2\pi i}\int_\infty^0 (-dt)\,(te^{-i\pi})^{z-1}\,\frac{(-at^r;q^r)_\infty}{(-t^r;q^r)_\infty} + \frac{1}{2\pi i}\int_0^\infty (-dt)\,(te^{i\pi})^{z-1}\,\frac{(-at^r;q^r)_\infty}{(-t^r;q^r)_\infty}\\ &= \frac{\sin\pi z}{\pi}\int_0^\infty dt\,t^{z-1}\,\frac{(-at^r;q^r)_\infty}{(-t^r;q^r)_\infty}\,.\end{aligned} \tag{3.8}$$

Comparing (3.6) with (3.8), one obtains

$$\frac{1}{2\pi i}\int_{-\infty}^{(0+)} dt\,t^{z-1}\,\frac{(at^r;q^r)_\infty}{(t^r;q^r)_\infty} = \frac{1}{r}\,\frac{\sin\pi z}{\sin\pi(z/r)}\,\frac{(q^{r-z};q^r)_\infty\,(a;q^r)_\infty}{(q^r;q^r)_\infty\,(aq^{-z};q^r)_\infty}\,, \tag{3.9}$$

which for $a = 0$ simplifies to

$$\frac{1}{2\pi i}\int_{-\infty}^{(0+)} dt\,\frac{t^{z-1}}{(t^r;q^r)_\infty} = \frac{1}{r}\,\frac{\sin\pi z}{\sin\pi(z/r)}\,\frac{(q^{r-z};q^r)_\infty}{(q^r;q^r)_\infty}\,. \tag{3.10}$$

By analytic continuation, these formulas hold for any complex z. With the help of (3.10), for noninteger n, one can rewrite (3.4) as

$$I_n^{(1)\,r,s}(z;q) = \frac{1}{r}\left(\frac{z}{r+s}\right)^{n/r}\sum_{k=0}^\infty \frac{(q^{n+sk+r};q^r)_\infty}{(q^s;q^s)_k\,(q^r;q^r)_\infty}\,\frac{\sin\pi(n+sk)}{\sin\pi\big((n+sk)/r\big)}\left(\frac{z}{r+s}\right)^{k(r+s)/r}\,. \tag{3.11}$$

Using this definition, one can now check that the relations (2.16) and the difference equation (2.17) remain valid for noninteger n.

In the previous section we have studied realizations of the representation $Q_{r,s}(\omega, m_0)$ of $\mathcal{G}_{r,s}$ on the space of functions of one complex variable. We shall now discuss realizations of this representation on spaces of functions of two complex variables, x and y. In particular, we shall look for a realization of $P_\pm$ and J in terms of generalized q-difference operators acting on the space generated by basis vectors of the form $f_m(x,y) = y^m\,F_m(x)$, $m = m_0 + n$, $n \in \mathbb{Z}$, such that (2.2) are satisfied. The constant ω is nonessential, since it can be changed by rescaling the generators P_+ and P_-; for simplicity, we shall henceforth set

$\omega = 1$. With the help of the relations (2.16), it is easy to check that such a realization is provided by the operators

$$P_+ = y^r\left(\frac{r+s}{z}\right)\left(1 - T_y^{\frac{r}{r+s}}\,T_x^{\frac{rs}{r+s}}\right)\,, \tag{3.12a}$$

$$P_- = y^{-r}\left(\frac{r+s}{z}\right)\left(1 - T_y^{-\frac{s}{r+s}}\,T_x^{\frac{rs}{r+s}}\right)\,, \tag{3.12b}$$

$$J = y\frac{\partial}{\partial y}\,, \tag{3.12c}$$

and the basis functions

$$f_m(x,y) = y^m\,I_{-m}^{(1)\,r,s}(x;q)\,, \tag{3.13}$$

with $m = m_0 + n$, $n \in \mathbb{Z}$. Note that the equation (2.2d) in this realization becomes the difference equation (2.17). Different two-variable realizations of $\mathcal{G}_{r,s}$ can be obtained by taking other generalized q-Bessel functions as basis vectors. Here, however, we shall concentrate on the realization (3.12) and derive from it addition formulas involving $I_m^{(1)\,r,s}(x;q)$ and $I_m^{(2)\,r,s}(x;q)$.

The basic ingredient is again Eq.(2.6), that we shall specialize for the operator (2.23); with $m = m_0 + n$, one obtains

$$U^{(2)}(\alpha,\beta)\,f_m(x,y) = \sum_{k=-\infty}^\infty U^{(2)}_{k\,m-m_0}(\alpha,\beta)\,f_{m_0+k}(x,y)\,, \tag{3.14}$$

where the model independent matrix elements $U^{(2)}_{kn}(\alpha,\beta)$ are still given by the formulas (2.25), with $\omega = 1$. To get addition formulas for q-Bessel functions, one now needs to evaluate explicitly the l.h.s. of (3.10), *i.e.* to compute directly the action of $U^{(2)}(\alpha,\beta)$ on the basis functions (3.13), when P_+ and P_- are realized as in (3.12). We shall not be able to give a closed form for this action; however, suitable integral representations will be provided.

We shall start by computing the action of $E_{q^s}(\beta P_-)$ on $f_m(x,y)$. With the help of the definition (1.5b) of the q-exponential and of (2.2b), one easily gets

$$E_{q^s}(\beta P_-)\,y^m\,I_{-m}^{(1)\,r,s}(x;q) = \sum_{l=0}^\infty q^{sl(l-1)/2}\,\frac{\beta^l}{(q^s;q^s)_l}\,y^{m-ls}\,I_{-m+ls}^{(1)\,r,s}(x;q) \tag{3.15a}$$

$$\begin{aligned}&= \frac{1}{r}\left(\frac{x}{r+s}\right)^{-m/r}\sum_{l,k=0}^\infty \beta^l\,y^{m-ls}\,\frac{q^{sl(l-1)/2}\,(q^{s(k+l)-m+r};q^r)_\infty}{(q^s;q^s)_l\,(q^s;q^s)_k\,(q^r;q^r)_\infty}\\ &\qquad\times\frac{\sin\pi((k+l)s-m)}{\sin\pi\big[((k+l)s-m)/r\big]}\left(\frac{x}{r+s}\right)^{k+s(k+l)/r}\,,\end{aligned} \tag{3.15b}$$

where in the last formula we have written for $I_{-m}^{(1)\,r,s}(x;q)$ its series expansion (3.11), assuming $m_0 \neq 0$. It is now convenient to introduce a new summation index n, replacing k trough the relation $l + k = n$. After using

$$(a;q)_{n-m} = \frac{(a;q)_n}{(q^{1-n}/a;q)_m}\left(-\frac{q}{a}\right)^m q^{\frac{1}{2}m(m-1)-mn}\,, \tag{3.16}$$

the sum over l in (3.15b) can be explicitly computed thanks to the q-binomial theorem (1.4). One finds,

$$E_{q^s}(\beta P_-)\, y^m\, I^{(1)\,r,s}_{-m}(x;q) = \frac{y^m}{r}\left(\frac{x}{r+s}\right)^{-m/r}$$
$$\times \sum_{n=0}^{\infty} \frac{\left(-\beta y^{-s}\left(\frac{r+s}{x}\right);q^s\right)_n \left(q^{sn-m+r};q^r\right)_\infty}{(q^s;q^s)_n\,(q^r;q^r)_\infty}\, \frac{\sin\pi(ns-m)}{\sin\pi((ns-m)/r)}\left(\frac{x}{r+s}\right)^{n(r+s)/r}. \tag{3.17}$$

One can give an integral representation of the r.h.s. of this formula by using again the result (3.10):

$$E_{q^s}(\beta P_-)\, y^m\, I^{(1)\,r,s}_{-m}(x;q)$$
$$= y^m\left(\frac{x}{r+s}\right)^{-m/r}\sum_{n=0}^{\infty}\frac{\left(-\beta y^{-s}\left(\frac{r+s}{x}\right);q^s\right)_n}{(q^s;q^s)_n}\,\frac{1}{2\pi i}\int_{-\infty}^{(0+)} dz\,\frac{z^{m-sn-1}}{(z^r;q^r)_\infty}\left(\frac{x}{r+s}\right)^{n(r+s)/r}. \tag{3.18}$$

By exchanging integral and sum, and using once more the q-binomial thorem, one obtains:

$$E_{q^s}(\beta P_-)\, y^m\, I^{(1)\,r,s}_{-m}(x;q)$$
$$= y^m\left(\frac{x}{r+s}\right)^{-m/r}\frac{1}{2\pi i}\int_{-\infty}^{(0+)} dz\, z^{m-1}\,\frac{\left(-\beta(yz)^{-s}\left(\frac{x}{r+s}\right)^{s/r};q^s\right)_\infty}{(z^r;q^r)_\infty\left(z^{-s}\left(\frac{x}{r+s}\right)^{(r+s)/r};q^s\right)_\infty}. \tag{3.19}$$

Recalling (3.15a) and introducing the new variable $t=\beta/y^s$, one finally gets

$$\left(\frac{x}{r+s}\right)^{m/r}\frac{1}{2\pi i}\int_{-\infty}^{(0+)} dz\, z^{-m-1}\,\frac{\left(-tz^{-s}\left(\frac{x}{r+s}\right)^{s/r};q^s\right)_\infty}{(z^r;q^r)_\infty\left(z^{-s}\left(\frac{x}{r+s}\right)^{(r+s)/r};q^s\right)_\infty}$$
$$= \sum_{l=0}^{\infty}\frac{q^{sl(l-1)/2}}{(q^s;q^s)_l}\, t^l\, I^{(1)\,r,s}_{m+sl}(x;q). \tag{3.20}$$

Analogous results are obtained by considering

$$E_{q^r}(\alpha P_+)\, y^m\, I^{(1)\,r,s}_{-m}(x;q) = \sum_{l=0}^{\infty} q^{rl(l-1)/2}\frac{\alpha^l}{(q^r;q^r)_l}\, y^{m+rl}\, I^{(1)\,r,s}_{-m-rl}(x;q). \tag{3.21}$$

By writing for $I^{(1)\,r,s}_{-m-rl}(x;q)$ its series expansion (assuming $m_0\neq 0$) and using again the q-binomial theorem one obtains

$$E_{q^r}(\alpha P_+)\, y^m\, I^{(1)\,r,s}_{-m}(x;q) = \frac{y^m}{r}\left(\frac{r+s}{x}\right)^{-m/r}$$
$$\times\sum_{k=0}^{\infty}\frac{\left(-\alpha y^r\left(\frac{x}{r+s}\right);q^r\right)_\infty\left(q^{sk+r-m};q^r\right)_\infty}{(q^s;q^s)_k\,(q^r;q^r)_\infty\left(-\alpha y^r q^{sk-m}\left(\frac{x}{r+s}\right);q^r\right)_\infty}\,\frac{\sin\pi(ks-m)}{\sin\pi((ks-m)/r)}\left(\frac{x}{r+s}\right)^{k(r+s)/r}. \tag{3.22}$$

This time, an integral representation for the r.h.s. of (3.22) can be obtained with the help of (3.9). Using also the summation formula (1.5a), one easily gets

$$E_{q^r}(\alpha P_+)\, y^m\, I^{(1)\,r,s}_{-m}(x;q)$$
$$= y^m\left(\frac{r+s}{x}\right)^{-m/r}\frac{1}{2\pi i}\int_{-\infty}^{(0+)} dz\, z^{m-1}\,\frac{\left(-\alpha(yz)^r\left(\frac{x}{r+s}\right);q^r\right)_\infty}{(z^r;q^r)_\infty\left(z^{-s}\left(\frac{x}{r+s}\right)^{(r+s)/r};q^s\right)_\infty}. \tag{3.23}$$

From (3.21), with the redefinition $\alpha y^r = t$, the following generating relation is obtained

$$\left(\frac{r+s}{x}\right)^{m/r}\frac{1}{2\pi i}\int_{-\infty}^{(0+)} dz\, z^{-m-1}\,\frac{\left(-tz^r\left(\frac{x}{r+s}\right);q^r\right)_\infty}{(z^r;q^r)_\infty\left(z^{-s}\left(\frac{x}{r+s}\right)^{(r+s)/r};q^s\right)_\infty}$$
$$= \sum_{l=0}^{\infty}\frac{q^{rl(l-1)/2}}{(q^r;q^r)_l}\, t^l\, I^{(1)\,r,s}_{m-rl}(x;q). \tag{3.24}$$

Finally, we shall assume $\alpha\beta\neq 0$ and compute the action of the operator $U^{(2)}(\alpha,\beta)$ as given in the representation (3.12) on the basis vectors (3.13). The details of the calculation are very similar to the ones already described and only the final formula will be given. One finds,

$$E_{q^r}(\alpha P_+)\, E_{q^s}(\beta P_-)\, y^m\, I^{(1)\,r,s}_{-m}(x;q)$$
$$= y^m\left(\frac{x}{r+s}\right)^{-m/r}\frac{1}{2\pi i}\int_{-\infty}^{(0+)} dt\, t^{m-1}\,\frac{\left(-\alpha(yt)^r\left(\frac{x}{r+s}\right);q^r\right)_\infty\left(-\beta(yt)^{-s}\left(\frac{x}{r+s}\right)^{s/r};q^s\right)_\infty}{(t^r;q^r)_\infty\left(t^{-s}\left(\frac{x}{r+s}\right)^{(r+s)/r};q^s\right)_\infty}. \tag{3.25}$$

Inserting this and the result (2.7) for the matrix elements $U^{(2)}_{kn}(\alpha,\beta)$ in (3.14), with $\omega=1$, one obtains

$$\left(\frac{x}{r+s}\right)^{m/r}\frac{1}{2\pi i}\int_{-\infty}^{(0+)} dt\, t^{-m-1}\,\frac{\left(-\alpha(yt)^r\left(\frac{x}{r+s}\right);q^r\right)_\infty\left(-\beta(yt)^{-s}\left(\frac{x}{r+s}\right)^{s/r};q^s\right)_\infty}{(t^r;q^r)_\infty\left(t^{-s}\left(\frac{x}{r+s}\right)^{(r+s)/r};q^s\right)_\infty}$$
$$= \sum_{k=-\infty}^{\infty}\left(\frac{\alpha}{\beta}\, q^{-(r-s)/2}\right)^{k/(r+s)} y^k\, I^{(2)\,r,s}_k\left((r+s)(\alpha^s\beta^r q^{-rs})^{1/(r+s)};q\right)\, I^{(1)\,r,s}_{m-k}(x;q). \tag{3.26}$$

This summation formula can be further simplified by letting first $\beta = q^s\alpha$ and then $w = y\sqrt{q}$, $(r+s)\alpha = z$:

$$\left(\frac{x}{r+s}\right)^{m/r}\frac{1}{2\pi i}\int_{-\infty}^{(0+)} dt\, t^{-m-1}\,\frac{\left(-(z/x)(tw\sqrt{q})^r;q^r\right)_\infty\left(-\left(\frac{x}{r+s}\right)\left(\frac{\sqrt{q}}{tw}\right)^s\left(\frac{x}{r+s}\right)^{s/r};q^s\right)_\infty}{(t^r;q^r)_\infty\left(t^{-s}\left(\frac{x}{r+s}\right)^{(r+s)/r};q^s\right)_\infty}$$
$$= \sum_{k=-\infty}^{\infty} w^k\, I^{(2)\,r,s}_k(z;q)\, I^{(1)\,r,s}_{m-k}(x;q). \tag{3.27}$$