UdeM-LPN-TH86

Fermilab Library

0 1160 0013446 4

UdeM-LPN-TH86
March 1992

ON THE QUANTUM GROUP AND QUANTUM ALGEBRA APPROACH TO q-SPECIAL FUNCTIONS

Roberto Floreanini
Istituto Nazionale di Fisica Nucleare, Sezione di Trieste
Dipartimento di Fisica Teorica, Università di Trieste
Strada Costiera 11, 34014 Trieste, Italy

Luc Vinet (*)
Laboratoire de Physique Nucléaire
and
Centre de Recherches Mathématiques
Université de Montréal
Montréal, Canada H3C 3J7

Abstract

Two interpretations of q-special functions based on quantum groups and algebras have been presented in the literature. The connection between these approaches is explained using as example the case where $\mathcal{U}_q(sl(2))$ is the basic structure.

JUN 2 1992

(*) Supported in part by the National Sciences and Engineering Research Council (NSERC) of Canada.

Quantum groups and algebras[1-5] have been shown recently to provide an algebraic setting for q-special functions. In fact, two approaches to this program have been presented. In one case,[6-14] one proceeds in close analogy with Lie theory[15] and considers the elements of the quantum algebra $\mathcal{U}_q(\mathcal{G})$ that are obtained upon replacing the exponential map from the algebra $\mathcal{G}$ into the group G by q-exponential mapping. Their matrix elements in irreducible representations are found to be expressible in terms of q-hypergeometric functions. (This interpretation has its origin in Refs.[16,17].)

In the other case,[18-25] one considers rather the Hopf algebra $A(G_q)$ associated to the quantum group G_q. This is a subalgebra of the dual of $\mathcal{U}_q(\mathcal{G})$, generated by the coordinates of G_q. In this approach, the matrix elements of corepresentations of G_q are seen to involve basic hypergeometric functions of these coordinates.

The purpose of this note is to discuss the relations between these two points of view. This will be done by comparing the interpretations of the q-hypergeometric series ${}_2\phi_1(a,b;c;q,z)$ based on the quantum algebra $\mathcal{U}_q(sl(2))$ and the quantum group $SL_q(2)$. After recalling standard definitions in q-analysis,[26] we shall first describe the method of Refs.[9,11], stressing how it straightforwardly leads to a connection between the ${}_2\phi_1$ series and $\mathcal{U}_q(sl(2))$. We shall then show how one can from this approach recover the results of Ref.[23].

The two q-exponential functions

$$e_q(z) = \sum_{n=0}^{\infty} \frac{1}{(q;q)_n}\, z^n = \frac{1}{(z;q)_\infty}\,, \qquad |z|<1\,, \tag{1a}$$

$$E_q(z) = \sum_{n=0}^{\infty} \frac{q^{\frac{1}{2}n(n-1)}}{(q;q)_n}\, z^n = (-z;q)_\infty\,, \tag{1b}$$

will play a central role in our analysis. Note that $e_q(z)E_q(-z)=1$, and that $\lim_{q\to 1^-} e_q\big(z(1-q)\big) = \lim_{q\to 1^-} E_q\big(z(1-q)\big) = e^z$. For a and α arbitrary complex numbers, the q-shifted factorial is given by

$$(a;q)_\alpha = \frac{(a;q)_\infty}{(aq^\alpha;q)_\infty}\,, \tag{2}$$

where

$$(a;q)_\infty = \prod_{k=0}^{\infty}(1-aq^k)\,, \qquad |q|<1\,. \tag{3}$$

We shall denote by T_z the q-dilatation operator which acts as

$$T_z\, g(z) = g(qz)\,, \tag{4}$$

on functions of the variable z; out of it, the q-difference operators

$$D_z^+ = z^{-1}(1-T_z)\,, \tag{5a}$$

$$D_z^- = z^{-1}(1-T_z^{-1})\,, \tag{5b}$$

are constructed. Observe that $\frac{1}{(1-q)}D_z^+ \to d/dz$ and $\frac{1}{(1-q^{-1})}D_z^- \to d/dz$ as $q \to 1$, and that the q-exponentials obey

$$T_z^{-1}\, e_q(z) = (1-q^{-1}z)^{-1} e_q(z)\ , \tag{6a}$$
$$T_z\, E_q(z) = (1+z)^{-1} E_q(z)\ , \tag{6b}$$
$$D_z^+\, e_q(\lambda z) = \lambda e_q(\lambda z)\ , \tag{6c}$$
$$D_z^-\, E_q(\lambda z) = -q^{-1}\lambda E_q(\lambda z)\ , \tag{6d}$$

with λ a complex parameter. We also record the definition of the basic hypergeometric function ${}_1\phi_2(a,b;c;q,z)$:

$${}_2\phi_1(a,b;c;q,z) = \sum_{n=0}^{\infty} \frac{(a;q)_n (b;q)_n}{(q;q)_n (c;q)_n}\, z^n\ , \tag{7}$$

in terms of which various matrix elements will be expressed. In the following we shall assume $|q| < 1$.

The quantum universal enveloping algebra $\mathcal{U}_q(sl(2))$ is the Hopf algebra generated by the elements k, k^{-1}, e and f satisfying the relations

$$k\, e\, k^{-1} = q^{1/2} e\ , \qquad k\, f\, k^{-1} = q^{-1/2} f\ , \qquad [e,f] = \frac{k^2 - k^{-2}}{q^{1/2} - q^{-1/2}}\ , \tag{8}$$

and $k\,k^{-1} = k^{-1}k = 1$. The coproduct $\Delta : \mathcal{U}_q(sl(2)) \to \mathcal{U}_q(sl(2)) \otimes \mathcal{U}_q(sl(2))$, antipode $S : \mathcal{U}_q(sl(2)) \to \mathcal{U}_q(sl(2))$ and counit $\varepsilon : \mathcal{U}_q(sl(2)) \to \mathbf{C}$ are defined by:

$$\begin{aligned} &\Delta(k) = k \otimes k\ , && \Delta(e) = e \otimes k + k^{-1} \otimes e\ , && \Delta(f) = f \otimes k + k^{-1} \otimes f\ , \\ &S(k) = k^{-1}\ , && S(e) = -q^{1/2}\, e\ , && S(f) = -q^{-1/2}\, f\ , \\ &\varepsilon(k) = 1\ , && \varepsilon(e) = 0\ , && \varepsilon(f) = 0\ . \end{aligned} \tag{9}$$

The algebra $\mathcal{U}_q(sl(2))$ has a Poincaré-Birkhoff-Witt basis given by

$$e^{\mu}\, k^{\rho}\, f^{\nu}\ , \qquad \rho \in \mathbf{Z}\ ,\ \mu,\nu \in \mathbf{N}\ . \tag{10}$$

We now introduce an infinite-dimensional left $\mathcal{U}_q(sl(2))$-module $V^{(\lambda,m_0)} = \bigoplus_{j\in I} \mathbf{C}\xi_j$, where $I = \{i \mid i = m_0 + n,\ n \in \mathbf{Z}\}$, and λ, m_0 are complex numbers.[9,11] This representation is characterized by the following action of the generators on the basis vectors ξ_j, $j \in I$:

$$\begin{aligned} k\,\xi_j &= q^{-j/2}\,\xi_j\ , \\ e\,\xi_j &= q^{(1-2\lambda)/4}\,\frac{1-q^{\lambda+j}}{1-q}\,\xi_{j-1}\ , \\ f\,\xi_j &= q^{(1-2\lambda)/4}\,\frac{1-q^{\lambda-j}}{1-q}\,\xi_{j+1}\ . \end{aligned} \tag{11}$$

Given any $a \in \mathcal{U}_q(sl(2))$, its matrix elements $W_{ij}(a)$ in this representation are defined by

$$a\,\xi_j = \sum_{i\in\mathbf{Z}+m_0} \xi_i\, W_{ij}(a)\ . \tag{12}$$

It clearly follows that

$$W_{ij}(ab) = \sum_{k\in\mathbf{Z}+m_0} W_{ik}(a)\, W_{kj}(b)\ , \tag{13}$$

and, for instance, one has

$$W_{ij}(e^{\mu}k^{\rho}f^{\nu}) = \left(\frac{q^{(1-2\lambda)/4}}{1-q}\right)^{\mu+\nu} q^{-\rho(\nu+j)/2} \left(q^{\lambda-j-\nu+1};q\right)_{\nu} \left(q^{\lambda+j+\nu-\mu+1};q\right)_{\mu} \delta_{i,j+\nu-\mu}\ . \tag{14}$$

In analogy with ordinary Lie theory,[17] it is natural to define the following elements of $\mathcal{U}_q(sl(2))$:

$$U(\alpha,\beta,\gamma) = e_q(\alpha\, e)\, E_q(\beta\, f)\, k^{\gamma}\ , \tag{15}$$

where α, β and γ are complex parameters. Indeed, set $k = q^{-h/2}$ and $q = e^{-\eta}$. In the limit $q \to 1^-$, $\eta \to 0^+$, the defining relations (8) become those of $sl(2)$: $[h,e] = -e$, $[h,f] = f$, $[e,f] = -2h$, and $U\big((1-q)\alpha, (1-q)\beta, 2\gamma/\eta\big)$ goes into the $SL(2)$ group element $e^{\alpha e} e^{\beta f} e^{\gamma h}$. (Other combinations of little and big q-exponentials could also be used. See, for instance Refs.[13,14].)

Remarkably, the matrix elements of $U(\alpha,\beta,\gamma)$ on $V^{(\lambda,m_0)}$ turn out to be expressible in terms of the function ${}_2\phi_1(a,b;c;q,z)$. In fact, using (14) and identities involving q-shifted factorials,[26] it is straightforward to show that[9,11]

$$\begin{aligned} W_{ij}\Big(U(\alpha,\beta,\gamma)\Big) &= q^{-\gamma j/2} \left(\alpha\,\frac{q^{(1-2\lambda)/4}}{1-q}\right)^{j-i} \frac{\left(q^{\lambda+i+1};q\right)_{j-i}}{(q;q)_{j-i}} \\ &\quad \times {}_2\phi_1\left(q^{\lambda+j+1}, q^{j-\lambda}; q^{j-i+1}; q, -\alpha\beta\,\frac{q^{(1-2j)/2}}{(1-q)^2}\right)\ , \quad \text{if } i \le j\ , \end{aligned} \tag{16a}$$

$$\begin{aligned} W_{ij}\Big(U(\alpha,\beta,\gamma)\Big) &= q^{-\gamma j/2} q^{(i-j)(i-j-1)/2} \left(\beta\,\frac{q^{(1-2\lambda)/4}}{1-q}\right)^{i-j} \frac{\left(q^{\lambda-i+1};q\right)_{i-j}}{(q;q)_{i-j}} \\ &\quad \times {}_2\phi_1\left(q^{\lambda+i+1}, q^{i-\lambda}; q^{i-j+1}; q, -\alpha\beta\,\frac{q^{(1-2j)/2}}{(1-q)^2}\right)\ , \quad \text{if } i \ge j\ , \end{aligned} \tag{16b}$$

with $i,j \in \mathbf{Z} + m_0$. This establishes most simply the connection between the basic hypergeometric series ${}_2\phi_1$ and $\mathcal{U}_q(sl(2))$ which can now be exploited to obtain properties of these special functions (see Refs.[9,11]).

Notice that it is possible to give a one-variable model for the representation (11), where the generators are expressed as q-difference operators in the complex variable z acting on the space of all linear combinations of the functions z^n, $n \in \mathbf{Z}$. Indeed, by taking

$$\begin{aligned} k &= q^{-m_0/2}\, T_z^{-1/2}\ , \\ e &= q^{(1-2\lambda)/4} \left(\frac{1}{1-q} D_z^+ + \frac{1-q^{\lambda+m_0}}{1-q}\,\frac{1}{z}\, T_z\right)\ , \\ f &= q^{(1-2\lambda)/4} \left(\frac{z^2}{1-q} D_z^- + \frac{1-q^{\lambda-m_0}}{1-q}\, z\, T_z^{-1}\right)\ , \end{aligned} \tag{17}$$

and $\xi_j = z^n$, $j = m_0 + n$, $n \in \mathbb{Z}$, for the basis vectors, one can check that the relations (11) are satisfied. This has been used in Ref.[9] to obtain from (12) and (16) generating relations for the function ${}_2\phi_1(a,b;c;q,z)$.

The matrix elements $W_{ij}(U(\alpha,\beta,\gamma))$ themselves define models of the module $V^{(\lambda,m_0)}$ and directly provide a three-variable realization of $\mathcal{U}_q(sl(2))$. For each element $a \in \mathcal{U}_q(sl(2))$ define the operator $\tilde{\pi}(a)$ acting on the variables α, β and γ such that $\tilde{\pi}(a)\,U(\alpha,\beta,\gamma) = U(\alpha,\beta,\gamma)\,a$. Upon operating with ξ_j, one obtains:

$$\tilde{\pi}(a)\,W_{ij}\Big(U(\alpha,\beta,\gamma)\Big) = \sum_{k\in\mathbb{Z}+m_0} W_{ik}\Big(U(\alpha,\beta,\gamma)\Big)\,W_{kj}(a)\,, \tag{18}$$

that is, the functions $W_{ij}(U(\alpha,\beta,\gamma))$ transform like the vectors ξ_j. From the properties of the q-exponentials, it is easy to construct the operators $\tilde{\pi}$ for k, e and f, and thus for any $a \in \mathcal{U}_q(sl(2))$ by composition. One finds

$$\tilde{\pi}(k) = \mathcal{E}_\gamma\,, \tag{19a}$$

$$\tilde{\pi}(e) = q^{\gamma/2}\left(D^+_\alpha + \frac{q^{1/2}}{(1-q)^2}\,\beta\,(\mathcal{E}^2_\gamma - T_\beta\,\mathcal{E}^{-2}_\gamma)\right)\,, \tag{19b}$$

$$\tilde{\pi}(f) = -q^{1-\gamma/2}\,D^-_\beta\,, \tag{19c}$$

where $\mathcal{E}_\gamma$ is the shift operator defined by $\mathcal{E}_\gamma\,g(\gamma) = g(\gamma+1)$ on any function of γ. That $\tilde{\pi}(k)$ and $\tilde{\pi}(f)$ are the required operators is readily verified. To derive $\tilde{\pi}(e)$, one first acts with D^+_α on $U(\alpha,\beta,\gamma)$ using (6c), and then shows that

$$e_q(-\beta f)\,e\,E_q(\beta f) = e - \frac{q^{1/2}}{(1-q)^2}\,\beta\,(k^2 - (1+\beta f)^{-1}k^{-2})\,, \tag{20}$$

with the help of the following formula[17]

$$e_q(\zeta X)\,Y\,E_q(-\zeta X) = \sum_{n=0}^{\infty}\frac{\zeta^n}{(q;q)_n}\,[X,Y]_n\,, \tag{21}$$

where

$$[X,Y]_0 = Y, \qquad [X,Y]_{n+1} = X\,[X,Y]_n - q^n[X,Y]_n\,X, \quad \text{for } n = 1,2,\ldots\,.$$

Recalling (6b) and (19a), one arrives at (19b).

It is a simple exercise to verify that $\tilde{\pi}(k)$, $\tilde{\pi}(e)$ and $\tilde{\pi}(f)$ obey the same commutation relations as k, e and f. The observation that the matrix elements $W_{ij}(U(\alpha,\beta,\gamma))$ are also basis vectors for representatins of $\mathcal{U}_q(sl(2))$ can be used to obtain addition formulas for the basic hypergeometric functions.

The realization $\tilde{\pi}$ can also be used to obtain the q-difference equation obeyed by ${}_2\phi_1(a,b;c;q,z)$. The Casimir element C, which belongs to the center of $\mathcal{U}_q(sl(2))$, is given by

$$C = \frac{q^{1/2}k^2 + q^{-1/2}k^{-2} - 2}{(q^{1/2} - q^{-1/2})^2} + fe\,, \tag{22}$$

and on $V^{(\lambda,m_0)}$ assumes the value

$$C^{(\lambda)} = \frac{q^{\lambda+1/2} + q^{-\lambda-1/2} - 2}{(q^{1/2} - q^{-1/2})^2}\,. \tag{23}$$

Then, using *e.g.* (16a), from

$$\tilde{\pi}(C)\,W_{ij}\Big(U(\alpha,\beta,\gamma)\Big) = C^{(\lambda)}\,W_{ij}\Big(U(\alpha,\beta,\gamma)\Big)\,, \tag{24}$$

one can work out the equation satisfied by ${}_2\phi_1\big(q^{\lambda+j+1}, q^{j-\lambda}; q^{j-i+1}; q, -\alpha\beta\,\frac{q^{(1-2j)/2}}{(1-q)^3}\big)$. With the obvious identifications for a, b, c and z, one finds[26]

$$\begin{aligned}\Big\{z(c-abqz)(D^+_z)^2 + \Big[(1-c) + ((1-a)(1-b) - (1-c))z\Big]D^+_z \\ -(1-a)(1-b)\Big\}\,{}_2\phi_1(a,b;c;q,z) = 0\,.\end{aligned} \tag{25}$$

We shall now make contact with the quantum group approach and indicate in particular how the results described so far enable one to recover the matrix elements of the $SL_q(2)$ corepresentation given in Ref.[23].

Let $\mathcal{A}$ be the space dual to $\mathcal{U}_q(sl(2))$. The coordinate ring $A(SL_q(2))$ of the quantum group $SL_q(2)$ is introduced as follows. Consider the fundamental representation $X:\ \mathcal{U}_q(sl(2)) \to \mathcal{E}nd\,\mathbf{C}^2$ which is defined by

$$X(k) = \begin{pmatrix} q^{1/4} & 0 \\ 0 & q^{-1/4}\end{pmatrix}\,, \qquad X(e) = \begin{pmatrix} 0 & 1 \\ 0 & 0\end{pmatrix}\,, \qquad X(f) = \begin{pmatrix} 0 & 0 \\ 1 & 0\end{pmatrix}\,. \tag{26}$$

Since X is a representation, we have

$$X(ab) = X(a)\,X(b)\,, \qquad a,b \in \mathcal{U}_q(sl(2))\,. \tag{27}$$

The matrix elements of X, that we shall denote as

$$X = \begin{pmatrix} x & u \\ v & y\end{pmatrix}\,, \tag{28}$$

define mappings $\mathcal{U}_q(sl(2)) \to \mathbf{C}$, via the evaluation map $a \to X(a)$; hence x, u, v and y are elements of $\mathcal{A}$. We shall call $A(SL_q(2))$ the subalgebra of $\mathcal{A}$ generated by these coordinate elements. This algebra is seen to inherit from $\mathcal{U}_q(sl(2))$ a Hopf structure. The coproduct $\Delta_{\mathcal{A}}:\ \mathcal{A} \to \mathcal{A}\otimes\mathcal{A}$, antipode $S_{\mathcal{A}}:\ \mathcal{A} \to \mathcal{A}$ and counit $\varepsilon_{\mathcal{A}}:\ \mathcal{A} \to \mathbf{C}$ are defined by

$$\Delta_{\mathcal{A}}(X)\,(a\otimes b) = X(ab)\,, \qquad a,b \in \mathcal{U}_q(sl(2))\,, \tag{29}$$

and

$$S_{\mathcal{A}}(X) = \begin{pmatrix} y & -q^{1/2}u \\ -q^{-1/2}v & x\end{pmatrix}\,, \qquad \varepsilon_{\mathcal{A}}(X) = \begin{pmatrix} 1 & 0 \\ 0 & 1\end{pmatrix}\,. \tag{30}$$

In terms of matrix elements, the definition (29) is equivalent to

$$\Delta_{\mathcal{A}}(X_{rs}) = \sum_{t=1,2} X_{rt}\, X_{ts}\ , \qquad r,s = 1,2\ , \tag{31}$$

with the understanding that $(\Phi \otimes \Psi)(a \otimes b) = \Phi(a)\Psi(b)$ for $\Phi, \Psi \in \mathcal{A}$, $a, b \in \mathcal{U}_q(sl(2))$. The product $\mu_{\mathcal{A}} :\ \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ is determined by

$$\mu_{\mathcal{A}}(\Phi \otimes \Psi)(a) \equiv \Phi\Psi(a) = \Phi \otimes \Psi\big(\Delta(a)\big)\ . \tag{32}$$

With these definitions one finds that

$$\begin{aligned} x^L u^M v^N \big(e^\mu k^\rho f^\nu\big) &= q^{-\frac{1}{4}\rho(\mu+\nu-L)-\frac{1}{4}L(\mu+\nu)-\frac{1}{4}(\mu(\mu-1)+\nu(\nu-1))+\frac{1}{2}(\mu-M)(\nu-N)} \\ &\qquad \times \frac{(q;q)_L}{(q;q)_{\mu-M}\,(q;q)_{L-\mu+M}}\,\frac{(q;q)_\mu\,(q;q)_\nu}{(1-q)^{\mu+\nu}}\ , \qquad \text{for } \mu - M = \nu - N,\ \ M \le \mu \le M+L\ , \\ &= 0\ , \qquad \text{otherwise}\ , \end{aligned} \tag{33a}$$

and

$$\begin{aligned} u^M v^N y^L \big(e^\mu k^\rho f^\nu\big) &= q^{-\frac{1}{4}\rho(\mu+\nu+L)-\frac{1}{4}L(\mu+\nu)-\frac{1}{4}(\mu(\mu-1)+\nu(\nu-1))} \\ &\qquad \times \frac{(q;q)_\mu\,(q;q)_\nu}{(1-q)^{\mu+\nu}}\ \delta_{\mu,M}\ \delta_{\nu,N}\ , \end{aligned} \tag{33b}$$

with M, N, and L nonnegative integers. Given these evaluations on the Poincaré-Birkhoff-Witt basis elements of $\mathcal{U}_q(sl(2))$, it is immediate to see that the coordinate elements obey the following commutation relations

$$q^{1/2}xu = ux \quad q^{1/2}xv = vx \quad q^{1/2}uy = yu \quad q^{1/2}vy = yv \quad uv = vu\ , \tag{34}$$

and

$$\det{}_q X = xy - q^{-1/2}uv = yx - q^{1/2}uv = 1_{\mathcal{A}}\ . \tag{35}$$

The matrix elements W_{ij} defined in (12) can also be viewed as elements of $\mathcal{A}$; they indeed provide linear mappings from $\mathcal{U}_q(sl(2))$ into $\mathbf{C}$, $a \to W_{ij}(a)$, $a \in \mathcal{U}_q(sl(2))$. Introducing an infinite matrix W, with elements W_{ij}, the composition relation (13) can be rewritten in the form

$$\Delta_{\mathcal{A}} W = W \otimes W\ , \tag{36}$$

and we conclude that W defines a corepresentation of $SL_q(2)$.

We would like now to derive the analytic expressions for the elements W_{ij} of W in terms of the coordinates x, u, v and y of $\mathcal{A}$, starting from the formulas (16), the evaluation of W_{ij} on $U(\alpha,\beta,\gamma)$. This is certainly possible since the element $U(\alpha,\beta,\gamma) = e_q(\alpha e)\, E_q(\beta f)\, k^\gamma$ generates the complete basis of $\mathcal{U}_q(sl(2))$:

$$(-1)^\nu q^{\nu(2-\rho)/2}\left(\left(D_\alpha^+\right)^\mu \left(D_\beta^-\right)^\nu U(\alpha,\beta,\rho)\right)\Big|_{\alpha=\beta=0} = e^\mu\, k^\rho\, f^\nu\ . \tag{37}$$

An arbitrary element of $\mathcal{A}$ is therefore completely specified once its evaluation on $U(\alpha,\beta,\gamma)$ is given. With the help of the following formulas,

$$\begin{aligned} x^L u^M v^N \Big(U(\alpha,\beta,\gamma)\Big) &= q^{-\frac{1}{4}\gamma(M-N-L)-\frac{1}{4}L(M+N)-\frac{1}{4}(M(M-1)-N(N-1))} \\ &\qquad \times \left(-\frac{\alpha\beta}{(1-q)^2}\, q^{\frac{1}{2}(N-M-L+1)};q\right)_L \frac{\alpha^M\beta^N}{(1-q)^{M+N}}\ , \end{aligned} \tag{38a}$$

$$\begin{aligned} u^M v^N y^L \Big(U(\alpha,\beta,\gamma)\Big) &= q^{-\frac{1}{4}\gamma(M-N+L)-\frac{1}{4}L(M+N)-\frac{1}{4}(M(M-1)-N(N-1))} \\ &\qquad \times \frac{\alpha^M\beta^N}{(1-q)^{M+N}}\ , \end{aligned} \tag{38b}$$

M, N and L nonnegative integers, and knowing (see (33)) that $x^L u^M v^N$ and $u^M v^N y^L$ span $A(SL_q(2))$, the expressions for W_{ij} can be easily abstracted from the matrix elements $W_{ij}(U(\alpha,\beta,\gamma))$ given before. One obtains[23]

$$W_{ij} = q^{(i-j)(\lambda-j)}\,\frac{\left(q^{\lambda+i+1};q\right)_{j-i}}{(q;q)_{j-i}}\ {}_2\phi_1\left(q^{\lambda+j+1}, q^{j-\lambda}; q^{j-i+1}; q, -q^{1/2}uv\right) u^{j-i} y^{i+j}\ , \qquad \text{if } i \le j\ ,\ i+j \ge 0\ , \tag{39a}$$

$$W_{ij} = q^{(i-j)(\lambda+i)}\,\frac{\left(q^{\lambda+i+1};q\right)_{j-i}}{(q;q)_{j-i}}\ x^{-(i+j)} u^{j-i}\ {}_2\phi_1\left(q^{\lambda-i+1}, q^{-i-\lambda}; q^{j-i+1}; q, -q^{1/2}uv\right)\ , \qquad \text{if } i \le j\ ,\ i+j \le 0\ , \tag{39b}$$

$$W_{ij} = q^{(j-i)(\lambda-i)}\,\frac{\left(q^{\lambda-i+1};q\right)_{i-j}}{(q;q)_{i-j}}\ {}_2\phi_1\left(q^{\lambda+i+1}, q^{i-\lambda}; q^{i-j+1}; q, -q^{1/2}uv\right) v^{i-j} y^{i+j}\ , \qquad \text{if } i \ge j\ ,\ i+j \ge 0\ , \tag{39c}$$

$$W_{ij} = q^{(j-i)(\lambda+j)}\,\frac{\left(q^{\lambda-i+1};q\right)_{i-j}}{(q;q)_{i-j}}\ x^{-(i+j)} v^{i-j}\ {}_2\phi_1\left(q^{\lambda-j+1}, q^{-j-\lambda}; q^{i-j+1}; q, -q^{1/2}uv\right)\ , \qquad \text{if } i \ge j\ ,\ i+j \le 0\ . \tag{39d}$$

These four cases need to be distinguished in order for the elements W_{ij} to be analytic in x, u, v and y.

Let us briefly explain how these formulas are derived from (16). We shall use as example the case $i \le j$. First, notice from (38) that α is paired with u, and β with v under the evaluation on $U(\alpha,\beta,\gamma)$. A glance at (16a) then tells us that W_{ij} has to involve u^{j-i} and a function of uv so as to yield α^{j-i} times a function of $\alpha\beta$ when evaluated on $U(\alpha,\beta,\gamma)$. Since $x^L u^M v^N$ and $u^M v^N y^L$ form a basis for $A(SL_q(2))$, W_{ij} must be either of the form $\varphi(uv)\, u^{j-i} y^L$ or of the form $x^{L'} u^{j-i}\, \psi(uv)$, with φ and ψ two analytic functions. To obtain

these functions, and the indices L and L', consider the following formulas, derived with the help of (38):

$$\varphi(uv)\,u^{j-i}y^L\Big(U(\alpha,\beta,\gamma)\Big) = q^{-\frac{1}{4}\gamma(j-i+L)-\frac{1}{4}(j-i)(j-i+L-1)}\left(\frac{\alpha}{1-q}\right)^{j-i} \times \varphi\Big(\frac{\alpha\beta}{(1-q)^2}q^{-\frac{1}{2}(j-i+L)}\Big)\,, \tag{40a}$$

$$x^{L'}u^{j-i}\,\psi(uv)\Big(U(\alpha,\beta,\gamma)\Big) = q^{-\frac{1}{4}\gamma(j-i-L')-\frac{1}{4}(j-i)(j-i+L'-1)}\left(\frac{\alpha}{1-q}\right)^{j-i} \times \Big(-\frac{\alpha\beta}{(1-q)^2}q^{-\frac{1}{2}(j-i+L'-1)};q\Big)_{L'}\,\psi\Big(\frac{\alpha\beta}{(1-q)^2}q^{-\frac{1}{2}(j-i+L')}\Big)\,, \tag{40b}$$

and compare them respectively with (16a) and with

$$W_{ij}\Big(U(\alpha,\beta,\gamma)\Big) = q^{-\gamma j/2}\left(\alpha\frac{q^{(1-2\lambda)/4}}{1-q}\right)^{j-i}\frac{(q^{\lambda+i+1};q)_{j-i}}{(q;q)_{j-i}}\left(-\alpha\beta\frac{q^{(1+2i)/2}}{(1-q)^2};q\right)_{-(i+j)} \times {}_2\phi_1\Big(q^{\lambda-i+1},q^{-i-\lambda};q^{j-i+1};q,-\alpha\beta\frac{q^{(1+2i)/2}}{(1-q)^2}\Big)\,,\quad i\le j\,, \tag{41}$$

which is obtained from (16a) by using Heine's q-analog of Euler's transformation fromula:[26]

$${}_2\phi_1(q^a,q^b;q^c;q,z) = (q^{a+b-c}z;q)_{c-a-b}\;{}_2\phi_1(q^{c-a},q^{c-b};q^c;q,q^{a+b-c}z)\,. \tag{42}$$

Observe now that L and L' are immediately determined from the fact that the dependence of $W_{ij}\,(U(\alpha,\beta,\gamma))$ on γ is all contained in the factor $q^{-\gamma j/2}$. Thus, from (40a) one gets $L=i+j$, while from (40b), $L'=-i-j$. Since we want W_{ij} to be analytic in the coordinates of $SL_q(2)$, we must take $W_{ij}=\varphi(uv)\,u^{j-i}y^{i+j}$ when $i+j\ge 0$ and $W_{ij}=x^{-(i+j)}u^{j-i}\,\psi(uv)$ when $i+j\le 0$. It is then straightforward to see by comparing (40a) with (16a) and (40b) with (41) that the functions $\varphi(uv)$ and $\psi(uv)$ must be those recorded in (39a) and (39b). The expressions for W_{ij} when $i\ge j$ are obtained in a similar way.

The matrix elements (39) are computed in a different way in Ref.[23]. First, the general form of W_{ij} is specified. For each of the four index subsets characterized by the signs of $i+j$ and $i-j$, this leaves in W_{ij} an analytic function $\varphi(uv)$ to be determined. In order to obtain these functions one introduces a $\mathcal{U}_q(sl(2))$-action π on $\mathcal{A}$, such that $(\pi(a)\,\Phi)(b)=\Phi(ba)$, with $\Phi\in\mathcal{A}$, and $a,b\in\mathcal{U}_q(sl(2))$. On the corepresentation W of $SL_q(2)$ one has

$$\pi(a)\,W = W\cdot W(a)\,, \tag{43}$$

implying that the entries in each column of W realize a basis for the module $V^{(\lambda,m_0)}$. Recalling that the Casimir operator of $\mathcal{U}_q(sl(2))$ assumes the value $C^{(\lambda)}$ on $V^{(\lambda,m_0)}$ (see (23)), one can write: $\pi(C)\,W_{ij}=C^{(\lambda)}\,W_{ij}$. This equation implies a second-order difference equation for the functions $\varphi(uv)$, that, using (25), can be now expressed in terms of the hypergeometric series ${}_2\phi_1$.

To establish the connection between basic hypergeometric functions and $SL_q(2)$ in this fashion, one needs to know explicitly how $\pi(k)$, $\pi(e)$ and $\pi(f)$ act *e.g.* on $x^Lu^Mv^N$ and $u^Mv^Ny^L$. This can be accomplished inductively. From the definition of the representation π, one sees that $\pi(a)\,\Phi\Psi=\sum_r\pi(a_r)\Phi\cdot\pi(a^r)\Psi$, for $\Phi,\Psi\in\mathcal{A}$, and $\Delta(a)=\sum_r a_r\otimes a^r$, $a\in\mathcal{U}_q(sl(2))$. With the twisted derivation rules that this entails, one can proceed by induction starting from (43) with W replaced by X. The results are given in Ref.[23].

The derivation of these actions can be simplified by using instead the three-variable realization of $\mathcal{U}_q(sl(2))$ given in (19). From the definition of π and $\tilde\pi$ it is clear that

$$\pi(a)\,\Phi\Big(U(\alpha,\beta,\gamma)\Big) = \tilde\pi(a)\Big(\Phi\big(U(\alpha,\beta,\gamma)\big)\Big)\,,\qquad \Phi\in\mathcal{A}\,,\quad a\in\mathcal{U}_q(sl(2))\,. \tag{44}$$

The following strategy can now be adopted to determine the action of $\pi(a)$ on Φ. It suffices of course to consider the cases $a=k,e,f$. First, evaluate Φ on $U(\alpha,\beta,\gamma)$ and then operate with $\tilde\pi(a)$ on the result; these steps are readily performed using (38) and (19). In view of (44), this gives the evaluation of $\pi(a)\Phi$ on $U(\alpha,\beta,\gamma)$, which, as we already pointed out, completely determines this functional. Comparing with the various evaluation formulas, it is immediate to read off the expression for $\pi(a)\Phi$.

Let us show for example how $\pi(e)u^n$ and $\pi(f)\varphi(uv)$ are obtained in this way. From (38) one has $u^n(U(\alpha,\beta,\gamma))=q^{-\gamma n/4-n(n-1)/4}\alpha^n/(1-q)^n$. We now act on this function of α and γ with the q-difference operator $\tilde\pi(e)$ given in (19b) to find

$$\begin{aligned}\pi(e)u^n\Big(U(\alpha,\beta,\gamma)\Big) &= \tilde\pi(e)\Big(u^nU(\alpha,\beta,\gamma)\Big)\\ &= q^{-\frac{1}{4}\gamma(n-2)-\frac{1}{4}n(n-1)}\frac{1-q^n}{1-q}\,\frac{\alpha^{n-1}}{(1-q)^{n-1}}\left(1+\frac{\alpha\beta}{(1-q)^2}q^{-(n-1)/2}\right)\,.\end{aligned} \tag{45}$$

From this, using again (38), one immediately obtains

$$\pi(e)u^n = q^{-\frac{1}{4}(n-1)}\frac{1-q^n}{1-q}\,zu^{n-1}\,. \tag{46}$$

It is also instructive to examine $\pi(f)\varphi(uv)$, with φ an analytic function. From (40) and (17c), one has

$$\begin{aligned}\tilde\pi(f)\Big(\varphi(uv)\big(U(\alpha,\beta,\gamma)\big)\Big) &= -\,q^{1-\gamma/2}\frac{1}{\beta}\,(1-T_\beta^{-1})\,\varphi\Big(\frac{\alpha\beta}{(1-q)^2}\Big)\\ &= \frac{q^{-\gamma/2}}{1-q}\,\frac{\alpha}{1-q}\,\Big(D_z^+\varphi(z)\Big)\Big|_{z=q^{-1}\alpha\beta/(1-q)^2}\,,\end{aligned} \tag{47}$$

and then using (40a) one arrives at

$$\pi(f)\varphi(uv) = \frac{q^{1/4}}{1-q}\,\Big(D_{uv}^+\,\varphi(uv)\Big)\,uy\,. \tag{48}$$

In conclusion, we see that the quantum algebra approach to q-special functions, developped in analogy with the standard Lie algebraic interpretation of special functions, is simple and fruitful. It allows one to recover the results obtained in the quantum group approach to q-special functions and often offers computational simplicity.

ACKNOWLEDGEMENTS

We are grateful to Willard Miller for discussions and for informing us of some of his results prior to publication. R.F. would like to thank the Université de Montréal for its hospitality and support.

REFERENCES

1. Drinfel'd, V.G., Quantum groups, in: *Proceedings of the International Congress of Mathematicians*, Berkeley (1986), vol. **1**, pp. 798-820, (The American Mathematical Society, Providence, 1987)
2. Jimbo, M., A q-difference analogue of $U(g)$ and the Yang-Baxter equation, Lett. Math. Phys. **10**, 63-69 (1985); A q-analogue of $U(gl(N+1))$, Hecke algebra and the Yang-Baxter equation, *ibid.* **11**, 247-252 (1986)
3. Woronowics, S.L., Compact matrix pseudogroups, Comm. Math. Phys. **111**, 613-665 (1987)
4. Faddeev, L.D., Reshetikhin, N.Yu. and Takhtajan, L.A., Quantization of Lie groups and Lie algebras, in *Algebraic analysis*, vol. 1, 129 (Academic Press, New York, 1988)
5. Manin, Yu.I., *Quantum Groups and Non-Commutative Geometry*, (Centre de Recherches Mathematiques, Montréal, 1988)
6. Floreanini, R. and Vinet, L., q-Orthogonal polynomials and the oscillator quantum group, Lett. Math. Phys. **22**, 45-54 (1991)
7. Floreanini, R. and Vinet, L., The metaplectic representation of $su_q(1,1)$ and the q-Gegenbauer polynomials, J. Math. Phys., to appear
8. Floreanini, R. and Vinet, L., q-Conformal quantum mechanics and q-special functions, Phys. Lett. B, to appear
9. Floreanini, R. and Vinet, L., Quantum algebras and q-special functions, University of Montreal-preprint, UdeM-LPN-TH54, 1991
10. Floreanini, R. and Vinet, L., Addition formulas for q-Bessel functions, University of Montreal-preprint, UdeM-LPN-TH60, 1991

11. Floreanini, R. and Vinet, L., Representations of quantum algebras and q-special functions, Proceedings of the *II International Wigner Symposium*, Dobrev, V. and Scherer, W., eds., (Springer-Verlag, Berlin, 1992), to appear
12. Floreanini, R. and Vinet, L., Generalized q-Bessel functions, University of Montreal-preprint, UdeM-LPN-TH87, 1992
13. Kalnins, E.G., Manocha, H.L. and Miller, W., Models of q-algebra representations: I. Tensor products of special unitary and oscillator algebras, J. Math. Phys., to appear
14. Kalnins, E.G, Miller, W., and Mukherjee, S., Models of q-algebra representations: the group of plane motions, University of Minnesota preprint, 1992
15. Miller, W., *Lie Theory and Special Functions*, (Academic Press, New York, 1968)
16. Miller, W., Lie theory and q-difference equations, SIAM J. Math. Anal. **1**, 171-188 (1970)
17. Agarwal, A.K., Kalnins, E.G. and Miller, W., Canonical equations and symmetry techniques for q-series, SIAM J. Math. Anal. **18**, 1519-1538 (1987)
18. Vaksman, L.L. and Soibelman, Ya.S., Algebra of functions of the quantum group $SU(2)$, Funct. Anal. Appl. **22**, 1-14 (1988)
19. Masuda, T., Mimachi, K., Nakagami, Y., Noumi, M. and Ueno, K., Representations of the quantum group $SU_q(2)$ and the little q-Jacobi polynomials, J. Funct. Anal. **99**, 357-386 (1991)
20. Koornwinder, T.H., Representations of the twisted $SU(2)$ quantum group and some q-hypergeometric orthogonal polynomials, Nederl. Akad. Wetensch. Proc. Ser. **A92**, 97-117 (1989)
21. Vaksman, L.L. and Korogodskii, L.I., An algebra of bounded functions on the quantum group of the motions of the plane, and q-analogues of Bessel functions, Soviet Math. Dokl. **39**, 173-177 (1989)
22. Koornwinder, T.H., The addition formula for little q-Legendre polynomials and the $SU(2)$ quantum group, SIAM J. Math. Anal. **22**, 295-301 (1991)
23. Masuda, T., Mimachi, K., Nakagami, Y., Noumi, M., Saburi, Y. and Ueno, K., Unitary representations of the quantum group $SU_q(1,1)$: structure of the dual space of $U_q(sl(2))$, Lett. Math. Phys. **19**, 187-194 (1990); Unitary representations of the quantum group $SU_q(1,1)$: II-matrix elements of unitary representations and the basic hypergeometric functions, *ibid.* **19**, 195-204 (1990)
24. Koelnik, H.T., On quantum groups and q-special functions, University of Leiden-thesis, 1991
25. Swarttouw, R., The Hahn-Exton q-Bessel functions, University of Delft-thesis, 1992
26. Gasper, G. and Rahman, M., *Basic Hypergeometric Series*, (Cambridge University Press, Cambridge, 1990)

ROUTE TO
NAME
LOCATION
A.Y