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DYNAMICAL PARASUPERALGEBRAS AND HARMONIC OSCILLATORS †

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Abstract

The dynamical algebra of the n-dimensional harmonic oscillator with one second-order parafermionic degree of freedom is exhibited. It provides an example and an explicit realization of a second-order parasuperalgebra. These constructs generalize the more familiar superalgebra structures. They possess a $\mathcal{Z}_2$ grading and involve a symmetric trilinear product among their odd elements. Particular attention is given to the one- and two-dimensional cases. It is pointed out that the one-dimensional oscillator Hamiltonian is parasupersymmetric, i.e., that it belongs to the class of Hamiltonians H for which one has a realization of the algebra $[H,Q]=[H,Q^\dagger]=0$, $Q^2Q^\dagger+QQ^\dagger Q+Q^\dagger Q^2=4QH$. It is further noted that the parasupersymmetric Hamiltonians constructed by Rubakov and Spiridonov always possess two additional conserved parasupercharges. It is finally explained how the two-dimensional oscillator parasuperalgebra provides a spectrum-generating algebra for the one-dimensional parasupersymmetric Morse Hamiltonian.

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Loosely speaking a quantum-mechanical system is said to possess a dynamical algebra if its states form representation spaces for this algebra. The simplest situation where this occurs is when the Hamilton operator can itself be expressed as an algebra element in a realization of some algebra [1]. There are also many interesting cases where the Hamiltonian can be written as a bilinear combination of algebra generators [2]. In both instances, there exist various techniques that allow for a group theoretic resolution of the dynamics. We shall here be mostly concerned with examples of the first type.

When systems involving both bosonic and fermionic canonical degrees of freedom are considered it is appreciated [3] that superalgebras can arise as dynamical algebras. This is so, when there are constants of motion that generate supersymmetry transformations, that is transformations mixing the bosonic and the fermionic variables.

It has been known [4] for a long time that the canonical quantization rules are not the only relations that can be imposed on the basic dynamical variables in order for the Heisenberg equations to be compatible with the "classical" equations of motion. There is in fact an infinity of consistent schemes and quantities satisfying the corresponding generalized quantization rules are said to be parabosonic or parafermionic [5].

As a matter of fact, there exist systems which involve dynamical variables both of the ordinary Bose type and of the para-Fermi type. In such a context, we may expect symmetry operations transforming the bosonic variables into the parafermionic ones (and vice-versa) to occur. These operations which generalize the familiar supersymmetry transformations have been called parasypersymmetries [6]. Their generators realize new algebraic structures, parasuperalgebras, of which superalgebras are a special case. The main feature of parasuperalgebras lies in the fact that they involve a multi-linear product rule for their fermionic elements. The purpose of this contribution is to give an example where this new type of dynamical algebra is encountered.

Let $\alpha_i^\dagger$ and α_i, $i = 1, ...n$, respectively stand for ordinary bosonic creation and annihilation operators satisfying

$$[\alpha_i, \alpha_j^\dagger] = \delta_{ij}, \qquad i, j = 1, ... n. \tag{1}$$

To these variables let us add one single set of second-order parafermionic creation and annihilation operators $a^\dagger$ and a. By definition these quantities obey the trilinear relations [5]

$$a^3 = 0, \quad aa^\dagger a = 2a, \quad a^2a^\dagger + a^\dagger a^2 = 2a. \tag{2}$$

We shall take the following Hamiltonian to govern the time evolution of these dynamical variables:

$$H = \sum_{i=1}^{n} \frac{\omega}{2}\{\alpha_i^\dagger, \alpha_i\} + \frac{\omega}{2}[a^\dagger, a]. \tag{3}$$

It is well known that the canonical commutation relations (1) are realized if one uses n coordinates x_i and takes

$$\alpha_i = \frac{1}{\sqrt{2\omega}}(p_i - i\omega x_i), \qquad \alpha_i^\dagger = \frac{1}{\sqrt{2\omega}}(p_i + i\omega x_i), \tag{4}$$

with $p_i = -i\partial/\partial x_i$. As for the parafermionic variables, it is not difficult to see that a and $a^\dagger$ are irreducibly represented by the following 3×3 matrices:

$$a = \sqrt{2}\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \qquad a^\dagger = \sqrt{2}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}. \tag{5}$$

Substitution of (4) and (5) in H gives

$$H = \sum_{i=1}^{n} \frac{1}{2}\left(p_i^2 + \omega^2 x_i^2\right) + \omega\Sigma_3, \tag{6}$$

with

$$\Sigma_3 = \frac{1}{2}[a^\dagger, a] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \tag{7}$$

making clear that this Hamiltonian describes a n-dimensional oscillator. We shall now present its dynamical algebra which as we shall see, is a fairly large second-order parasuperalgebra. (This explains our interest in this otherwise rather trivial system.) Aspects and applications that are specific to the one and two-dimensional cases will be discussed in the conclusions. In particular, we shall indicate what is meant by a parasupersymmetric Hamiltonian and shall explain how the invariance

parasuperalgebra of the 2-dimensional oscillator provides a spectrum-generating algebra for the parasupersymmetric Morse Hamiltonian.

Let us introduce the following constants of motion:

$$P_i = \sqrt{\omega} e^{-i\omega t} \alpha_i^\dagger, \qquad P_i^\dagger = \sqrt{\omega} e^{i\omega t} \alpha_i, \quad i = 1, \dots n \tag{8}$$

$$T_\mu = \begin{pmatrix} a\, e^{i\omega t} \\ -b^\dagger e^{-i\omega t} \end{pmatrix}_\mu, \quad \bar{T}_\mu = \begin{pmatrix} b\, e^{i\omega t} \\ a^\dagger e^{-i\omega t} \end{pmatrix}_\mu = (-i\sigma_2 T^*)_\mu, \quad \mu = 1, 2 \tag{9}$$

with

$$b = \frac{1}{2}(a^\dagger a^2 - a^2 a^\dagger). \tag{10}$$

Their conservation is easily verified by checking that they belong to the kernel of $d/dt = \partial/\partial t + i[H, \ ]$.

Let us also identify as follows all bilinear monomials in P_i, $P_i^\dagger$, T_μ and $\bar{T}_\mu$ that can be constructed:

$$A_{ij} = P_i P_j, \quad H_{ij}^{(0)} = \frac{1}{2}\{P_i^\dagger, P_j\}, \quad A_{ij}^\dagger = P_i^\dagger P_j^\dagger, \tag{11a}$$

$$L_a = \frac{1}{4} T^\dagger \sigma_a T, \qquad a = 1, 2, 3, \tag{11b}$$

$$Y = \frac{1}{2} T^\dagger T, \tag{11c}$$

$$\begin{aligned} Q_{i\mu} &= P_i T_\mu, \qquad \bar{Q}_{i\mu} = P_i \bar{T}_\mu, \\ S_{i\mu} &= P_i^\dagger T_\mu, \qquad \bar{S}_{i\mu} = P_i^\dagger \bar{T}_\mu. \end{aligned} \tag{11d}$$

(In equation (11b), σ_a, $a = 1, 2, 3$, stands for the standard Pauli matrices.) This exhausts the list of the independent bilinears; note that

$$\frac{1}{4}\bar{T}^\dagger \sigma_a T = \frac{1}{4} T^\dagger \sigma_a \bar{T} = \frac{1}{4}\bar{T}^\dagger \sigma_a \bar{T} = L_a, \tag{12a}$$

$$\frac{1}{2} T^\dagger \bar{T} = \frac{1}{2}\bar{T}^\dagger T = 4I - 3Y, \tag{12b}$$

$$\frac{1}{2}\bar{T}^\dagger \bar{T} = Y. \tag{12c}$$

As the structure relations that we are about to give will show, T, Q and S (and $\bar{T}$, $\bar{Q}$ and $\bar{S}$ as well) transform as 2-spinors under the $SU(2)$ generated by the L_a's. We claim that the operators defined in (8), (9) and (11) form a closed algebraic

set V, if one includes the identity operator I. Note that V possesses a natural $\mathcal{Z}_2$ grading with $(P_i, P_i^\dagger, A_{ij}, A_{ij}^\dagger, H_{ij}^{(0)}, L_a, Y, I) \in V^{(B)}$ as the bosonic basis elements and $(T_\mu, \bar{T}_\mu, Q_{i\mu}, \bar{Q}_{i\mu}, S_{i\mu}, \bar{S}_{i\mu}) \in V^{(PF)}$ as the parafermionic ones. We thus have the decomposition $V = V^{(B)} + V^{(PF)}$. The structure relations of this algebra are of the following form:

$$\begin{aligned} &[V^{(B)}, V^{(B)}] \subset V^{(B)}, \\ &[V^{(B)}, V^{(PF)}] \subset V^{(PF)}, \\ &\{V^{(PF)}, V^{(PF)}, V^{(PF)}\} \subset V^{(B)} V^{(PF)}, \end{aligned} \tag{13}$$

with [,] the usual commutator and { , , } defined by

$$\{X_1, X_2, X_3\} = X_1(X_2X_3 + X_3X_2) + \textit{cyclic permutations of } (1,2,3). \tag{14}$$

This fully symmetric trilinear product arises because we have chosen the parafermionic variables a and $a^\dagger$ of second order. (A p-linear symmetric product would need to be used if the parafermionic variables were taken to be of order p.) It is straightforward to determine the various products of the basis elements (in our realization), they are given below.

The subspace $V^{(B)}$ is itself a Lie algebra. It is identified as $Sp(n) \supset\!\!\!\!+ h(n) \oplus SU(2) \oplus U(1)$, i.e. the semi-direct sum of the rank n symplectic algebra $Sp(n)$ (generated by $H_{ij}^{(0)}$, A_{ij} and $A_{ij}^\dagger$) and of the Heisenberg algebra, $h(n)$, in n dimensions [7] (generated by P_i, $P_i^\dagger$ and I) to which a $SU(2)$ (generated by L_a) and a $U(1)$ (generated by Y) are added as direct summands. In our basis, the non-vanishing commutators are:

$$\begin{aligned} &[H_{ij}^{(0)}, A_{kl}] = \omega\delta_{ik}A_{jl} + \omega\delta_{il}A_{jk}, \\ &[H_{ij}^{(0)}, A_{kl}^\dagger] = -\omega\delta_{jk}A_{il}^\dagger - \omega\delta_{jl}A_{ik}^\dagger, \\ &[A_{ij}^\dagger, A_{kl}] = \omega\delta_{ik}H_{jl}^{(0)} + \omega\delta_{il}H_{jk}^{(0)} + \omega\delta_{jk}H_{il}^{(0)} + \omega\delta_{jl}H_{ik}^{(0)}, \\ &[H_{ij}^{(0)}, H_{kl}^{(0)}] = 2\omega\delta_{il}H_{kj}^{(0)} - 2\omega\delta_{jk}H_{il}^{(0)}, \end{aligned} \tag{15a}$$

$$[P_i^\dagger, P_j] = \omega\delta_{ij}I, \tag{15b}$$

$$\begin{aligned} &[H_{ij}^{(0)}, P_k] = \omega\delta_{ik}P_j, \\ &[A_{ij}^\dagger, P_k] = \omega\delta_{ik}P_j^\dagger + \omega\delta_{jk}P_i^\dagger, \\ &[H_{ij}^{(0)}, P_k^\dagger] = -\omega\delta_{jk}P_i^\dagger, \\ &[A_{ij}, P_k^\dagger] = -\omega\delta_{ik}P_j - \omega\delta_{jk}P_i, \end{aligned} \tag{15c}$$

$$[L_a, L_b] = i\epsilon_{abc} L_c. \tag{15d}$$

The transformation properties of the parafermionic charges under the bosonic symmetries are given by:

$$\begin{aligned}
&[H^{(0)}_{ij}, Q_{k\mu}] = \omega\delta_{ik} Q_{j\mu}, && [H^{(0)}_{ij}, S_{k\mu}] = -\omega\delta_{jk} S_{i\mu},\\
&[A_{ij}, Q_{k\mu}] = 0, && [A_{ij}, S_{k\mu}] = -\omega\delta_{ik} Q_{j\mu} - \omega\delta_{jk} Q_{i\mu},\\
&[A^\dagger_{ij}, Q_{k\mu}] = \omega\delta_{ik} S_{j\mu} + \omega\delta_{jk} S_{i\mu}, && [A^\dagger_{ij}, S_{k\mu}] = 0,\\
&[P_i, Q_{j\mu}] = 0, && [P_i, S_{j\mu}] = -\omega\delta_{ij} T_\mu,\\
&[P^\dagger_i, Q_{j\mu}] = \omega\delta_{ij} T_\mu, && [P^\dagger_i, S_{j\mu}] = 0,\\
&[I, Q_{i\mu}] = 0, && [I, S_{i\mu}] = 0,\\
&[Y, Q_{i\mu}] = \bar{Q}_{i\mu}, && [Y, S_{i\mu}] = \bar{S}_{i\mu},\\
&[L_a, Q_{i\mu}] = -\tfrac{1}{2}(\sigma_a)_{\mu\nu} Q_{i\nu}, && [L_a, S_{i\mu}] = -\tfrac{1}{2}(\sigma_a)_{\mu\nu} S_{i\nu},
\end{aligned} \tag{16a}$$

$$\begin{gathered}
[H^{(0)}_{ij}, T_\mu] = [A_{ij}, T_\mu] = [A^\dagger_{ij}, T_\mu] = [P_i, T_\mu] = [P^\dagger_i, T_\mu] = [I, T_\mu] = 0,\\
[Y, T_\mu] = \bar{T}_\mu, \qquad [L_a, T_\mu] = -\tfrac{1}{2}(\sigma_a)_{\mu\nu} T_\nu.
\end{gathered} \tag{16b}$$

The commutation relations involving $\bar{Q}$, $\bar{S}$ and $\bar{T}$ are obtained from the above formulas by effecting the substitutions $Q \leftrightarrow \bar{Q}$, $S \leftrightarrow \bar{S}$, $T \leftrightarrow \bar{T}$. Finally, the trilinear products of the parafermionic basis elements read as follows with $\epsilon_{12} = -\epsilon_{21} = 1$ and $\mu, \nu, \rho = 1, 2$:

$$\begin{aligned}
&\{Q_{i\mu}, Q_{j\nu}, Q_{k\rho}\} = 0,\\
&\{S_{i\mu}, S_{j\nu}, S_{k\rho}\} = 0,\\
&\{T_\mu, T_\nu, T_\rho\} = 0,
\end{aligned} \tag{17a}$$

$$\begin{aligned}
&\{Q_{i\mu}, Q_{j\nu}, S_{k\rho}\} = 2\omega\epsilon_{\mu\rho}\delta_{ik} Q_{j\nu} + 2\omega\epsilon_{\nu\rho}\delta_{jk} Q_{i\mu},\\
&\{S_{i\mu}, S_{j\nu}, Q_{k\rho}\} = -2\omega\epsilon_{\mu\rho}\delta_{ik} S_{j\nu} - 2\omega\epsilon_{\nu\rho}\delta_{jk} S_{i\mu},\\
&\{Q_{i\mu}, Q_{j\nu}, T_\rho\} = 0,\\
&\{T_\mu, T_\nu, Q_{i\rho}\} = 0,\\
&\{S_{i\mu}, S_{j\nu}, T_\rho\} = 0,\\
&\{T_\mu, T_\nu, S_{i\rho}\} = 0,
\end{aligned} \tag{17b}$$

$$\{Q_{i\mu}, S_{j\nu}, T_\rho\} = 2\omega\epsilon_{\mu\nu}\delta_{ij} T_\rho, \tag{17c}$$

$$
\begin{aligned}
&\{Q_{i\mu}, Q_{j\nu}, \bar{Q}_{k\rho}\} = 4\epsilon_{\mu\rho}[A_{ij}Q_k]_\nu + (\mu \leftrightarrow \nu),\\
&\{S_{i\mu}, S_{j\nu}, \bar{S}_{k\rho}\} = 4\epsilon_{\mu\rho}[A^\dagger_{ij}S_k]_\nu + (\mu \leftrightarrow \nu),\\
&\{T_\mu, T_\nu, \bar{T}_\rho\} = 4\epsilon_{\mu\rho}[I\,T]_\nu + (\mu \leftrightarrow \nu),
\end{aligned}
\tag{17d}
$$

$$
\begin{aligned}
\{Q_{i\mu}, Q_{j\nu}, \bar{S}_{k\rho}\} &= 2\epsilon_{\mu\rho}[(H^{(0)}_{ki} + \tfrac{2}{3}\omega\delta_{ki}L_a\sigma_a)Q_j + (i \leftrightarrow j)]_\nu + (\mu \leftrightarrow \nu)\\
&\quad - \omega\epsilon_{\mu\nu}[\delta_{ki}\bar{Q}_{j\rho} - (i \leftrightarrow j)],\\
\{S_{i\mu}, S_{j\nu}, \bar{Q}_{k\rho}\} &= 2\epsilon_{\mu\rho}[(H^{(0)}_{jk} - \tfrac{2}{3}\omega\delta_{jk}L_a\sigma_a)S_i + (i \leftrightarrow j)]_\nu + (\mu \leftrightarrow \nu)\\
&\quad - \omega\epsilon_{\mu\nu}[\delta_{jk}\bar{S}_{i\rho} - (i \leftrightarrow j)],\\
\{Q_{i\mu}, Q_{j\nu}, \bar{T}_\rho\} &= 4\epsilon_{\mu\rho}[P_iQ_j]_\nu + (\mu \leftrightarrow \nu),\\
\{T_\mu, T_\nu, \bar{Q}_{i\rho}\} &= 4\epsilon_{\mu\rho}[P_iT]_\nu + (\mu \leftrightarrow \nu),\\
\{S_{i\mu}, S_{j\nu}, \bar{T}_\rho\} &= 4\epsilon_{\mu\rho}[P^\dagger_iS_j]_\nu + (\mu \leftrightarrow \nu),\\
\{T_\mu, T_\nu, \bar{S}_{i\rho}\} &= 4\epsilon_{\mu\rho}[P^\dagger_iT]_\nu + (\mu \leftrightarrow \nu),
\end{aligned}
\tag{17e}
$$

$$
\begin{aligned}
&\{Q_{i\mu}, S_{j\nu}, \bar{Q}_{k\rho}\} = 2\epsilon_{\mu\rho}[A_{ki}S_j + (H^{(0)}_{jk} - \tfrac{2}{3}\omega\delta_{jk}L_a\sigma_a)Q_i]_\nu + (\mu \leftrightarrow \nu) - \omega\epsilon_{\mu\nu}\delta_{jk}\bar{Q}_{i\rho},\\
&\{Q_{i\mu}, S_{j\nu}, \bar{S}_{k\rho}\} = 2\epsilon_{\mu\rho}[A^\dagger_{jk}Q_i + (H^{(0)}_{ki} + \tfrac{2}{3}\omega\delta_{ki}L_a\sigma_a)S_j]_\nu + (\mu \leftrightarrow \nu) - \omega\epsilon_{\mu\nu}\delta_{ki}\bar{S}_{j\rho},\\
&\{Q_{i\mu}, T_\nu, \bar{Q}_{j\rho}\} = 2\epsilon_{\mu\rho}[A_{ij}T + P_iQ_j]_\nu + (\mu \leftrightarrow \nu),\\
&\{Q_{i\mu}, T_\nu, \bar{T}_\rho\} = 2\epsilon_{\mu\rho}[P_iT + IQ_i]_\nu + (\mu \leftrightarrow \nu),\\
&\{S_{i\mu}, T_\nu, \bar{S}_{j\rho}\} = 2\epsilon_{\mu\rho}[A^\dagger_{ij}T + P^\dagger_iS_j]_\nu + (\mu \leftrightarrow \nu),\\
&\{S_{i\mu}, T_\nu, \bar{T}_\rho\} = 2\epsilon_{\mu\rho}[P^\dagger_iT + IS_i]_\nu + (\mu \leftrightarrow \nu),
\end{aligned}
\tag{17f}
$$

$$
\begin{aligned}
&\{Q_{i\mu}, S_{j\nu}, \bar{T}_\rho\} = 2\epsilon_{\mu\rho}[P^\dagger_jQ_i + P_iS_j]_\nu + (\mu \leftrightarrow \nu),\\
&\{S_{i\mu}, T_\nu, \bar{Q}_{j\rho}\} = 2\epsilon_{\mu\rho}[P_jS_i + (H^{(0)}_{ij} - \tfrac{2}{3}\omega\delta_{ij}L_a\sigma_a)T]_\nu + (\mu \leftrightarrow \nu) + \omega\epsilon_{\mu\nu}\delta_{ij}\bar{T}_\rho,\\
&\{T_\mu, Q_{i\nu}, \bar{S}_{j\rho}\} = 2\epsilon_{\mu\rho}[P^\dagger_jQ_i + (H^{(0)}_{ji} + \tfrac{2}{3}\omega\delta_{ji}L_a\sigma_a)T]_\nu + (\mu \leftrightarrow \nu) + \omega\epsilon_{\mu\nu}\delta_{ji}\bar{T}_\rho.
\end{aligned}
\tag{17g}
$$

The relations of the form $\{A, \bar{B}, \bar{C}\}$ or $\{\bar{A}, \bar{B}, \bar{C}\}$ can be obtained from the above using

$$
\{\bar{A}, \bar{B}, \bar{C}\} = -\overline{\{A, B, C\}} \tag{18a}
$$

and

$$
\{\bar{A}, \bar{B}, C\} = -\overline{\{A, B, \bar{C}\}}. \tag{18b}
$$

Investigating the symmetries of oscillators with a single parafermionic degree of freedom has thus revealed a remarkable algebraic structure. Two obvious generalizations would lead to even richer parasuperalgebras. First, one could consider oscillators with more than one parafermionic degree of freedom; second, one could allow for parafermionic variables of degree greater than two (see refs. [5] and [6]). These situations remain to be analysed.

We shall now close this paper by discussing some aspects of the one-dimensional case and by explaining how the invariance parasuperalgebra of the two-dimensional oscillator provides a spectrum generating albebra for the parasupersymmetric Morse Hamiltonian [8].

In one dimension the Hamiltonian (6) reduces to

$$\begin{aligned} H &= H_{11}^{(0)} + 2\omega L_3 \\ &= \frac{\omega}{2}\left(\{\alpha^\dagger, \alpha\} + [a^\dagger, a]\right). \end{aligned} \tag{19}$$

Let us single out the following two odd elements from its dynamical parasuperalgebra basis:

$$Q \equiv Q_{11} = \sqrt{\omega}\alpha^\dagger a, \qquad Q^\dagger = \bar{S}_{12} = \sqrt{\omega}\alpha a^\dagger. \tag{20}$$

These constants commute with the Hamiltonian in accordance with eqs. (16):

$$[H, Q] = [H, Q^\dagger] = 0. \tag{21}$$

Their trilinear products are obtained by specializing formulas (17) and using the fact that

$$(L_a \sigma_a T)_\mu = 3(L_3 \sigma_3 T)_\mu. \tag{22}$$

One finds:

$$Q^3 = 0, \qquad {Q^\dagger}^3 = 0, \tag{23a}$$

$$\{Q, Q, Q^\dagger\} = 8HQ, \tag{23b}$$

$$\{Q, Q^\dagger, Q^\dagger\} = 8HQ^\dagger. \tag{23c}$$

This parasuperalgebra is the second-order generalization of the superalgebra

$$\begin{aligned} Q^2 = {Q^\dagger}^2 = 0, \qquad \{Q, Q^\dagger\} = 2H, \\ [H, Q] = [H, Q^\dagger] = 0. \end{aligned} \tag{24}$$

It was first introduced by Rubakov and Spiridonov [6].

A Hamiltonian H is said to be parasupersymmetric (of order 2) if there exists two parasupercharges Q and $Q^\dagger$ such that the parasuperalgebra specified by the relations (21) and (23) is realized. The one-dimensional harmonic oscillator Hamiltonian (19) is therefore parasupersymmetric.

As shown in reference [6], one-dimensional parasupersymmetric Hamiltonian can be constructed quite generally in terms of two functions $W_1(x)$ and $W_2(x)$, that satisfy

$$W_2^2(x) - W_1^2(x) + W_2'(x) + W_1'(x) = k, \tag{25}$$

with k a real constant and the prime standing for differentiation. (The "potentials" W_1 and W_2 are otherwise arbitrary.) Given two such functions, one can check that the relations (21) and (23) are realized for

$$H = \frac{1}{2}\left(p^2 + \frac{1}{2}(W_1^2 + W_2^2) + \frac{3}{2}(W_1' - W_2') + W_2' a^\dagger a - W_1' a a^\dagger\right) \tag{26a}$$

and

$$Q = \frac{1}{2\sqrt{2}}\left[(p + iW_1)a^\dagger a^2 + (p + iW_2)a^2 a^\dagger\right]. \tag{26b}$$

In our oscillator special case $W_1 = W_2 = \omega x$ and $k = 2\omega$.

We would here like to point out that the above Hamiltonians always possess two further bosonic constants, Σ_3 and Y, and two additional conserved parasupercharges, $\bar{Q}$ and $\bar{Q}^\dagger$ [9]. Indeed, let $\bar{Q}$ be the parasupercharge which is is obtained by commmuting Q with $Y = \frac{1}{2}(aa^\dagger + a^\dagger a)$, i.e. let

$$\begin{aligned} \bar{Q} &= [Y, Q] \\ &= \frac{1}{2\sqrt{2}}\left[(p + iW_1)b^\dagger b^2 + (p + iW_2)b^2 b^\dagger\right] \\ &= \frac{1}{2\sqrt{2}}\left[(p + iW_1)a^\dagger a^2 - (p + iW_2)a^2 a^\dagger\right]. \end{aligned} \tag{27}$$

One finds that

$$\begin{gathered} [H, \bar{Q}] = [H, \bar{Q}^\dagger] = [H, \Sigma_3] = [H, Y] = 0, \\ [\Sigma_3, Q] = -Q, \quad [\Sigma_3, \bar{Q}] = -\bar{Q}, \quad [\Sigma_3, Q^\dagger] = Q^\dagger, \quad [\Sigma_3, \bar{Q}^\dagger] = \bar{Q}^\dagger, \\ [Y, \bar{Q}] = Q, \quad [Y, Q^\dagger] = -\bar{Q}^\dagger, \quad [Y, \bar{Q}^\dagger] = -Q^\dagger. \end{gathered} \tag{28}$$

The trilinear products involving Q, $Q^\dagger$, $\bar{Q}$ and $\bar{Q}^\dagger$ are straightforwardly evaluated and in addition to those given in (23) one has:

$$\begin{aligned}
&\{\bar{Q},\bar{Q},\bar{Q}^\dagger\} = 8H\bar{Q},\\
&\{Q,Q,\bar{Q}^\dagger\} = -2kQ,\\
&\{\bar{Q},\bar{Q},Q^\dagger\} = -2k\bar{Q},\\
&\{Q,\bar{Q},\bar{Q}^\dagger\} = 4HQ - k\bar{Q},\\
&\{\bar{Q},Q,Q^\dagger\} = 4H\bar{Q} - kQ,\\
&\{Q,Q,\bar{Q}\} = \{Q,\bar{Q},\bar{Q}\} = \{\bar{Q},\bar{Q},\bar{Q}\} = 0,\\
&(\textit{plus hermitian conjugated relations}).
\end{aligned} \tag{29}$$

In our oscillator example,

$$\bar{Q} = \bar{Q}_{11}, \qquad \bar{Q}^\dagger = -S_{12}. \tag{30}$$

One may verify that the trilinear relations (17) among Q_{11}, $\bar{S}_{12}$, $\bar{Q}_{11}$ and S_{12} correspondingly reduce to those given in (29) with $k = 2\omega$.

The last point that we would like to make is to indicate the connection between the parasupersymmetric one-dimensional Morse Hamiltonian and the two-dimensional oscillator, hence showing that the dynamical parasuperalgebra of the latter system is also a spectrum-generating algebra for the former [8].

The parasupersymmetric Morse Hamiltonian is obtained by substituting

$$W_1 = -\frac{h}{2}e^{-x} + \frac{1}{2}(h-1) \quad \textit{and} \quad W_2 = W_1 + 1 \tag{31}$$

in Eq. (26a). One finds

$$2H_M = -\frac{d^2}{dx^2} + \frac{h^2}{4}(e^{-x}-1)^2 + he^{-x}\Sigma_3 + \frac{1}{4}. \tag{32}$$

The eigenstates of the Hamiltonian

$$H = H_{11}^{(0)} + H_{22}^{(0)} + \Sigma_3 \tag{33}$$

can be obtained by constructing the unitary representations of its dynamical parasuperalgebra; more precisely, by determining basis for the corresponding representation space. (We have set $\omega = 1$.) A natural set of quantum numbers is provided

by the following eigenvalue equations:

$$(H_{11}^{(0)} + H_{22}^{(0)})|h^{(0)}, \delta, m_2\rangle = h^{(0)}|h^{(0)}, \delta, m_2\rangle, \qquad h^{(0)} = 1, 2..., \tag{34a}$$

$$\Sigma_3|h^{(0)}, \delta, m_2\rangle = \delta|h^{(0)}, \delta, m_2\rangle, \qquad \delta = 1, 0, -1, \tag{34b}$$

$$M_2|h^{(0)}, \delta, m_2\rangle = m_2|h^{(0)}, \delta, m_2\rangle, \quad -\tfrac{1}{2}(h^{(0)} - 1) \leq m_2 \leq \tfrac{1}{2}(h^{(0)} - 1), \tag{34c}$$

with

$$M_2 = \frac{i}{2}(\alpha_2^\dagger \alpha_1 - \alpha_1^\dagger \alpha_2). \tag{35}$$

The range of values for m_2 is determined by observing that $M_1 = \frac{1}{2}(\alpha_1^\dagger \alpha_2 + \alpha_2^\dagger \alpha_1)$, M_2 and $M_3 = \frac{1}{2}(\alpha_1^\dagger \alpha_1 - \alpha_2^\dagger \alpha_2)$ form an $SU(2)$ algebra with $M^2 = M_1^2 + M_2^2 + M_3^2 = \frac{1}{4}[(H_{11}^{(0)} + H_{22}^{(0)})^2 - 1]$. Now let

$$\tilde{H} = H_{11}^{(0)} + H_{22}^{(0)} + 2\Sigma_3. \tag{36}$$

Instead of $h^{(0)}$, we can equivalently use as quantum number the eigenvalue $h = h^{(0)} + 2\delta$ of $\tilde{H}$, identify the basis states as $|h, \delta, m_2\rangle$ and replace (34a) by

$$\tilde{H}\,|h, \delta, m_2\rangle = h\,|h, \delta, m_2\rangle, \qquad h = -1, 0, 1, \ldots. \tag{37}$$

In terms of the polar coordinates

$$x_1 = r cos\phi, \qquad x_2 = r sin\phi, \tag{38}$$

$M_2 = -\frac{i}{2}\partial/\partial\phi$. Equation (34$c$) is then immediately integrated and one gets

$$\langle r, \phi|h, \delta, m_2\rangle = e^{2im_2\phi}\Psi_{h,\delta,m_2}(r). \tag{39}$$

Separating the variables in

$$\langle r, \phi|\tilde{H}|h, \delta, m_2\rangle = h\langle r, \phi|h, \delta, m_2\rangle, \tag{40}$$

one finds

$$\left[\frac{1}{2}\left(-\frac{1}{r}\frac{d}{dr}r\frac{d}{dr} + \frac{4m_2^2}{r^2} + r^2\right) + 2\delta\right]\Psi_{h,\delta,m_2}(r) = h\Psi_{h,\delta,m_2}(r). \tag{41}$$

The solutions to this equation can be obtained either directly or by applying ladder operators on the ground state wave functions. They are given by

$$\Psi_{h,\delta,m_2}(r) = C_\delta r^{2m_2} e^{-r/2} F(m_2 - \tfrac{1}{2}(h - 2\delta - 1), 2m_2 + 1; r^2), \tag{42}$$

with C_δ normalization constants and F a confluent hypergeometric function.

The eigenfunctions of the Morse Hamiltonian can now be obtained from (42) by a mere change of variable. Indeed, setting $r^2 = he^{-x}$ in (41) and multiplying this equation on both sides by r^2 yields

$$H_M \Psi_{h,\delta,m_2}(x) = \left(-\frac{m_2^2}{2} + \frac{1}{8}(h^2+1)\right) \Psi_{h,\delta,m_2}(x), \tag{43}$$

with H_M as given in (32).

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