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False Vacuum Decay: A Supersymmetric Quantum Mechanical Approach

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ABSTRACT

The decay of false vacua is formulated using supersymmetric quantum mechanics. The rate of decay depends critically on the energy splitting between the two lowest energy levels, which is found via a systematic, rapidly converging perturbation expansion. Perturbative calculations to any order can be easily carried out using logarithmic perturbation theory. Our approach yields substantially better results than alternative widely used semiclassical analyses.

I. INTRODUCTION

In a double-well potential, a classical particle can remain indefinitely in the neighborhood of a higher local minimum that is separated by a barrier from a lower minimum, provided the kinetic energy of the particle is less than the height of the barrier. However, a quantum mechanical particle with its wavefunction initially localized in the higher minimum has a finite probability of being found in the lower minimum due to tunneling through the potential barrier. This is known as the decay of the false vacuum or the metastability of a local minimum. It is a very important process in many branches of physics and chemistry, and has been widely studied[1-12]. Langer[6] constructed one of the first theories of decay of metastable states of a quantum mechanical system, which was extended by Voloshin, Kobzarev and Okun[7] to quantum field theory. It was corrected by Coleman[8], Callan and Coleman[9], Stone[10] and Frampton[11]. A recent preprint by Boyanovsky, Willey and Holman[12] has further improved the earlier work. References [8], [9], provide a detailed semiclassical treatment of quantum metastability.

In this paper, we study the metastability of a one dimensional system employing the techniques of supersymmetric quantum mechanics(SUSY-QM). In particular, we analyze the energy difference between the two lowest states, which is intimately connected with the slippage of the wavefunction from the higher potential well to the neighboring lower well.

In Section II, we briefly review the formalism of supersymmetric quantum mechanics; for a more detailed description the reader is referred to Ref.[13]. In Section III, we discuss the relevance of the energy splitting to the problem of the decay of the false vacuum. One way to determine this splitting of the energy levels is to solve the Schrödinger differential equation numerically. However, that becomes increasingly difficult and less reliable as the barrier

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gets higher and the splitting gets smaller. The SUSY-QM based approach described in this paper gives an extremely reliable and accurate determination of the energy splitting. In fact, the reliability increases for smaller splitting. We compare our SUSY-QM results with numerically computed values, and with those coming from a recent semi-classical analysis[12].

II. SUPERSYMMETRIC QUANTUM MECHANICS

For a quantum mechanical problem with a potential $V_-(x)$, supersymmetry allows one to construct a partner potential $V_+(x)$ whose energy eigenvalues E_n^+ are in one to one correspondence with the excited states of $V_-(x)$; i.e. $E_{n-1}^+ = E_n^-$, where E_n^- are eigenvalues of $V_-(x)$ and n is a positive integer (i.e. $n \in Z_+$). In the arena of SUSY-QM, one often describes $V_-(x)$ in terms of its ground state wavefunction. Hence, let us assume that the wavefunction $\psi_0^{(-)}$ of $V_-(x)$ is known, and the corresponding ground state energy E_0^- has been adjusted to be zero. The relevant Schrödinger equation is given by

$$H_-\psi_0(x) \equiv \left(-\frac{d^2}{dx^2} + V_-(x)\right)\psi_0 = 0\,, \tag{1}$$

and we are using units with $\hbar = 2m = 1$. The above Hamiltonian can also be written in terms of the ground state wavefunction $\psi_0(x)$ as

$$H_- = \left(-\frac{d^2}{dx^2} + \frac{\psi_0''}{\psi_0}\right), \tag{2}$$

where ψ_0'' represents the second spatial derivative of the wavefunction. We now define two operators

$$A = \left(\frac{d}{dx} - \frac{\psi_0'}{\psi_0}\right)\,, \quad A^+ = \left(-\frac{d}{dx} - \frac{\psi_0'}{\psi_0}\right)\,. \tag{3}$$

In terms of A and A^+, the Hamiltonian H_- is simply given by A^+A. However, one can define another operator $H_+ \equiv AA^+ = -\frac{d^2}{dx^2} + V_+(x)$, where

$$V_+(x) = V_-(x) - 2\frac{d}{dx}\left(\frac{\psi_0'}{\psi_0}\right) = -V_-(x) + 2\left(\frac{\psi_0'}{\psi_0}\right)^2.$$

By construction, H_+ is a Hermitian and positive semi-definite operator. The potentials $V_-(x)$ and $V_+(x)$ are known as supersymmetric partners. We shall show shortly that they have the same eigenvalues except for the ground state energy E_0^-. The superpotential $W(x)$ is related to $\psi_0^{(-)}$ by $W(x) = -\frac{\psi_0'}{\psi_0}$ or equivalently $\psi_0(x) = exp(-\int^x W(x)\,dx)$. Operators A and A^+ can now be written as

$$A = \left(\frac{d}{dx} + W(x)\right), \quad A^+ = \left(-\frac{d}{dx} + W(x)\right); \tag{4}$$

and the potentials $V_-(x)$ and $V_+(x)$ are given by

$$V_\pm = W^2(x) \pm W'(x); \quad W'(x) = \frac{dW}{dx}. \tag{5}$$

The commutator $[A, A^+]$ is equal to $2W'(x)$.

Now we shall explicitly show the correspondence between E_n^- and E_n^+. Let us denote the eigenfunctions of $H_\pm$ that correspond to eigenvalues $E_n^\pm$, by $\psi_n^{(\pm)}$. One discovers that for $n \neq 0$;

$$H_+\left(A\psi_n^{(-)}\right) = AA^+\left(A\psi_n^{(-)}\right) = A\left(A^+A\psi_n^{(-)}\right) = AH_-\left(\psi_n^{(-)}\right) = E_n^-\left(A\psi_n^{(-)}\right)\,; \tag{6}$$

and thus for positive integral values of n, $A\psi_n^{(-)}$ is an eigenfunction of H_+, and we shall call it the supersymmetric partner state of $\psi_n^{(-)}$. Since the ground state of $V_-(x)$ does not have a SUSY partner ($A\psi_0^{(-)} = 0$); one finds $E_{n-1}^+ = E_n^-$, where $n \in Z_+$. Thus, if the eigenvalues and the eigenfunctions of H_- were known, one automatically learns the eigenvalues and the eigenfunctions of, what in general is, a completely different Hamiltonian H_+.

III. FALSE VACUUM DECAY

In a potential with a symmetric double well structure, eigenstates with energy substantially below the peak of the barrier, necessarily occur in pairs with small energy splitting. This splitting of energy levels gets smaller as the barrier increases. The same is also true for asymmetric double wells, provided the deviation from symmetry is small.

A wavefunction that is localized in the higher minimum (say, in the right well) of a double well, can be approximated by the linear combination $\frac{|\psi_1\rangle+|\psi_0\rangle}{\sqrt{2}}$, of the ground state and the first excited state eigenfunctions. Similarly a wavefunction localized in the other well is approximated by the orthogonal combination $\frac{|\psi_1\rangle-|\psi_0\rangle}{\sqrt{2}}$. For an initially localized particle, the probability for tunneling after a time τ is given by

$$P(\tau) = |\langle\frac{\psi_1 - \psi_0}{\sqrt{2}}|e^{-iH\tau}|\frac{\psi_1 + \psi_0}{\sqrt{2}}\rangle|^2 = sin^2[(\delta E)\tau]\,; \tag{7}$$

where δE represents the difference between the energy eigenvalues of the ground state and the first excited state. Thus, knowledge of δE is a crucial ingredient for determining the probability of tunneling. In the rest of this section, we will evaluate this energy splitting for an asymmetric well using SUSY-QM and compare with results obtained from the WKB formalism of Boyanovsky, Willey, and Holman[12] and with the numerical values resulting from solving the Schrödinger equation directly. We will extend the formalism developed by Keung, Kovacs and Sukhatme[14] for finding the energy difference for symmetric double wells.

In order to use the SUSY-QM method one needs to know the ground state wavefunction $\psi_0^{(-)}$. A specific example of an asymmetric double well which we will consider in detail corresponds to a ground state wave function which is the sum of two Gaussians centered at $\pm x_0$,

$$\psi_0^{(-)} = e^{-(x+x_0)^2} + e^{-a(x-x_0)^2}. \tag{8}$$

The positive parameter a is a measure of the asymmetry, and $a = 1$ corresponds to a symmetric wavefunction. The corresponding superpotential and potential are given by

$$W(x) \;=\; -\frac{\psi_0'^{(-)}(x)}{\psi_0^{(-)}(x)} \;=\; \left(\frac{-2a(x-x_0)e^{-a(x-x_0)^2} + 2(x+x_0)e^{-(x+x_0)^2}}{e^{-(x+x_0)^2} + e^{-a(x-x_0)^2}}\right)$$

and

$$V_-(x) \;=\; \frac{\psi_0''^{(-)}(x)}{\psi_0^{(-)}(x)} \;=\; \left(\frac{[-2a + 4a^2(x-x_0)^2]e^{-a(x-x_0)^2} + [-2 + 4(x+x_0)^2]e^{-(x+x_0)^2}}{e^{-(x+x_0)^2} + e^{-a(x-x_0)^2}}\right)$$

respectively. In Fig. 1, we have plotted $V_-(x)$ for various values of the asymmetry parameter a. For $a = 1$, they reduce to the symmetric case treated in Ref.[14], i.e.

$$W(x) = 2[x - x_0\,\tanh(2xx_0)]$$

and

$$V_-(x) = 4[x - x_0\,\tanh(2xx_0)]^2 - 2[1 - 2x_0^2\,sech^2(2xx_0)].$$

Increasing x_0 increases the height of the barrier, and also increases the distance between valleys.

To find δE, since the ground state energy of $V_-(x)$ is zero, we have to determine the energy of the first excited state of $V_-(x)$, and that, thanks to supersymmetry, happens to be the same as the ground state energy of the potential $V_+(x)$. Frequently, it is easier to determine numerically the ground state energy of $V_+(x)$ than the first excited state of $V_-(x)$[15].

From the ground state wavefunction $\psi_0^{(-)}$ of H_-, we can generate a function $\frac{1}{\psi_0^{(-)}}$ that solves the Schrödinger equation for H_+ with an eigenvalue zero; i.e. $H_+\left(\frac{1}{\psi_0^{(-)}}\right) = 0$. However, $\frac{1}{\psi_0^{(-)}}$ is not a normalizable function since $\frac{1}{\psi_0^{(-)}} \to \infty$ as $|x| \to \infty$. But from this function, one can construct a normalizable function $\phi(x)$,

$$\phi(x) \;=\; \frac{\int_x^\infty \left(\psi_0^{(-)}(y)\right)^2 dy}{2I_+\psi_0^{(-)}(x)} \qquad for\;\; x > 0 \tag{9}$$

$$= \frac{\int_{-\infty}^{x} \left(\psi_0^{(-)}(y)\right)^2 dy}{2I_- \psi_0^{(-)}(x)} \qquad for \;\; x < 0,$$

with $I_\pm$ given by,

$$\begin{aligned} I_- &= \left[\int_{-\infty}^{0} \left(\psi_0^{(-)}(y)\right)^2 dy\right] \\ I_+ &= \left[\int_{0}^{\infty} \left(\psi_0^{(-)}(y)\right)^2 dy\right] \end{aligned} \tag{10}$$

respectively. The function $\phi(x)$ is well defined for all values of x. It is easy to see that $\phi(x)$ is continuous at $x = 0$ with a value $\phi(0) = \frac{1}{2\psi_0^{(-)}(0)}$. One can easily show $H_+\phi(x) = 0$ for $x \neq 0$. However, $\phi(x)$ is also not an eigenfunction of H_+, as its derivative has a discontinuity at the origin; the discontinuity given by

$$\phi'|_{+\epsilon} - \phi'|_{-\epsilon} = -\psi_0^{(-)}(0)\left[\frac{1}{I_+} + \frac{1}{I_-}\right]. \tag{11}$$

However, $\phi(x)$ can be viewed as the ground state wavefunction of the singular Hamiltonian H_0 given by

$$H_0 = H_+ - \left(\psi_0^{(-)}(0)\right)^2 \left[\frac{1}{I_+} + \frac{1}{I_-}\right] \delta(x),$$

which allows eigenfunctions with discontinuous derivative owing to the presence of the Dirac δ-function term. Equivalently, one can write

$$\begin{aligned} H_+ &= H_0 + \delta H \\ &= H_0 + \left(\psi_0^{(-)}(0)\right)^2 \left[\frac{1}{I_+} + \frac{1}{I_-}\right] \delta(x); \end{aligned} \tag{12}$$

and hence the ground state eigenvalue and the eigenfunction of H_+ can be determined perturbatively by considering $\left(\psi_0^{(-)}(0)\right)^2 \left[\frac{1}{I_+} + \frac{1}{I_-}\right] \delta(x)$ as a perturbation on the unperturbed Hamiltonian H_0. Here we should like to point out that $\psi_0^{(-)}(0)$ plays the role of a small expansion parameter of this problem. This is an especially good choice for a symmetric double well potential with a high barrier since the value of $\psi_0^{(-)}(0)$ is very small. For the asymmetric case,

there is nothing particularly special about the origin and the discontinuity of $\phi(x)$ described in the eq. (11) can be chosen at another point, say $x = \bar{x}$. If $\bar{x}$ is chosen at the peak of the barrier (where $\psi_0^{(-)}$ has a very small value), the perturbation expansion converges even more rapidly. All the equations in this paper have been written with the choice $\bar{x} = 0$, but a generalization to an arbitrary value of $\bar{x}$ is straightforward.

The first order correction to the energy is

$$\begin{aligned} E^{(1)} &= \frac{\int_{-\infty}^{\infty} \delta H \left(\phi(x)\right)^2 dx}{\int_{-\infty}^{\infty} \phi^2(x) dx} \\ &= \frac{\int_{-\infty}^{\infty} \left(\psi_0^{(-)}(0)\right)^2 \left[\frac{1}{I_+} + \frac{1}{I_-}\right] \delta(x) \left(\phi(x)\right)^2 dx}{\int_{-\infty}^{\infty} \phi^2(x) dx} \\ &= \frac{1}{4}\left[\frac{1}{I_+} + \frac{1}{I_-}\right] \frac{1}{\int_{-\infty}^{\infty} \phi^2(x) dx}. \end{aligned} \tag{13}$$

Higher order corrections can be computed using the familiar Rayleigh-Schrödinger perturbation expansion, but this involves summations over all intermediate unperturbed eigenstates, which in general are not known. A much simpler alternative approach is to use logarithmic perturbation theory[16], which only requires knowledge of the unperturbed ground state wavefunction. The second order correction to the energy is given by

$$E^{(2)} = -\left[\int_{-\infty}^{0} dx \left(\frac{E^{(1)} \int_{-\infty}^{x} \phi^2(y)\, dy}{\phi(x)}\right)^2 + \int_{0}^{\infty} dx \left(\frac{E^{(1)} \int_{x}^{\infty} \phi^2(y)\, dy}{\phi(x)}\right)^2\right]; \tag{14}$$

and the energy splitting δE, correct to second order, is given by $E^{(1)} + E^{(2)}$. Similarly, even higher order corrections can be readily computed.

As mentioned before, an alternative analytic approach for computing δE is the WKB method. Ref.[12] is one of the most recent references on this formalism. It gives the following expression for the energy difference δE:

$$\delta E = \frac{2}{T_1}\left[(\delta W)^2 + e^{-2W_2}\right]^{\frac{1}{2}}; \tag{15}$$

where

$$T_1 = \int_{x_1}^{x_2} dx\,[E - V_-(x)]^{-\frac{1}{2}},$$

$$W_2 = \int_{x_2}^{x_3} dx\,[V_-(x) - E]^{\frac{1}{2}},$$

and δW is equal to $W_3 - W_1$. W_1 and W_3 are themselves given by

$$W_1 = \int_{x_1}^{x_2} dx\,[E - V_-(x)]^{\frac{1}{2}}, \qquad (16)$$
$$W_3 = \int_{x_3}^{x_4} dx\,[E - V_-(x)]^{\frac{1}{2}}.$$

The limits of integrations x_1, x_2, x_3 and x_4 ($x_1 < x_2 < x_3 < x_4$) are the roots of the equation $[E - V_-(x)] = 0$. In general, four classical turning points are to be expected for double well potentials.

To check the performance of the SUSY-QM and WKB methods, we computed δE by solving Schrödinger equation numerically using fourth order Runge-Kutta method. This was done for the specific example of eq. (8). We call it the numerical result in Fig. 2. We also calculated δE using WKB method, and in Fig. 2, plotted all three values of δE (obtained from eq. (13), (15), and from the numerical solution), as functions of x_0. We find that, as x_0 increases the barrier gets higher, and the SUSY-QM generated result for δE approaches extremely close to the numerical answer. In fact for $x_0 > 1.5$ the SUSY-QM generated result agrees so well with the numerical result, that their graphs fuse with each other in Fig. 2. In Fig. 3a and 3b, we plot δE calculated by different methods against the asymmetry parameter a, keeping x_0 fixed at the values of 1.2, and 2.0 respectively. Agreement of SUSY-QM generated output with the numerical result is better in Fig. 3b than in Fig. 3a. This is to be expected as the perturbation parameter $\psi_0^{(-)}(0)$ has a smaller value in the first case. We find that the results based on the WKB method deviate much faster from the numerical solution as the asymmetry parameter is decreased ($a = 1$ is the symmetric case) than SUSY-QM generated

answer. Again we find that, for almost symmetric situations the SUSY-QM generated output agrees extremely well with the numerical result. We also note that, for very asymmetric cases ($a \approx .4$), SUSY-QM approach provides results that are again, in excellent agreement with the numerical solution. However, as we have stated before, the supersymmetry based approach is much easier than solving the Schrödinger equation with an extremely small eigenvalue lying very close to the ground state with vanishing energy.

IV. CONCLUSION

Decay of false vacua is a process of considerable interest in many research areas that deal with phase transitions due to quantum tunneling. It appears in cosmology[1], solid state physics[2], particle physics[3], chemistry[4], statistical mechanics[5], and in many other fields. Much of the analytical study of metastability has been anchored around the semiclassical WKB approximation.

Here we employ another very attractive idea that has drawn much attention in recent years – supersymmetric quantum mechanics. We demonstrate how SUSY-QM can be used to determine the probability of quantum tunneling, and the consequent decay of the false vacuum. This is a fully quantum mechanical analysis, and can be easily carried out to any desired level of accuracy using logarithmic perturbation theory. We find that this method works very well in giving the energy splitting δE, even for exceedingly asymmetric double wells. The perturbation series converges especially rapidly when the potential barrier is high.

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FIGURE CAPTIONS:

Fig. 1. This set of graphs depict the form of the potentials $V_-(x)$ for three different values of the asymmetry parameter a. These potentials correspond to the ground state wavefunctions given in eq. (8).

Fig. 2. A plot of the energy splitting δE vs. x_0.

Fig. 3a. A plot of δE vs. a.

Fig. 3b. A plot of δE vs. a. This graph clearly shows the excellent agreement between the results obtained from the SUSY-QM procedure (upto first order) and the "exact" (numerical) results. Results obtained from WKB method also have been plotted for comparision.

Figure 1

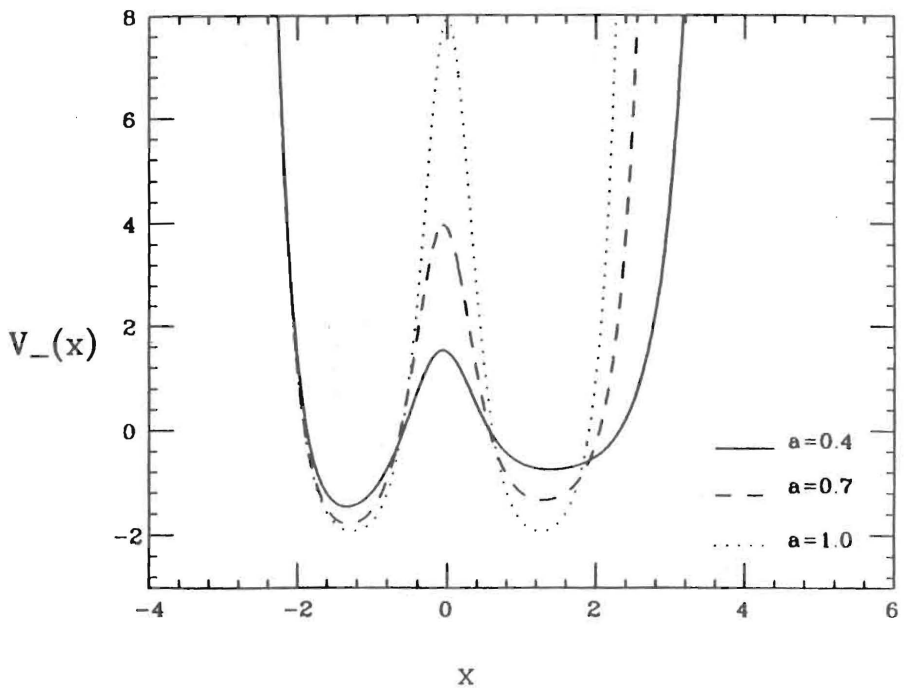

Figure 2

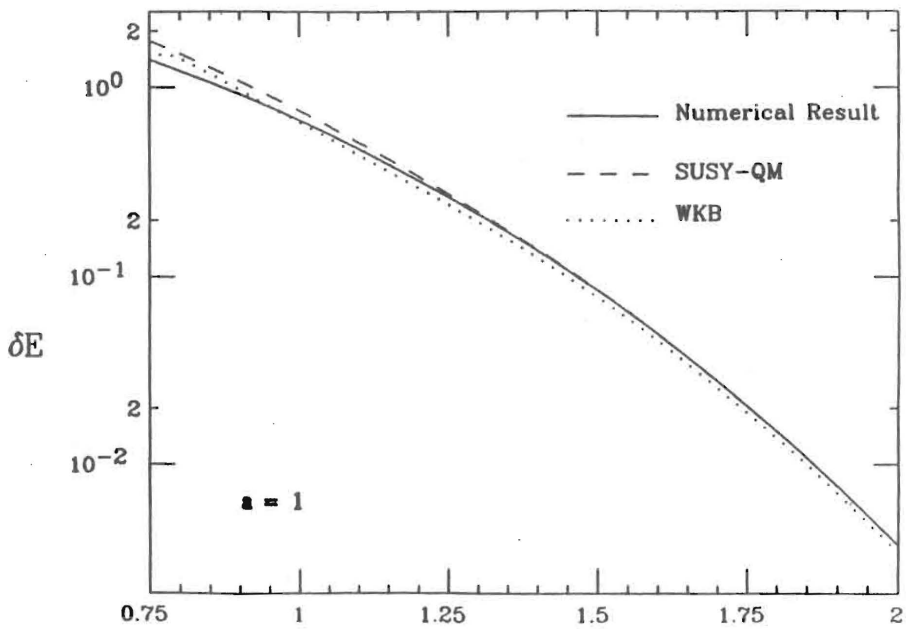

Figure 3a

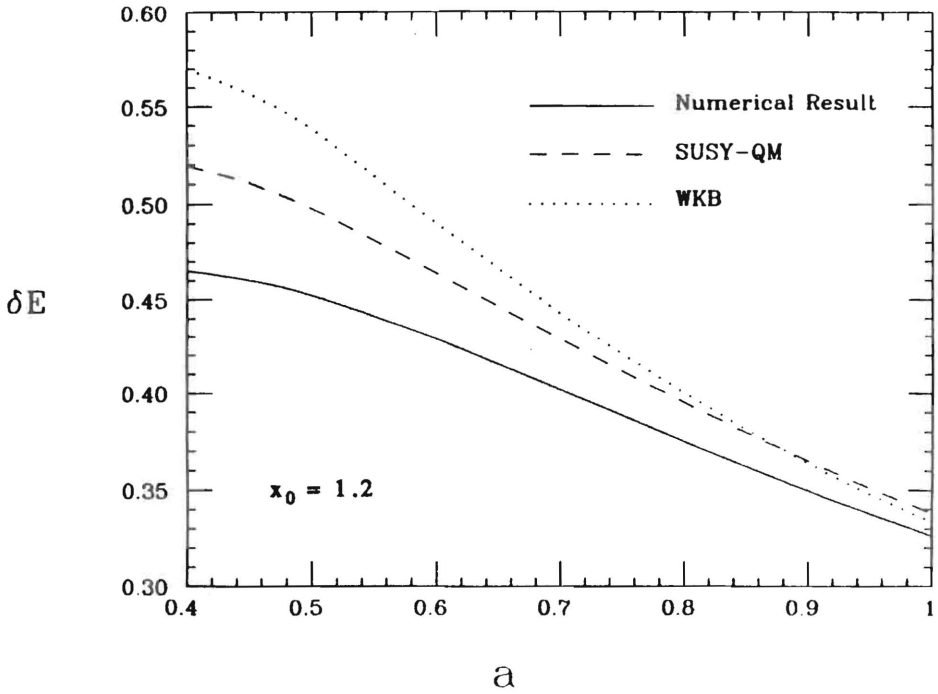
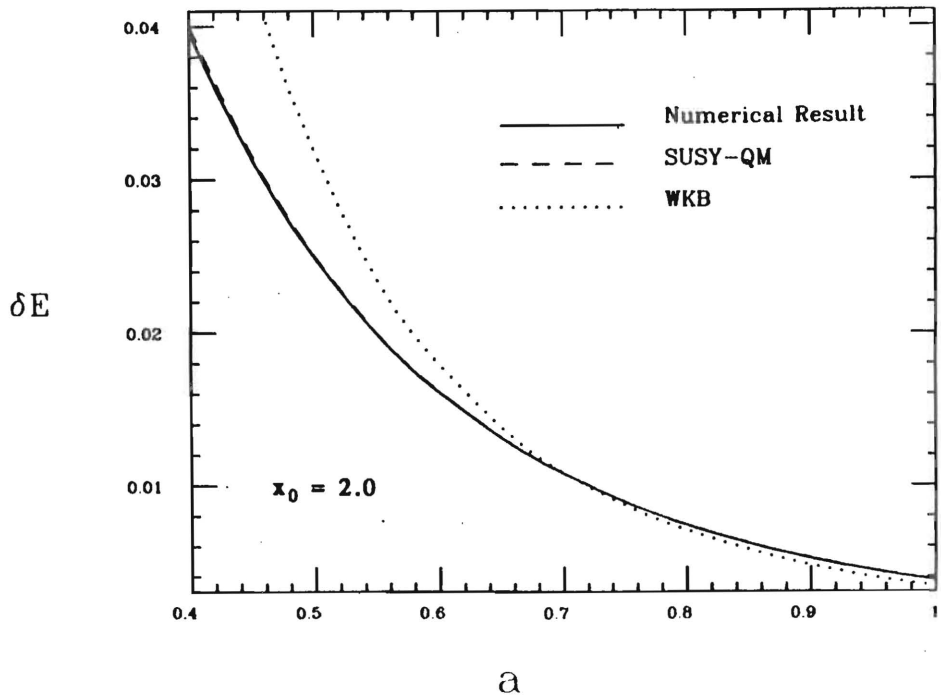
Figure 3b