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Solutions to the Bethe Ansatz Equations in the XXX-Heisenberg Model

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The XXX-Heisenberg model appears to be fundamental in giving rise to many of the recent interesting developments in modern quantum field theory and mathematical physics. Integrable models, conformal field theories and quantum groups are seen to arise from various limits, extensions and generalizations[1]. It is, in fact, the original model considered by Bethe[2], which gave rise to the celebrated "Bethe ansatz" solutions. Recently there appeared an article[3], studying the Bethe ansatz equations in the two particle sector of the spin one-half model. It was shown numerically that as the number of lattice sites, N, increases past 21, a new pair of real solutions appear. These do not fit into the conventional scheme of classification of the solutions to the Bethe ansatz equations causing some anxiety as to the verity of the completeness of the full set of $SU(2)$ extended Bethe ansatz states, which just by counting should be 2^N states. This however, is not a problem, since exactly two complex states disappear and form the real pair which appears, conserving the total number of states. We analyze the Bethe ansatz equations analytically and confirm the numerical result.

The spin $\frac{1}{2}$ XXX-Heisenberg model with N spins corresponds to the Hamiltonian

$$H = \frac{J}{4}\sum_{i=1}^{N}(\vec{\sigma}_i \cdot \vec{\sigma}_{i+1} - 1)$$

with $\vec{\sigma}_{N+1} = \vec{\sigma}_1$, and the $\vec{\sigma}_i$ are just the Pauli matrices for each i. The Bethe ansatz in the M particle sector corresponds to eigenfunctions of the form

$$|\Psi\rangle = \sum_{x_1<x_2<\cdots<x_M} \psi(x_1,\cdots,x_M)\sigma^-_{x_1}\sigma^-_{x_2}\sigma^-_{x_3}\cdots\sigma^-_{x_M}|0\rangle$$

with

$$\psi(x_1,\cdots,x_M) = \sum_{P\in S_M} e^{\{i\sum_{j=1}^{M} k_{P(j)}x_j + i\sum_{\substack{j<l\\ P(j)>P(l)}} \phi_{P(j),P(l)}\}},$$

$|0\rangle$ is the ferromagnetic state with all spins up, P is a permutation on M objects and $\phi_{i,j}$ satisfies

$$2\cot(\frac{\phi_{i,j}}{2}) = \cot(\frac{k_i}{2}) - \cot(\frac{k_j}{2}).$$

$k_1 \cdots k_M \in [0, 2\pi]$ are the spectral parameters. The wave-function is symmetric under interchange of any two coordinates x_j.

Imposing periodic boundary conditions

$$\psi(x_1, \cdots, x_{M-1}, N+1) = \psi(1, x_1, \cdots, x_{M-1})$$

implies the M coupled equations

$$e^{ik_j N} = \prod_{\substack{l=1 \\ l \neq j}}^{M} e^{i\phi_{j,l}} \qquad j = 1, 2, \cdots, M.$$

With $\Lambda_j = \cot(\frac{k_j}{2})$ we get the equivalent form for the equations

$$(e(\Lambda_j))^N = \prod_{\substack{l=1 \\ l \neq j}}^{M} e\left(\frac{\Lambda_j - \Lambda_i}{2}\right) \qquad j = 1, 2, \cdots, M,$$

with

$$e(\Lambda) = \frac{\Lambda + i}{\Lambda - i}.$$

The energy of the state is then given by

$$E(\Lambda_1, \cdots, \Lambda_M) = \sum_{j=1}^{M} \frac{-2J}{\Lambda_j^2 + 1}.$$

The usual hypothesis states that in the large N limit, for fixed M, any solution $\Lambda_1, \cdots, \Lambda_M$ consists of "strings" of the form

$$\Lambda_\alpha^{n,j} = \Lambda_\alpha^n + i(n + 1 - 2j) + o(e^{-\delta N}) \qquad j = 1, \cdots, n$$

where $n \geq 1$ gives the length of the string, α labels strings of a given length, j specifies the imaginary part of Λ and $\delta > 0$. With such a hypothesis, the Bethe ansatz equations only involve the real parts, Λ_α^n. The solutions are parametrized by (half-odd) integer numbers I_α^n for $N - M_n$ (even) odd, where M_n is the total number of string of length n. It is generally believed that there is a $1-1$ correspondence between solutions of the Bethe ansatz equations and sets of independent, non-repeating integers I_α^n, ($I_\alpha^n \neq I_\beta^n$ for $\alpha \neq \beta$, such that no two strings of the same length have the same integer)[4]. This is essential for all proofs of the completeness of the $SU(2)$ extended Bethe ansatz solutions for the XXX-Heisenberg model. The new real solutions shown below violate this correspondence, for $N > 21$ there exist fewer complex string solutions than predicted by counting integers.

There is presumably no problem with completeness, two complex solutions drop out to form a real pair.

The Bethe ansatz equations for $M = 2$ are

$$(e(\Lambda_1))^N = e(\frac{\Lambda_1 - \Lambda_2}{2})$$
$$(e(\Lambda_2))^N = e(\frac{\Lambda_2 - \Lambda_1}{2}).$$

Now

$$e(\frac{\Lambda_1 - \Lambda_2}{2}) = \frac{\frac{\Lambda_1 - \Lambda_2}{2} + i}{\frac{\Lambda_1 - \Lambda_2}{2} - i} = \frac{\Lambda_1 + i - (\Lambda_2 - i)}{\Lambda_1 - i - (\Lambda_2 + i)}.$$

Replacing for Λ_i with $\Lambda_i = \frac{i(e(\Lambda_i)+1)}{e(\Lambda_i)-1}$ gives

$$e(\frac{\Lambda_1 - \Lambda_2}{2}) = -\left(\frac{e(\Lambda_1)e(\Lambda_2) - 2e(\Lambda_1) + 1}{e(\Lambda_1)e(\Lambda_2) - 2e(\Lambda_2) + 1}\right).$$

We define the variables $X_i = e(\Lambda_i)$, then we get

$$X_1^N = -\left(\frac{X_1X_2 - 2X_1 + 1}{X_1X_2 - 2X_2 + 1}\right) \quad \text{and} \quad X_2^N = -\left(\frac{X_1X_2 - 2X_2 + 1}{X_1X_2 - 2X_1 + 1}\right).$$

Multiplying these equations together gives,

$$(X_1X_2)^N = 1, \quad \text{hence} \quad X_1X_2 = \omega$$

where ω is an Nth root of unity. Thus replacing $X_2 = \frac{\omega}{X_1}$ above, yields

$$X_1^N = -\left(\frac{\omega - 2X_1 + 1}{\omega - 2\frac{\omega}{X_1} + 1}\right) = -X_1\left(\frac{\omega - 2X_1 + 1}{\omega X_1 - 2\omega + X_1}\right).$$

Assuming $X_1 \neq \frac{2\omega}{\omega+1}$, or 0, we get

$$X_1^{(N-1)}((\omega+1)X_1 - 2\omega) + \omega - 2X_1 + 1 = 0$$

ie.

$$(\omega+1)X_1^N - 2\omega X_1^{(N-1)} - 2X_1 + (\omega+1) = 0.$$

We want the roots of this polynomial equation, for each choice of ω. We are actually interested in the real roots in terms of Λ_1 which are mapped to roots on the unit circle in terms of X_1. Furthermore

$$\left|\frac{2\omega}{(\omega+1)}\right|^2 = \frac{4}{(\omega+1)(\omega^*+1)} = \frac{4}{2 + 2\mathrm{Re}\omega} = \frac{2}{1 + \mathrm{Re}\omega} > 1$$

for $\omega \neq 1$. Thus the denominator that we multiplied by may only vanish for $\omega = 1$. In this case, $X_1 = 1 = X_2$ which gives no Bethe ansatz wave function since it vanishes due to the Pauli principle.

The equation is cast in a more symmetric form with the replacement $X_1 \to \sqrt{\omega}\tilde{X}_1$. We first consider the case N odd, where it is always possible to have $\sqrt{\omega}^N = -1$. Then we get

$$\begin{aligned}
0 &= (\omega+1)\omega^{\frac{N}{2}}\tilde{X}_1^N - 2\omega\omega^{\frac{N-1}{2}}\tilde{X}_1^{(N-1)} - \omega^{\frac{1}{2}}2\tilde{X}_1 + (\omega+1) \\
&= -(\omega+1)\tilde{X}_1^N + 2\sqrt{\omega}\tilde{X}_1^{(N-1)} - 2\sqrt{\omega}\tilde{X}_1 + (\omega+1) \\
&= -\sqrt{\omega}\left((\sqrt{\omega}+\frac{1}{\sqrt{\omega}})\tilde{X}_1^N - 2\tilde{X}_1^{N-1} + 2\tilde{X}_1 - (\sqrt{\omega}+\frac{1}{\sqrt{\omega}})\right)
\end{aligned}$$

which yields

$$\cos(\frac{\theta}{2})\tilde{X}_1^N - \tilde{X}_1^{(N-1)} + \tilde{X}_1 - \cos(\frac{\theta}{2}) = 0,$$

with the definition $\sqrt{\omega} = e^{i(\frac{\theta}{2})}$. Now we look for solutions on the unit circle, $\tilde{X}_1 = e^{i\alpha}$. The equation becomes

$$\cos(\frac{\theta}{2})e^{iN\alpha} - e^{i(N-1)\alpha} + e^{i\alpha} - \cos(\frac{\theta}{2}) = 0,$$

which simplifies magically to

$$\begin{aligned}
0 &= e^{\frac{iN\alpha}{2}}\cos(\frac{\theta}{2})(e^{\frac{iN\alpha}{2}} - e^{\frac{-iN\alpha}{2}}) - e^{i\alpha}e^{\frac{i(N-2)\alpha}{2}}(e^{\frac{i(N-2)\alpha}{2}} - e^{\frac{-i(N-2)\alpha}{2}}) \\
&= e^{\frac{iN\alpha}{2}}2i\left(\cos(\frac{\theta}{2})\sin(\frac{N\alpha}{2}) - \sin(\frac{(N-2)\alpha}{2})\right).
\end{aligned}$$

Thus

$$\cos(\frac{\theta}{2}) = \frac{\sin(\frac{(N-2)\alpha}{2})}{\sin(\frac{N\alpha}{2})}.$$

This equation is easily studied graphically. $\alpha \in [0, 2\pi]$. The right hand side has zeros at $\alpha = \frac{2k\pi}{N-2}$ for $k = 1, 2, 3, \cdots, N-3$ and poles at $\alpha = \frac{2k\pi}{N}$ for $k = 1, 2, 3, \cdots, N-1$. The value at $\alpha = 0$ is $\frac{N-2}{N}$ but at $\alpha = \pi$ it is -1. Graphically we get a root for each intersection of a horizontal line with $y = \cos(\frac{\theta}{2})$ with the curve $y = \frac{\sin(\frac{(N-2)\alpha}{2})}{\sin(\frac{N\alpha}{2})}$:

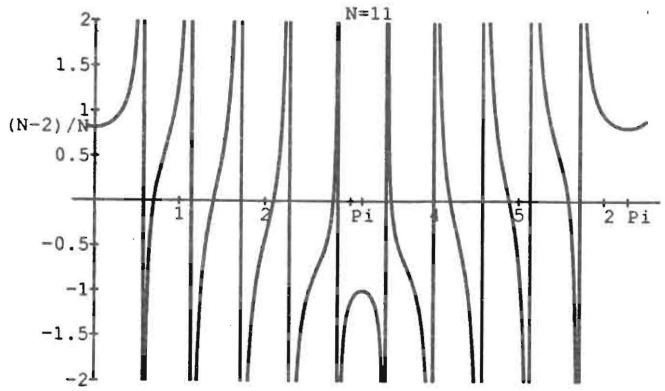

Figure 1: $\frac{\sin(\frac{(N-2)\alpha}{2})}{\sin(\frac{N\alpha}{2})}$, $N = 11$

Just counting the intersections it is evident that there is a root for each zero. There are always $N-3$ such roots. An additional pair of real roots can appear if

$$\cos(\frac{\theta}{2}) \geq \frac{N-2}{N},$$

giving $N-1$ real roots, but $\tilde{X}_1 = 1$ is also a root of the original equation, giving totally N roots. This exhausts all roots of the polynomial and then there are no complex conjugate pairs (for Λ). Thus we find

$$\cos(\frac{\theta_{\text{critical}}}{2}) = \frac{N-2}{N}.$$

For large N, this will have solutions for $\frac{\theta}{2}$ near 0 or 2π, but because of reflection symmetry about $\alpha = \pi$ we need only look around $\alpha = 0$. Assuming θ is small and making an expansion in θ and $\frac{1}{N}$, we get

$$1 - \frac{1}{2}(\frac{\theta_{\text{critical}}}{2})^2 + \cdots = 1 - \frac{2}{N}$$

giving

$$\theta_{\text{critical}} = \frac{4}{\sqrt{N}}.$$

The values taken by $\frac{\theta}{2}$ (the condition that $\omega^{\frac{N}{2}} = -1$ must be satisfied) give

$$\frac{\theta}{2} = \frac{\pi m}{N} \quad \text{for} \quad m = 1, 3, 5, \cdots, N-2,$$

or

$$\frac{\theta}{2} = \frac{\pi m}{N} \pm \pi \quad \text{for} \quad m = 2, 4, 6, \cdots, N-1,$$

(N is odd). Hence for the odd series in m, we get

$$m < \frac{2}{\pi}\sqrt{N}$$

and for the even series in m, we must take the minus sign, giving

$$|m - N| < \frac{2}{\pi}\sqrt{N}.$$

We see that the number of new solutions behave like $\sqrt{N}$. The first N for which we have a new solution is $N = 23$, when $m = 3$ is allowed, next at $N = 65$, $m = 5$ is allowed and so on.

For the case N even, we get the following modifications. First of all, for either choice for the square root, $\sqrt{\omega} = \pm e^{i\frac{\pi m}{N}}$, we have $\sqrt{\omega}^{\frac{N}{2}} = (-1)^m$. Thus for the case m odd, we get the condition as above,

$$\cos(\frac{\theta}{2}) = \frac{\sin(\frac{(N-2)\alpha}{2})}{\sin(\frac{N\alpha}{2})},$$

but here $N = 2n$. Thus,

$$\cos(\frac{\theta}{2}) = \frac{\sin((n-1)\alpha)}{\sin(n\alpha)}.$$

The R.H.S. is a function which has zeros at $\alpha = \frac{k\pi}{(n-1)}$, $k = 1, 2, 3, \cdots, 2n-3$, excluding $k = n-1$, i.e. $2n-4$ zeros, and poles at $\alpha = \frac{k\pi}{n}$, $k = 1, 2, \cdots, 2n-1$, excluding $k = n$. The value at $\alpha = 0$ is $\frac{n-1}{n}$. The situation is as before except at $\alpha = \pi$, the value is $-\frac{n-1}{n}$, not -1, and the function "turns over". The function is reflection symmetric about $\alpha = \pi$. We therefore obtain at least $2n-4$ real roots (the number of zeros), but we get two additional real roots, making a total of $2n-2 = N-2$, for

$$|\cos(\frac{\theta}{2})| > \frac{n-1}{n}.$$

This condition can be satisfied for $\frac{\theta}{2}$ near zero or near π. We also of course get the reflections of these about π. We get two new solutions for $m < \frac{2}{\pi}\sqrt{N}$ and for $|m - \frac{N}{2}| < \frac{2}{\pi}\sqrt{N}$, N.B. m is odd. Thus for the case N even, but m odd there are two solutions which violate the string hypothesis correpondence. We also always have solutions of the original polynomial corresponding to $\tilde{X}_1 = \pm 1$, (N even) which gives totally N roots, hence there are no complex roots for critical values of m.
For m even, the equation for $\tilde{X}_1$ becomes

$$\begin{aligned} 0 &= (\omega+1)\omega^{\frac{N}{2}}\tilde{X}_1^N - 2\omega\omega^{\frac{N-1}{2}}\tilde{X}_1^{(N-1)} - \omega^{\frac{1}{2}}2\tilde{X}_1 + (\omega+1) \\ &= (\omega+1)\tilde{X}_1^N - 2\sqrt{\omega}\tilde{X}_1^{(N-1)} - 2\sqrt{\omega}\tilde{X}_1 + (\omega+1) \\ &= \sqrt{\omega}\left((\sqrt{\omega} + \frac{1}{\sqrt{\omega}})\tilde{X}_1^N - 2\tilde{X}_1^{N-1} - 2\tilde{X}_1 + (\sqrt{\omega} + \frac{1}{\sqrt{\omega}})\right). \end{aligned}$$

Replacing $\tilde{X}_1 = e^{i\alpha}$ yields

$$\cos(\frac{\theta}{2})e^{iN\alpha} - e^{i(N-1)\alpha} - e^{i\alpha} + \cos(\frac{\theta}{2}) = 0,$$

(with $\sqrt{\omega} = e^{i\frac{\theta}{2}}$) which simplifies to

$$\begin{aligned} 0 &= e^{\frac{iN\alpha}{2}}\cos(\frac{\theta}{2})(e^{\frac{iN\alpha}{2}} + e^{\frac{-iN\alpha}{2}}) - e^{i\alpha}e^{\frac{i(N-2)\alpha}{2}}(e^{\frac{i(N-2)\alpha}{2}} + e^{\frac{-i(N-2)\alpha}{2}}) \\ &= e^{\frac{iN\alpha}{2}}2\left(\cos(\frac{\theta}{2})\cos(\frac{N\alpha}{2}) - \cos(\frac{(N-2)\alpha}{2})\right), \end{aligned}$$

yielding

$$\cos(\frac{\theta}{2}) = \frac{\cos(\frac{(N-2)\alpha}{2})}{\cos(\frac{N\alpha}{2})}.$$

This equation is also easily studied graphically. The R.H.S. has zeros at $\alpha = \frac{k\pi}{(N-2)}$ for $k = 1, 3, 5, \cdots, 2N-5$ and poles at $\alpha = \frac{k\pi}{N}$ for $k = 1, 3, 5, \cdots, 2N-1$. We can check that at $\alpha = 0$ the function is -1 and at $\alpha = \pi$ it is -1. Graphically we get a root for each intersection of a horizontal line with $y = \cos(\frac{\theta}{2})$ with the curve $y = \frac{\cos(\frac{(N-2)\alpha}{2})}{\cos(\frac{N\alpha}{2})}$:

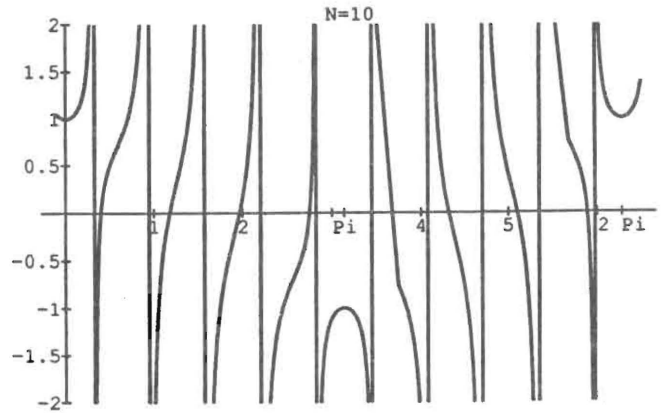

Figure 2: $\frac{\cos(\frac{(N-2)\alpha}{2})}{\cos(\frac{N\alpha}{2})}$, $N = 10$

Counting the intersections it is evident that there is a root for each zero. There are $N-2$ zeros, thus there are always only $N-2$ real roots, $\tilde{X}_1 = \pm 1$ are not roots in this case, in fact it is easy to see that $\cos\left(\frac{N\alpha}{2}\right)$ and $\cos\left(\frac{(N-2)\alpha}{2}\right)$ never vanish simultaneously. Hence there

are always two complex roots and no new real roots which violate the string hypothesis correspondence.

Our method can be generalized to the M particle case. In general we can reduce the Bethe ansatz equations to a set of coupled polynomial equations in terms of our variables X_i. These can in turn, in principle, be reduced to a single polynomial equation in just one variable, however of higher order. We find it is a dramatic simplification to deal with even, coupled polynomial equations than the original transcendental equations. We hope to extend our analysis to other models where the Bethe ansatz has proven useful.

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Instantons and the Ground State of the Massive Schwinger Model

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Abstract

We study the massive Schwinger model, quantum electrodynamics of massive, Dirac fermions, in 1+1 dimensions; with space compactified to a circle. In the limit that transitions to fermion–anti-fermion pairs can be neglected, we study the full ground state. We focus on the effect of instantons which mediate tunnelling transitions in the induced potential for the dynamical degree of freedom in the gauge field.

I. Introduction and Summary

The massive Schwinger model is the direct analog of the quantum electrodynamics of electrons and positrons in $1+1$ dimensions. It was studied[1] in a non-perturbative analysis of the effects of the mass term on the phenomena which manifest themselves in its massless relative[2], the usual Schwinger model, quark trapping (confinement) and spontaneous symmetry breaking without massless scalars (Higg's phenomena). The usual Schwinger model, is exactly solvable and has led to much insight concerning the actual structure of quantum field theories. It has afforded the first reconciliation of gauge invariance and the absence of massless states[2]. It also provided a scenario of how confinement could manifest itself at long distances while asymptotic freedom was valid at short distances[3]. The massive model, further modified with a current-current interaction was rigorously proven to exist by Fröhlich and Seiler[4]. Recently, it was shown that its fermionic determinant is directly relevant to the fermionic determinant in four dimensional Q.E.D., in the presence of non-constant, though uni-directional electromagnetic fields[5]. Thus the two dimensional model has direct bearing on a physical four dimensional theory.

The massive Schwinger model however, is not exactly solvable. Bosonization yields a scalar field with a $m\cos(\phi)$ self-interaction term and an electromagnetic interaction which is no easier to solve. The mass term prohibits the possibility of performing chiral gauge

transformations, even at the classical level, hence the gauge field cannot be removed from the Lagrangean. Imposition of Gauss' law however, allows for the elimination of all non-zero momentum modes of the gauge field at the expense of introducing the Coulomb interaction into the Hamiltonian. On the circle, we are left with one dynamical degree of freedom, the Wilson loop of the gauge field. For the infinite line the Wilson loop degree of freedom is still there, however it becomes infinitesimal[6].

Yang-Mills theories defined on a circle have been studied by Manton[6] (Schwinger model), by Rajeev[7] (pure Yang-Mills) and more recently by Langmann and Semenoff[8] including massless fermions. Some of the results in [8] pre-empt some of our results, however there are no dynamical calculations done there, and the normal ordering ground state energy that has been left out there is very important in the massive case. Topologically non-trivial gauge transformations on the circle, render the Wilson loop variable compact, in fact, also a circle. The resulting Hamiltonian contains a kinetic term for the loop variable, its interaction with the fermions and the properly normal ordered fermionic kinetic term. The normal ordering introduces an induced potential for the loop degree of freedom. This potential tends to localize the loop variable at its minimum. Instantons, however, mediate tunnelling transitions around the circle on which the loop variable is defined, which tend to delocalize the loop variable.

We focus on the effect of these instantons. We calculate the semi-classical correction to the ground state energy. When these corrections become appreciable, the spectrum must change from that of a localized particle to that of an essentially free particle which is constrained to be on a circle. This signals the breakdown of the semi-classical limit, however, the corresponding energy gives an estimate of the energy of the transition regime. We therefore find two regimes in the low energy spectrum of the massive Schwinger model. At low temperature, the spectrum is essentially a harmonic oscillator spectrum with a frequency ω. This gives a specific heat which is constant for high temperature, $\beta\omega \to 0$, but vanishing exponentially for low temperature, $\beta\omega \to \infty$. Heating the system will eventually move it into a new regime, where the loop variable delocalizes. Here the energy spectrum behaves like $\omega' n^2$, yielding a specific heat which behaves as $\sqrt{\frac{\omega'}{\beta}}$ as $\beta\omega' \to 0$.

II. Hamiltonian

The massive Schwinger model is governed by the Lagrangean density

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}(x,t)F^{\mu\nu}(x,t) + \Psi^\dagger(x,t)(i\partial_t - h(x,t))\Psi(x,t), \tag{1}$$

where $F_{\mu\nu}(x,t)$ is the electromagnetic field strength and $h(x,t)$ is the Hamiltonian of the massive Dirac fermion with a minimal electromagnetic interaction. We fix the gauge by taking $A_0(x,t) = 0$. This leaves one gauge field $A_1(x,t)$, and hence

$$F_{01} = \partial_t A_1(x,t) = \dot{v}(x,t) = -F_{10}. \tag{2}$$

The fermionic Hamiltonian is

$$h(x,t) = -i\gamma_5(\partial_x + iev(x,t)) + m\gamma^0, \tag{3}$$

where $\gamma_5 = i\gamma^0\gamma^1$, and we take the representation $\gamma_5 = \sigma^3$, $\gamma^0 = \sigma^1$ in terms of the Pauli matrices.

The equations of motion resulting from the Lagrangean are Ampère's Law,

$$\ddot{v}(x,t) = e\Psi^\dagger(x,t)\gamma_5\Psi(x,t), \tag{4}$$

and the Dirac equation,

$$i\partial_t\Psi(x,t) = h(x,t)\Psi(x,t) = \left(-i\gamma_5(\partial_x + iev(x,t)) + m\gamma^0\right)\Psi(x,t), \tag{5}$$

Gauss' Law, however, is absent

$$\partial_x\dot{v}(x,t) = e\Psi^\dagger(x,t)\Psi(x,t). \tag{6}$$

Invariance of the Lagrangean under static, local gauge transformations

$$\begin{aligned} v(x,t) &\to v(x,t) - \partial_x\Lambda(x) \\ \Psi(x,t) &\to e^{ie\Lambda(x)}\Psi(x,t) \end{aligned} \tag{7}$$

yields a local conserved charge $\mathcal{G}(x)$,

$$\mathcal{G}(x) = \partial_x\dot{v}(x,t) - e\Psi^\dagger(x,t)\Psi(x,t) \tag{8}$$

that is, the time derivative of Gauss' Law is zero,

$$\partial_t\mathcal{G}(x) = 0. \tag{9}$$

The canonical formalism is straightforward, yeilding the Hamiltonian density

$$\mathcal{H} = \frac{1}{2}(\dot{v}(x))^2 + \Psi^\dagger(x,t)h(x,t)\Psi(x,t), \tag{10}$$

with canonically conjugate variable pairs

$$\begin{aligned} v(x,t), &\quad \Pi_{v(x,t)}(x,t) = \dot{v}(x,t) \\ \Psi(x,t), &\quad \Pi_{\Psi(x,t)}(x,t) = i\Psi^\dagger(x,t) \end{aligned} \tag{11}$$

and Poisson brackets

$$\begin{aligned} \{v(x,t_x), \dot{v}(y,t_y)\}_{t_x=t_y} &= \delta(x-y) \\ \{\Psi(x,t_x), i\Psi^\dagger(y,t_y)\}_{t_x=t_y} &= \delta(x-y). \end{aligned} \tag{12}$$

For the quantum theory we work in the Schrödinger picture with time independent operators but time dependent states. The classical Poisson brackets are replaced with commutators or anti-commutators,

$$\begin{aligned}[v(x,t_x),\dot{v}(y,t_y)]_{t_x=t_y} &= i\delta(x-y),\\ \{\Psi(x,t_x),\Psi^\dagger(y,t_y)\}_{t_x=t_y} &= \delta(x-y).\end{aligned} \tag{13}$$

The quantization proceeds essentially without subtlety, other than operator ordering ambiguities. Once the normal ordering infinities are subtracted, this yields a finite well defined Hamiltonian and Gauss operator, requiring no regularization. The normal ordering constants are fixed by gauge invariance and locality[9].

The fermions can be quantized in the Hilbert space of free, massive Dirac fermions, while the gauge fields can be quantized in their corresponding free Hilbert space. The free Hamiltonian is

$$\hat{H}^0 = \int dx \left(\frac{1}{2}(\dot{\hat{v}}(x))^2 + :\Psi^\dagger(x)h^0(x)\Psi(x):\right) \tag{14}$$

where

$$\begin{aligned}
\Psi(x) &= \sum_{p\in Z} \psi^0_+(x,p)a_p + \psi^0_-(x,p)b^\dagger_p\\
\hat{v}(x) &= \sum_{p\in Z} v(p)e^{i\frac{xp}{L}}\\
\dot{\hat{v}}(x) &= \sum_{p\in Z} \frac{e^{-i\frac{xp}{L}}}{2\pi L}\frac{-id}{dv(p)}\\
h^0(x) &= -i\gamma^5\partial_x + m\gamma^0\\
\psi^0_\pm(x,p) &= \frac{e^{i\frac{xp}{L}}}{\sqrt{2\pi L}}\psi^0_\pm(p)\\
\psi^0_\pm(p) &= \frac{1}{\sqrt{\left(2(\pm\sqrt{p^2+(mL)^2})(\pm\sqrt{(p)^2+(mL)^2}-p)\right)}}\begin{pmatrix}(mL)\\ \pm\sqrt{p^2+(mL)^2}-p\end{pmatrix}
\end{aligned} \tag{15 – 20}$$

and

$$\{a_p,a^\dagger_q\} = \{b_p,b^\dagger_q\} = \delta_{p,q}, \quad \text{all others zero.} \tag{21}$$

The normal ordering in (14) is with respect to these operators, the vacuum state defined by

$$a_p|0>=b_p|0>=0, \tag{22}$$

for all p. Explicitly the free fermionic Hamiltonian is

$$\hat{H}^0_F = \frac{1}{L}\sum_{p\in Z}\sqrt{p^2+(mL)^2}(a^\dagger_p a_p + b^\dagger_p b_p). \tag{23}$$

The gauge field Hilbert space is presumably a "wave functional" of the variables $v(p)$. This is rather formal here, since the ground state wave functional would be the infinite fold product of normalized ground state wave functions for each independent variable, which does not exist. The problem arises simply because the infinite fold tensor product is not isomorphic to the space of normalizable complex valued wave functionals of an infinite number of variables, with the momentum operator represented by the functional derivative. This is in contra-distinction to the analogous case for a finite number of variables. Since we will be able to eliminate all but a single gauge degree of freedom on imposing Gauss' law, we will not overly concern ourselves with the precise definition of the gauge field Hamiltonian and Hilbert space.

Introducing the interaction at the first quantized level yields the Hamiltonian

$$h(x) = -i\gamma^5(\partial_x + iev(x)) + m\gamma^0 \tag{24}$$

with putative second quantized version

$$\hat{H}_F = \hat{H}^0_F + \int dx v(x)(\Psi^\dagger(x)\gamma^5\Psi(x)). \tag{25}$$

We must define the last term in the R.H.S., the interaction current density

$$j^1(x) = \Psi^\dagger(x)\gamma^5\Psi(x) = \bar{\Psi}(x)\gamma^1\Psi(x) \tag{26}$$

and we will also need the charge density

$$\rho(x) = \Psi^\dagger(x)\Psi(x) = \bar{\Psi}(x)\gamma^0\Psi(x). \tag{}$$

Precisely, we will define, their corresponding momentum components, for $p \neq 0$

$$\begin{aligned}
j^\mu(p) = \sum_{q\in Z}\Big(&\langle\psi^0_+(q)|\gamma^0\gamma^\mu|\psi^0_+(p+q)\rangle a^\dagger_q a_{p+q} + \langle\psi^0_+(q)|\gamma^0\gamma^\mu|\psi^0_-(p+q)\rangle a^\dagger_q b^\dagger_{p+q}\\
&+ \langle\psi^0_-(q)|\gamma^0\gamma^\mu|\psi^0_+(p+q)\rangle b_q a_{p+q} - \langle\psi^0_-(q)|\gamma^0\gamma^\mu|\psi^0_-(p+q)\rangle b^\dagger_{p+q} b_q\Big),
\end{aligned} \tag{28}$$

where the bracket $\langle\cdot|\cdot\rangle$ is between the spinors in equation (20). These are well defined operators with their domain consisting of states corresponding to finitely many excitations above the free vacuum state. For $p = 0$ we must actually normal order by subtracting infinite constants, we take

$$\begin{aligned}
Q &= \int dx j^0(x) = \sum_{q\in Z}(a^\dagger_q a_q - b^\dagger_q b_q)\\
Q_5 &= \int dx j^1(x) = \sum_{q\in Z}\Bigg(\frac{q}{\sqrt{q^2+(mL)^2}}a^\dagger_q a_q + \frac{(mL)}{\sqrt{q^2+(mL)^2}}a_q b_q\\
&\quad + \frac{(mL)}{\sqrt{q^2+(mL)^2)}}b^\dagger_q a^\dagger_q - \frac{q}{\sqrt{q^2+(mL)^2)}}b^\dagger_q b_q\Bigg).
\end{aligned} \tag{29}$$

These charges clearly have arbitrary definitions up to c-numbers, as far as the fermionic variables are concerned. We can fix these c-numbers by imposing gauge invariance and locality[9]. These c-numbers will only affect the fermionic Hamiltonian by c-numbers if we take equations (28) as the basic building blocks for constructing the Hamiltonian (25).

Gauge invariance manifests itself with the condition that the spectrum of (25) is independent of $v(x)$, we can gauge away $v(x)$ with the gauge transformation

$$e^{ie\int_0^x dy v(y)}. \tag{30}$$

Actually we may not remove all of the constant part of $v(x)$, the gauge transformation (30) must be single valued on the circle. Thus we must modify (30) to

$$e^{ie\int_0^x dy(v(y)-v)} \tag{31}$$

with

$$v = \frac{1}{2\pi L}\int_0^{2\pi L} dx v(x). \tag{32}$$

As explained in reference 9, this gives the gauge covariant Hamiltonian

$$\hat{H}_F = \hat{H}_F^0 + e\sum_{\substack{p\in Z\\ p\neq 0}} v(-p)j^1(p) + Le^2\sum_{p\in Z} v(-p)v(p) + evQ_5. \tag{33}$$

The gauge covariance is manifest if we remove $v(-p)$ dependence by a unitary transformation

$$\hat{H}_F = \mathcal{U}^\dagger(\hat{H}_F^0 + evQ_5 + Le^2v^2)\mathcal{U} = \mathcal{U}^\dagger\bar{H}_F\mathcal{U} \tag{34}$$

where

$$\mathcal{U} = e^{-e\sum_{p\neq 0}\frac{L}{p}v(-p)j^0(p)}. \tag{35}$$

This follows from the commutation relations, which can be rigorously established,

$$\begin{aligned}
&[j^0(p), j^1(q)] = 2p\delta p, -q\\
&[\hat{H}_F^0, j^0(p)] = -\frac{p}{L}j^1(p)\\
&[Q, Q_5] = [Q, j^\mu(p)] = [Q_5, j^\mu(p)] = [j^0(p), j^0(q)] = [j^1(p), j^1(q)] = 0.
\end{aligned} \tag{36}$$

The full Hamiltonian then is

$$\begin{aligned}
\hat{H} &= \sum_{p\in Z}\frac{-1}{2\pi L}\frac{1}{2}\frac{d^2}{dv(p)dv(-p)} + \hat{H}_F\\
&= \mathcal{U}^\dagger\left(\frac{-1}{2\pi L}\sum_{\substack{p\in Z\\ p\neq 0}}\frac{1}{2}\left(\frac{d}{dv(-p)} + \frac{eLj^0(p)}{p}\right)\left(\frac{d}{dv(p)} - \frac{eLj^0(-p)}{p}\right) + \frac{-1}{4\pi L}\frac{d^2}{dv^2} + \bar{H}_F\right)\mathcal{U}\\
&= \mathcal{U}^\dagger\bar{H}\mathcal{U}
\end{aligned} \tag{37}$$

and we are looking for the ground state of $\bar{H}$ or equivalently $\bar{H}$.

We must not forget Gauss' law, which actually simplifies matters. At the quantum level we impose Gauss' law as a constraint on physical states. Physical states are those which are annihilated by the Gauss operator:

$$\hat{\mathcal{G}}(x)|\text{physical} >= 0. \tag{38}$$

We can write $\hat{\mathcal{G}}(x)$ by Fourier decomposition as

$$\hat{\mathcal{G}}(x) = \sum_{p\in Z}\frac{e^{i\frac{px}{L}}}{2\pi L}\hat{G}(p) \tag{39}$$

where

$$\begin{aligned}
\hat{G}(p) &= \frac{p}{L}\frac{d}{dv(-p)} - ej^0(p)\\
&= \begin{cases}\mathcal{U}^\dagger\frac{p}{L}\frac{d}{dv(-p)}\mathcal{U} & p\neq 0\\ \mathcal{U}^\dagger Q\mathcal{U} & p = 0\end{cases}
\end{aligned} \tag{40}$$

with Q the charge operator.

Thus we look for eigenstates

$$\bar{H}|\mathcal{E} >= \mathcal{E}|\mathcal{E} > \tag{41}$$

subject to the simpler conditions

$$\begin{cases}\frac{p}{L}\frac{d}{dv(-p)}|\mathcal{E} >= 0 & p\neq 0\\ Q|\mathcal{E} >= 0 & p = 0.\end{cases} \tag{42}$$

Clearly the conditions are trivial to satisfy, $|\mathcal{E} >$ is a charge zero state, that is indepedent of $v(p)\forall p$. Then the eigenvalue problem for $|\mathcal{E} >$ reduces to

$$\left(\frac{-1}{2\pi L}\sum_{\substack{p\in Z\\ p\neq 0}}\frac{1}{2}e^2L^2\frac{j^0(p)j^0(-p)}{p^2} - \frac{1}{4\pi L}\frac{d^2}{dv^2} + \bar{H}_F\right)|\mathcal{E} >= \mathcal{E}|\mathcal{E} >. \tag{43}$$

The first term is just the Coulomb energy of the fermions while the v is the only physical degree of freedom in the gauge field.

We must still deal with topologically non-trivial gauge transformations

$$g_k(x) = e^{-i\frac{kx}{L}} \qquad k\in Z. \tag{44}$$

The effect on v is to shift it by a constant,

$$v \to v + \frac{k}{eL}. \tag{45}$$

Hence v is really a circular variable with circumference $\frac{1}{eL}$. The fermionic part of the Hamiltonian transforms covariantly under these gauge transformations, the unitary operator V which implements these transformations for $k = 1$, effects the mapping on the annihilation and creation operators which diagonalize $\bar{H}_F$,

$$\begin{aligned} \bar{a}_p(v) &= V^\dagger \bar{a}_{p+1}(v - \frac{1}{eL})V \\ \bar{b}_p(v) &= V^\dagger \bar{b}_{p+1}(v - \frac{1}{eL})V. \end{aligned} \tag{46}$$

The expression for V is simple,

$$V = \prod_{p \in Z} e^{-\frac{\pi}{2}\left(\bar{a}_p(v)\bar{a}^\dagger_{p+1}(v-\frac{1}{eL}) - \bar{a}_{p+1}(v-\frac{1}{eL})\bar{a}^\dagger_p(v)\right)} e^{-\frac{\pi}{2}\left(\bar{b}_p(v)\bar{b}^\dagger_{p+1}(v-\frac{1}{eL}) - \bar{b}_{p+1}(v-\frac{1}{eL})\bar{b}^\dagger_p(v)\right)}. \tag{47}$$

V is in fact independent of v, as can be seen by reexpressing $\bar{a}_p(v)$ in terms of the free annihilation and creation operators,

$$\begin{aligned} &\langle\psi_+(p,v)| \left(|\psi^0_+(p)\rangle a_p + |\psi^0_-(p)\rangle b^\dagger_p\right) \\ &= V^\dagger \langle\psi_+(p+1, v - \frac{1}{eL})| \left(|\psi^0_+(p+1)\rangle a_{p+1} + |\psi^0_-(p+1)\rangle b^\dagger_{p+1}\right) V. \end{aligned} \tag{48}$$

Therefore, if

$$\begin{aligned} a_p &= V^\dagger \langle\psi^0_+(p)| \left(|\psi^0_+(p+1)\rangle a_{p+1} + |\psi^0_-(p+1)\rangle b^\dagger_{p+1}\right) V \\ b^\dagger_p &= V^\dagger \langle\psi^0_-(p)| \left(|\psi^0_+(p+1)\rangle a_{p+1} + |\psi^0_-(p+1)\rangle b^\dagger_{p+1}\right) V \end{aligned} \tag{49}$$

equation (46) will be satisfied. This is a (unitarily implementable) Bogoliubov transformation that is completely independent of v. The Hamiltonian satisfies

$$V^\dagger \bar{H}_F(v) V = \hat{H}^0_F + e(v + \frac{1}{eL})Q_5 + e^2 L(v + \frac{1}{eL})^2 = \bar{H}_F(v + \frac{1}{eL}). \tag{50}$$

Thus the spectrum of the fermionic Hamiltonian is invariant after going around the circle in v, however the eigenstates form sections of a bundle over v, the fermionic states satisfy

$$V|v + \frac{1}{eL}> = |v>. \tag{51}$$

The physical state condition, equation (38), which corresponds to implementing Gauss' law, makes the states invariant under infinitesimal gauge transformations. The integrated version of equation (38), corresponding to finite gauge transformations that are still continuously connected to the identity (small), simply implies that the states are invariant under these small gauge transformations

$$e^{i\int dx \Lambda(x)\hat{G}(x)}|\text{physical}> = |\text{physical}>. \tag{52}$$

In $\hat{G}(x)$, the part corresponding to the divergence of the electric field, effects the transformation on the gauge field, while the fermionic charge density effects the appropriate transformation on the fermions. For the topologically non-trivial gauge transformations (large), we can only work with the finite form of these transformations, evidently an infinitesimal generator does not exist. We should impose invariance of the states under these transformations also. The transformation on the gauge field is a translation operator, $\mathcal{K}$,

$$\mathcal{K}(v) = v + \frac{1}{eL} \tag{53}$$

while the fermions are transformed by V, equation (46). Thus invariance of the states under large gauge transformations implies

$$\mathcal{K}V|\text{physical}> = |\text{physical}>. \tag{54}$$

The eigenstates of the total Hamiltonian then must have the form

$$|\mathcal{E}> = \sum_n \psi_n(v)|n, v>. \tag{55}$$

Here $|n, v>$ correspond to a complete set of fermionic states satisfying equation (51) and $\psi_n(v)$ is the bosonic wave function which is periodic under translation of v by $\mathcal{K}$,

$$\mathcal{K}(\psi_n(v)) = \psi_n(v + \frac{1}{eL}) = \psi_n(v). \tag{56}$$

If we express $\bar{H}$ in terms of fermionic annihilation and creation operators which diagonalize $\bar{H}_F$ we get

$$\begin{aligned} \bar{H} = &\frac{-1}{4\pi L}\frac{d^2}{dv^2} + \sum_{p \in Z} \sqrt{(\frac{p}{L} + ev)^2 + m^2}\left(\bar{a}^\dagger_p(v)\bar{a}_p(v) + \bar{b}^\dagger_p(v)\bar{b}_p(v)\right) - g(v) \\ &+ \frac{-1}{2\pi L}\sum_{\substack{p \in Z \\ p \neq 0}} \frac{1}{2} e^2 L^2 \frac{j^0(p) j^0(-p)}{p^2} \end{aligned} \tag{57}$$

and $g(v)$ is the induced potential from the fermions for the gauge degree of freedom[9],

$$g(v) = \frac{-1}{L}\frac{2mL}{\pi}\sum_{n=1}^{\infty}\frac{K_1(\pi n m L)}{n}(\cos(2\pi n e L v) - 1). \tag{58}$$

III. Approximations

The basic approximation that we make, which is not exact[10], is that excitations to fermion–anti-fermion pairs (corresponding to $\bar{H}_F(v)$) are suppressed. Standard perturbation theory

shows that corrections to the wave function and energy levels coming from intermediate states are not only suppressed by explicit factors of some coupling constant, but also due to powers of the ratio of the characteristic energy scale of the interaction Hamiltonian with the energy difference between the unperturbed initial state and the intermediate state[11]. If the Hamiltonian is a free part plus a perturbation,

$$H = H^0 + \lambda H' \tag{59}$$

and the full wave function admits an expansion of the form

$$\psi = \psi_0 + \lambda \sum_{n=1}^{\infty} \alpha_n \psi_n \tag{60}$$

with

$$H^0 \psi_n = E_n \psi_n \quad n = 0, 1, 2, \cdots, \tag{61}$$

we get

$$\begin{aligned} H\psi &= (H^0 + \lambda H')(\psi_0 + \lambda \sum_{n=1}^{\infty} \alpha_n \psi_n) \\ &= E_0 \psi_0 + \lambda \sum_{n=1}^{\infty} \alpha_n E_n \psi_n + \lambda H' \psi_0 + \lambda^2 \sum_{n=1}^{\infty} \alpha_n H' \psi_n \\ &= E\psi = E(\psi_0 + \lambda \sum_{n=1}^{\infty} \alpha_n \psi_n). \end{aligned} \tag{62}$$

This implies

$$E = E_0 + \lambda < \psi_0 | H' | \psi_0 > + o(\lambda^2) \tag{63}$$

and

$$\lambda \alpha_n = \lambda \frac{< \psi_n | H' | \psi_0 >}{E_0 - E_n} + o(\lambda^2). \tag{64}$$

In our case we consider excitations from the fermionic ground state $|0>>$ which is annihilated by $\bar{a}_p(v)$ and $\bar{b}_p(v)$. $j^0(p)$ can be expressed as a bilinear in these operators, hence the Coulomb energy term mediates transitions to intermediate states with zero, one or two fermion–anti-fermion pairs, at zero total momentum. Then

$$\lambda H' = \frac{(eL)^2}{4\pi} \left(\frac{1}{L} \sum_{\substack{p \in Z \\ p \neq 0}} \frac{j^0(p) j^0(-p)}{p^2} \right), \tag{65}$$

and the coefficients of states involving fermion–anti-fermion pairs, first order in the perturbative expansion of the full ground state are

$$\begin{aligned} &\frac{(eL)^2}{4\pi} \frac{\left(\frac{1}{L} \sum_{\substack{p \in Z \\ p \neq 0}} \frac{<\text{pairs}|j^0(p) j^0(-p)|0>>}{p^2} \right)}{(E_0 - E_{\text{pairs}})} \\ &\leq \frac{(eL)^2}{4\pi 2mL} \left(\frac{1}{L} \sum_{\substack{p \in Z \\ p \neq 0}} \frac{< \text{pairs}|j^0(p) j^0(-p)|0>>}{p^2} \right). \end{aligned} \tag{66}$$

This behaviour continues in each order. Thus the coupling constant emerges as $(eL)(\frac{e}{m})$, which we take to be arbitrarily small. Therefore we can neglect the contribution of pair states, arising because of the Coulomb term, to the full ground state. This makes intuitive sense in the following way: on a circle, we cannot separate charges to infinity, there is a maximum separation that we can separate charges. The Coulomb energy, which is linear in the separation, is bounded. It is a little more work to see that the corresponding operator is relatively bounded in comparison to the fermionic Hamiltonian about which we perturb. Hence in the limit that its coefficient goes to zero, we can rigorously neglect it.

The shift in the ground state energy has the leading contribution

$$\begin{aligned} &<< 0|\lambda H'|0>> \\ &= \frac{(eL)^2}{4\pi} \left(\frac{1}{L} \sum_{\substack{p \in Z \\ p \neq 0}} \frac{<< 0|j^0(p) j^0(-p)|0>>}{p^2} \right) \\ &= \frac{(eL)^2}{4\pi} \left(\frac{1}{L} \sum_{\substack{p \in Z \\ p \neq 0}} \frac{\sum_{q \in Z} \left| \langle \psi_-(q,v) | \psi_+(p+q,v) \rangle \right|^2}{p^2} \right) \\ &= \frac{(eL)^2}{4\pi} \frac{1}{L} \sum_{\substack{p,q \in Z \\ p \neq 0}} \frac{1}{p^2} \left(\frac{1}{2\sqrt{(q+eLv)^2 + (mL)^2}(\sqrt{(q+eLv)^2 + (mL)^2} + q + eLv)} \right) \times \\ &((mL)^2 \\ &-(\sqrt{(q+eLv)^2 + (mL)^2} + q + eLv)(\sqrt{(p+q+eLv)^2 + (mL)^2} - (p+q+eLv)))^2 \times \\ &\left(\frac{1}{2\sqrt{(p+q+eLv)^2 + (mL)^2}(\sqrt{(p+q+eLv)^2 + (mL)^2} - (p+q+eLv))} \right). \end{aligned} \tag{67}$$

This has as coefficient $(eL)^2$ multiplying a function of eLv, which is of order 1. This shift in the ground state energy is suppressed relative to $-g(v)$, the normal ordering contribution to the ground state energy, by again a factor of $(eL)(\frac{e}{m})$.

This yields the truncated Hamiltonian

$$H = \frac{-1}{4\pi L} \frac{d^2}{dv^2} + \frac{1}{L} \sum_{p \in Z} \sqrt{(p+ev)^2 + (mL)^2} \left(\bar{a}_p^\dagger(v) \bar{a}_p(v) + \bar{b}_p^\dagger(v) \bar{b}_p(v) \right) - g(v) \tag{68}$$

and the ground state will be of the form

$$|\mathcal{E}_0> = \psi(v) |0>> . \tag{69}$$

The state $|0>>$ depends on v, hence the derivative $\frac{d}{dv}$ will give transitions to other fermionic states. We have

$$|0>> = \mathcal{W}^\dagger |0> \tag{70}$$

with $|0>$ the free fermion vacuum, and

$$\mathcal{W} = \prod_{p\in Z} e^{\theta(p,v)(a_p^\dagger b_p^\dagger - b_p a_p)}. \tag{71}$$

$\mathcal{W}$ is the unitary operator implementing the Bogoliubov transformation

$$\begin{aligned} \bar{a}_p(v) &= \cos(\theta(p,v))a_p + \sin(\theta(p,v))b_p^\dagger = \mathcal{W}^\dagger a_p \mathcal{W} \\ \bar{b}_p^\dagger(v) &= -\sin(\theta(p,v))a_p + \cos(\theta(p,v))b_p^\dagger = \mathcal{W}^\dagger b_p^\dagger \mathcal{W}, \end{aligned} \tag{72}$$

with

$$\begin{aligned} \cos(\theta(p,v)) &=< \psi_+(p,v)|\psi_+^0(p) > \\ \sin(\theta(p,v)) &=< \psi_+(p,v)|\psi_-^0(p) > . \end{aligned} \tag{73}$$

Then,

$$\frac{d}{dv}|0>>= -\sum_{p\in Z} \frac{d(\theta(p,v))}{dv} \bar{a}_p^\dagger(v)\bar{b}_p^\dagger(v)|0>> . \tag{74}$$

The derivative can be expressed as

$$\frac{d(\theta(p,v))}{dv} = \frac{1}{\cos(\theta(p,v))}\frac{d\sin(\theta(p,v))}{dv} = \frac{1}{< \psi_+(p,v)|\psi_+^0(p) >}\frac{d < \psi_+(p,v)|\psi_-^0(p) >}{dv}. \tag{75}$$

The denominator is a smooth function of order 1 in eLv, and also in mL. Note that eLv is always in $[0,1]$. The numerator is

$$\begin{aligned} &\langle\psi_+(p,v)|\psi_-^0(p)\rangle = \\ &\frac{(mL)^2 - (\sqrt{(p+eLv)^2+(mL)^2} - (p+eLv))(\sqrt{p^2+(mL)^2}+p)}{2\sqrt{\sqrt{(p+eLv)^2+(mL)^2}(\sqrt{(p+eLv)^2+(mL)^2}-(p+eLv))}} \times \\ &\times \frac{1}{\sqrt{\sqrt{p^2+(mL)^2}(\sqrt{p^2+(mL)^2}+p)}} \end{aligned} \tag{76}$$

It is easy to see that $< \psi_+(p,v)|\psi_-^0(p) >$ behaves like $\frac{eLv}{mL}$ for $mL >> p$. For $mL << p$ it behaves like $\frac{mLeLv}{p^2}$, which is then much smaller than $\frac{eLv}{mL}$ and for $mL = \alpha p$, with $\alpha \approx 1$, we can directly factor the powers of mL out of the expression leaving a function of $\frac{eLv}{mL}$ and α. Thus differentiating with respect to v in each case gives a factor of $\frac{eL}{mL}$. Thus when this parameter is small we can neglect the action of the derivative $\frac{d}{dv}$ on the fermionic state, yielding the equation for $\psi(v)$

$$(\frac{-1}{4\pi L}\frac{d^2}{dv^2} - g(v))\psi(v) = \mathcal{E}^0\psi(v), \tag{77}$$

a simple, one-dimensional quantum mechanics problem on a circle with a periodic potential. For $m \neq 0$, $g(v)$ is a smooth periodic potential with a single, symmetric well at $v = \frac{1}{2eL}$.

It is clear what the excitation spectrum will be. For low energies, the variable will be localized in the bottom of the approximately harmonic well. The energy will be

$$\mathcal{E}_n = \hbar\omega(n+\frac{1}{2}) \quad n = 0,1,2,\cdots, \tag{78}$$

where ω is the curvature at the bottom of the well. Then for high energies, the variable v will hardly notice the small potential $-g(v)$, but will be essentially constrained by the size of the circle upon which it must sit. The circumference is $\frac{1}{eL}$, giving rise to an energy spectrum

$$\mathcal{E}_n = \hbar\omega'(n')^2 \quad n' = 1,2,\cdots. \tag{79}$$

ω is given by

$$\omega = \sqrt{\frac{1}{2\pi L}\left(\frac{d^2}{dv^2}g(v)\Big|_{v=\frac{1}{2eL}}\right)}, \tag{80}$$

while

$$\omega' = \pi e^2 L. \tag{81}$$

IV. Instantons

We proceed along the lines pioneered by Langer[12], and popularized by Coleman[13], for using the Euclidean path integral to calculate the effects of tunnelling. The idea is simple, the matrix element of e^{-TH} in the "position" eigenstate $|v = \frac{1}{2eL}>$ has a representation in terms of a Euclidean path integral

$$< v = \frac{1}{2eL}|e^{-TH}|v = \frac{1}{2eL} >= \int_{v(\frac{-T}{2})=v(\frac{T}{2})=\frac{1}{2eL}} \mathcal{D}v(\tau)e^{-\int_{\frac{-T}{2}}^{\frac{T}{2}} d\tau L^E(v(\tau))}, \tag{82}$$

where $L^E(v(\tau))$ is the continuation to (dimensionless) Euclidean time τ of the usual Lagrangean. In this simple case it corresponds to

$$L^E(v(\tau)) = \frac{1}{2}(\dot{v}(\tau))^2 - 2\pi L g(v(\tau)) \tag{83}$$

which can be thought of simply as the Lagrangean describing motion of a particle in minus the original potential, $-(-g(v))$. The matrix element equation (82) has the expansion

$$< v = \frac{1}{2eL}|e^{-TH}|v = \frac{1}{2eL} >= e^{-T\mathcal{E}_0} < v = \frac{1}{2eL}|\mathcal{E}_0 >< \mathcal{E}_0|v = \frac{1}{2eL} > + \cdots \tag{84}$$

thus in the limit that $T \to \infty$ we can extract $\mathcal{E}_0$, and the amplitudes $| < \mathcal{E}_0|v = \frac{1}{2eL} > |^2$; contributions from higher states will be exponentially suppressed.

The Euclidean functional integral can be evaluated in a saddle point approximation. The first step is to identify the saddle point, called an instanton here, and then perform the

functional integral in a Gaussian approximation about the saddle point. The Gaussian functional integral simply gives rise to $e^{-S_0^E}$ where S_0^E is the Euclidean action for an instanton, multiplied by the inverse square root of a functional determinant. The determinant is the product of all the eigenvalues of the functional operator corresponding to the second variation of the Euclidean action about the instanton. This product has two main problems, it is of course infinite, but it is also zero!

The infinite product of a continuum of eigenvalues that become arbitrarily large is formally infinite, but in fact completely ill-defined. It is, however, only the ratio of this product relative to the correponding (infinite) product for the free case that is important. This ratio is finite. The other problem comes from vanishing eigenvalues. These render the determinant zero. Such zero modes correspond to degeneracies of the original instanton. There usually exist a whole set of instantons with the same action. We should sum over the contribution from all saddle points (instantons) with the same, minimal action. When we perform this sum, we have already taken into account the direction in function space corresponding to the zero modes. Thus in the Gaussian integral, we should exclude the integration along the zero modes, the result being the determinant with the zero eigenvalues removed. There is a Jacobian factor which must be taken into account since the measure corresponding to summing over the contribution from degenerate instantons is different from that corresponding to integrating over the zero modes directions in the Gaussian functional integral.

Actually for large but finite T, there are no exact zero modes, corresponding to invariance under translation of the instanton in Euclidean time. The corresponding eigenvalue however, is exponentially small in T, thus the infinite T calculations will be exponentially close to those for finite, but large T. Furthermore, we must recognize that in this case, there are other approximate critical points, corresponding to N widely separated instantons which must also be considered. The corresponding action is N times the action of one instanton, implying naively that their contribution is suppressed by $N-1$ powers of $e^{-S_0^E}$ relative to the contribution for one instanton. The degeneracy factor of these approximate critical points is, however, $\frac{T^N}{N!}$, corresponding to independent translation in Euclidean time of each instanton. This factor can be arbitrarily large compensating the suppression from the exponential factor, until N surpasses T. T must always of course be sufficiently large so that the space per instanton, $\frac{T}{N}$, is still much larger that the size of the instanton. The size of the instanton is determined by the parameters that appear in the Lagrangean, hence has nothing to do with T and N. Thus it is always possible to satisfy this constraint. We should sum over N until it is of the same order as T. However, once N is of this size, the contribution of further terms in the expansion is exponentially small, due to the $\frac{1}{N!}$, thus we make only a negligible error to continue the sum up to ∞.

For mL sufficiently large we can keep only the first term in the series for $g(v)$,

$$-g(v) \to \frac{-4\pi mL}{\pi L} K_1(2\pi mL)\sin^2(\pi eLv) \tag{85}$$

and

$$\omega = 4\pi eL\sqrt{mLK_1(2\pi mL)}, \tag{86}$$

is the relevant frequency for a (dimensionless) Euclidean time, τ. The corresponding instanton equation, obtained by varying the Euclidean Lagrangean is

$$\frac{d^2}{d\tau^2}\bar{v}(\tau) = -8\pi eLmLK_1(2\pi mL)\sin(2\pi eL\bar{v}(\tau)). \tag{87}$$

This equation is easily integrated to give

$$\bar{v}(\tau) = \frac{2}{\pi eL}\tan^{-1}\left(e^{\pm\omega(\tau-\tau_0)}\right) - \frac{1}{2eL}, \tag{88}$$

the $\pm$ choosing an instanton or an anti-instanton. The action for either is given by

$$S_0 = \frac{8\sqrt{mLK_1(2\pi mL)}}{\pi eL}. \tag{89}$$

A method which we follow here, for calculating the ratio $\frac{\det'}{\det_0}$, where $\det'$ is the determinant with the zero modes excluded and $\det_0$ is the free determinant, is given in Coleman[13]. Here it is shown that

$$\frac{\det'}{\det_0} = \frac{\psi_0(\frac{T}{4\pi L})}{\lambda_0\psi_0^0(\frac{T}{4\pi L})} \tag{90}$$

where $\psi_0(\frac{T}{4\pi L})$ is the eigenfunction (evaluated at $\frac{T}{4\pi L}$) with the smallest eigenvalue λ_0, for the differential equation corresponding to the Schrödinger operator on the interval $[-\frac{T}{4\pi L}, \frac{T}{4\pi L}]$, with potential $+g(\bar{v}(\tau))''$ satisfying the boundary conditions $\psi_0(-\frac{T}{2}) = 0$, $\psi_0'(-\frac{T}{4\pi L}) = 1$. $\psi_0^0(\frac{T}{4\pi L})$ is the analogous solution for the free problem. As $T \to \infty$, these are easy to find. We find

$$\frac{\det'}{\det_0} = \frac{S_0(\pi eL)^2}{8\omega^3} = \frac{1}{4\omega^2}. \tag{91}$$

The contribution from the instantons and the anti-instantons sums separately. There is no constraint on the order in which they must appear, either tunnels from the same initial and final state. This gives two times the same contribution. The factor[13] K, which takes into account the Jacobian factor and the ratio of the determinant in the presence of one instanton and the free determinant then is

$$K = \left(\frac{S_0}{2\pi}\right)^{\frac{1}{2}}\left(\frac{\det'}{\det_0}\right)^{-\frac{1}{2}} = (\frac{\omega}{\pi})^{\frac{1}{2}}\frac{2\omega}{eL\pi}. \tag{92}$$

Then we find the path integral, equation (82), is given by (in dimensionless Euclidean time)

$$(\det_0)\sum_{N_1,N_2=0}^{\infty}\frac{(\frac{T}{2\pi L}Ke^{S_0})^{N_1}(\frac{T}{2\pi L}Ke^{S_0})^{N_2}}{N_1!N_2!} = (\frac{\omega}{\pi})^{\frac{1}{2}}e^{-\frac{T}{2\pi L}\frac{1}{2}\omega}e^{2\frac{T}{2\pi L}Ke^{-S_0}}. \tag{93}$$

The sum over N_1 is for the instantons while that over N_2 is for the anti-instantons, the upshot is the factor of 2 in the exponent. The free determinant is calculated in Coleman[13].

Finally we find,

$$
\begin{aligned}
< v = \frac{1}{2eL}|e^{-TH}|v = \frac{1}{2eL} > &= e^{-T\mathcal{E}_0} < v = \frac{1}{2eL}|\mathcal{E}_0 >< \mathcal{E}_0|v = \frac{1}{2eL} > + \cdots \\
&= \left(\frac{\omega}{\pi}\right)^{\frac{1}{2}} e^{-\frac{T}{2eL}\left(\frac{1}{2}\omega - 2\left(\frac{\omega}{e}\right)^{\frac{1}{2}} \frac{2\omega}{eL} e^{-S_0}\right)}.
\end{aligned}
\tag{94}
$$

Hence

$$
\begin{aligned}
\mathcal{E}_0 &= \frac{1}{2\pi L}\frac{1}{2}\omega\left(1 - \frac{16}{\pi}\left(\frac{\sqrt{mLK_1(2\pi mL)}}{eL}\right)^{\frac{1}{2}} e^{-S_0}\right) - \frac{1}{2\pi L}8mLK_1(2\pi mL) \\
&= \frac{1}{2\pi L}\frac{1}{2}\omega\left(1 - \frac{4\sqrt{2}}{\sqrt{\pi}}\sqrt{S_0}e^{-S_0}\right) - \frac{1}{2\pi L}8mLK_1(2\pi mL),
\end{aligned}
\tag{95}
$$

where we have included the offset due to the value of the minimum of the potential $-g(v)$.

The correction that we have calculated is non-perturbative. Our approximations, $eL\frac{e}{m} \to 0$, $\frac{e}{m} \to 0$ and $mL \to \infty$ still leave S_0 arbitrary (as can be seen from its expression). It can take values from 0 to ∞. The function $e^{-x}\sqrt{x}$ is maximum at $x = \frac{1}{2}$, where it is equal to $\frac{1}{\sqrt{2e}}$. Then $\frac{4\sqrt{2}}{\sqrt{\pi}}\frac{1}{\sqrt{2e}} \cong 1.3$. Hence as S_0 approaches $\frac{1}{2}$ from above or below, the effects of the instantons become non-negligible.

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