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QUADRATIC ALGEBRAS IN QUANTUM MECHANICS

Pascal Létourneau and Luc Vinet

Laboratoire de Physique Nucléaire
Université de Montréal
Montréal, Canada H3C 3J7

INTRODUCTION

Dynamical symmetries prove extremely useful and have applications in many physical situations[1]. Customarily, these symmetries are described mathematically in terms of Lie groups and algebras. It has been appreciated howewer that more general structures can sometimes be required. A celebrated example is that of Lie superalgebras; these arise when supersymmetries are encountered. Another type of algebras, that of quantum groups, is also being recognized as potentially relevant to describe dynamical symmetries, especially in the context of integrable models. These quantum groups belong to the class of quadratic algebras which are defined by subjecting the generators to quadratic relations. The purpose of this paper is to provide a simple example where the symmetries are most naturally discussed using such a quadratic algebra.

We shall consider a two-dimensional anisotropic harmonic oscillator with a 2 to 1 frequency ratio. This problem is known to admit accidental degeneracies that can be explained with the help of Lie algebraic techniques [2,3]. It is first shown that the associated Halmiltonian possesses two constants of motion that are quadratic in the momenta. A realization of the su(2) algebra is then constructed using these conserved quantities. Since the dimensions of the unitary representations of su(2) coincide with the degree of the degeneracies, these are thus accounted for. This resolution is not completely satisfactory howewer. The reason is that under commutation the constants of motion do not actually form a Lie algebra; one needs in fact, to divide these quantities by the square root of certain operators to obtain closure. As we shall explain, such difficulties do not show up if a quadratic algebra framework is used instead of the Lie algebraic one. It therefore seems that quadratic (and polynomial) algebras could be useful tools to deal with dynamical symmetries. Such a claim has been made also in Refs. [4-6], where other interesting examples are discussed.

We shall also make another interesting observation from this analysis of the anisotropic harmonic oscillator. Recently, quasi-exactly solvable problems have attracted a lot of attention. (For a review see Ref. [7].) These quantum systems are characterized by the fact that part of their spectra can be obtained algebraically. This is made possible by the existence of a hidden symmetry algebra generated by first

order differential operators. Hamiltonians that can be expressed as bilinear combinations of these operators will leave invariant any finite-dimensional representation space of the symmetry algebra and as a result, the corresponding part of their spectra can be computed through a finite-matrix eigenvalue problem. In one dimension, more specifically, one uses the su(2) realization

$$T^0 = -j + z\frac{\partial}{\partial z}, \qquad T^+ = 2jz - z^2\frac{\partial}{\partial z}, \qquad T^- = \frac{\partial}{\partial z}, \qquad [T^0, T^\pm] = \pm T^\pm, \qquad [T^+, T^-] = 2T^0. \tag{1}$$

which for j as emi-integer, entails a $(2j+1)-$dimensional representation over the space spanned by the monomials $1, z, \ldots, z^{2j}$. One then takes for Hamiltonians

$$\tilde{H} = \sum_{a,b=0,\pm} C_{ab} T^a T^b + \sum_{a=0,\pm} C_a T^a . \tag{2}$$

Through a similarity transformation and possibly, a change of variable $x = x(z)$, these Hamiltonians can be cast in the form $-\frac{1}{2}\frac{d^2}{dx^2} + V(x)$ and a list of quasi-exactly solvable Schrödinger operators is thus obtained. We may remark that exactly solvable systems result if the parameter j does not appear in the Hamiltonian. This happens for instance in the case of the Morse potential $V(x) = A(e^{-2ax} - 2e^{-x})$ which plays an important rôle in molecular physics. The simplest example of a quasi-exactly solvable problem in one dimension is provided by the anharmonic oscillator with potential $V(x) = \frac{1}{2}\omega^2 x^6 - 2\beta\omega^2 x^4 + [2\beta^2\omega^2 - \omega(4j + \frac{3}{2})]x^2$. This system, as it turns out, is related to the two-dimensional anisotropic oscillator with Hamiltonian $H = -\frac{1}{2}(p_1^2 + p_2^2) + \frac{1}{2}\omega^2[x_1^2 + 4(x_2 - \beta)^2]$, $p_i = -i\frac{\partial}{\partial x_i}$ $i = 1, 2$. Indeed, owing to the higher symmetry of the Hamiltonian, the Schrödinger equation $H\psi(x_1, x_2) = E\psi(x_1, x_2)$ separates in parabolic coordinates as well in Cartesian coordinates[3]. We shall see that when parabolic coordinates are used, the separated equations coincide exactly with the one-dimensional equation associated to the anharmonic oscillator with potential $V(x)$.

We take great pleasure in dedicating this paper to Franco Iachello who has so brilliantly applied algebraic methods to domains as diverse as nuclear, high energy, atomic, and molecular physics. We hope that he will find interesting the occurrence of quadratic algebras in quantum mechanics. Let us also recall that Professor Iachello himself studied (among others) the dynamical symmetries of the Morse Hamiltonian[5]. He has shown that these can be traced back to the su(2) symmetry of the two-dimensional isotropic harmonic oscillator Schrödinger equation. In this case, one separates the variables in polar coordinates and after a change of variables, identifies the radial part as the Schrödinger equation for the particle in the one-dimensional Morse potential. It is in close analogy with this approach that the relation between the quasi-exactly anharmonic oscillator and the two-dimensional anisotropic oscillator with a 2 to 1 frequency ratio is established here.

The remainder of the paper is organized as follows. We first set our notation and identify the accidental degenaracies of a generalized 2-d anisotropic oscillator. We then present the quadratic dynamical algebra and show how the dynamics is resolved by constructing the representations of this algebra. We indicate in particular how the degeneracies are explained in this framework. Finally, we make the connection between the 2-d anisotropic oscillator and the quasi-exactly solvable anharmonic oscillator. Conclusions follow.

THE GENERALIZED 2-D ANISOTROPIC OSCILLATOR

We shall be considering a two-dimensional quantum system with Hamiltonian

$$H = H_1 + H_2 , \tag{3}$$

where

$$H_1 = \frac{1}{2}(p_1^2 + \omega^2 x_1^2) + \frac{1}{8}(\alpha^2 - 1)\frac{1}{x_1^2} , \tag{4a}$$

$$H_2 = \frac{1}{2}\left[(p_2^2 + 4\omega^2(x_2 - \beta)^2\right] , \tag{4b}$$

$p_i = -i\frac{\partial}{\partial x_i}$, $i = 1, 2$, and $\alpha, \beta, \omega \in R$. In the special case $\alpha = 1$, H governs the dynamics of an anisotropic oscillator with a 2 to 1 frequency ratio.

It will be convenient to introduce the annihilation and creation operators

$$a_1 = \frac{1}{\sqrt{2\omega}}(\omega x_1 + ip_1), \qquad a_1^\dagger = \frac{1}{\sqrt{2\omega}}(\omega x_1 - ip_1), \tag{5a}$$

$$a_2 = \sqrt{2\omega}\left[(x_2 - \beta) + \frac{i}{2\omega}p_2)\right], \qquad a_2^\dagger = \sqrt{2\omega}\left[(x_2 - \beta) - \frac{i}{2\omega}p_2)\right], \tag{5b}$$

that satisfy the commutation relations

$$[a_k, a_l^\dagger] = k\delta_{kl} \qquad [a_k, a_l] = [a_k^\dagger, a_l^\dagger] = 0 \quad k, l = 1, 2. \tag{6}$$

In terms of these operators,

$$\frac{H_1}{\omega} = a_1^\dagger a_1 + \frac{1}{4}(\alpha^2 - 1)(a_1 + a_1^\dagger)^{-2} + \frac{1}{2}, \tag{7a}$$

$$\frac{H_2}{\omega} = a_2^\dagger a_2 + 1. \tag{7b}$$

The spectrum of $H = H_1 + H_2$ is easily obtained by separating the Schrödinger equation in Cartesian coordinates. With $\psi(x_1, x_2) = \phi(x_1)\chi(x_2)$, the equation $H\psi = E\psi$ amounts to $H_1\phi(x_1) = E_1\phi(x_1)$ and $H_2\chi(x_2) = E_2\chi(x_2)$ with $E = E_1 + E_2$. We recognized in H_2, the Hamiltonian of a 1-d harmonic oscillator; its eigenfunctions are of the form $< x_2|n_2 > \propto < x_2|(a_2^\dagger)^{n_2}|0 >$ and have energy $E_2 = 2n_2 + 1$. The eigenvalues of H_1 can be determined by exploiting the conformal symmetry of this operator. Indeed, define the operators

$$\begin{aligned} B_1^+ &= (a_1^\dagger)^2 - \frac{1}{4}(\alpha^2 - 1)(a_1 + a_1^\dagger)^{-2} \\ &= a_1^\dagger(a_1 + a_1^\dagger) - \frac{H_1}{\omega} + \frac{1}{2} \end{aligned} \tag{8a}$$

$$B_1^- = (B_1^+)^\dagger . \tag{8b}$$

Clearly,

$$[H_1, B_1^\pm] = \pm 2\omega B_1^\pm . \tag{9}$$

It is also straightforward to check that

$$B_1^\pm B_1^\mp = \left[(\frac{H_1}{\omega} \mp 1) + \frac{|\alpha|}{2}](H_1 \mp 1) - \frac{\alpha}{2}\right] , \tag{10}$$

from where it follows that

$$[B_1^+, B_1^-] = -4\frac{H_1}{\omega}\,. \tag{11}$$

One immediatly identifies (9) and (11) as the commutation relations of $so(2,1)$. The relevant representations of this algebra are easily constructed. We known that H_1 is bounded below. There must therefore be states annihilated by B_1^- from which the eigenstates of H_1 are to be obtained by repeated action with B_1^+; correspondingly, the energy eigenvalue will raise above the minimal values in steps of 2ω. One thus sets

$$H_1|E_1>= E_{E_1}|E_1>\,, \tag{12a}$$

$$B_1^+|E_1>= s_{E_1+2\omega}|E_1+2\omega>\,, \tag{12b}$$

$$B_1^-|E_1>= s_{E_1}|E_1-2\omega>\,. \tag{12c}$$

From (12), one has $B_1^+B_1^-|E_1>= s^2_{E_1}|E_1>$ and with the help of (10) one obtains

$$s^2_{E_1} = \left(\frac{E_1}{\omega} - 1 + \frac{|\alpha|}{2}\right)\left(\frac{E_1}{\omega} - 1 - \frac{|\alpha|}{2}\right). \tag{13}$$

The lowest energies are the zeros of $s^2_{E_1}$ and following our discussion the spectrum of H_1 is found to be

$$E_1 = \left(2n_1 + 1 \pm \frac{|\alpha|}{2}\right)\omega \qquad n_1 = 0,1,\ldots \tag{14}$$

Note that the set $(2n_1 + 1 - |\alpha|/2)\omega$ is admissible only if $0 \le |\alpha| < 2$, otherwise there would be states with non-positive norms. When $\alpha = 1$, the two series in (14) combine to give the standard oscillator spectrum $E_1 = (n_1 + \frac{1}{2})\omega$, $n_1 = 0,1,\ldots$ The total energies are thus

$$\begin{aligned} E = E_1 + E_2 &= \left[2(n_1+n_2+1) \pm \frac{|\alpha|}{2}\right]\omega \\ &= (2N \pm \frac{|\alpha|}{2})\omega \qquad n_1,n_2 = 0,1,\ldots, \quad N = 1,2,\ldots \end{aligned} \tag{15}$$

Since the integer N is made out of the sum of the two integer-valued quantum numbers n_1 and n_2, accidental degeneracies occur and their multiplicity is in fact given by N. We shall now show that a dynamical quadratic algebra can be used to explain these degeneracies.

THE DYNAMICAL QUADRATIC ALGEBRA

Consider the operators

$$D = H_1 - H_2\,, \tag{16a}$$

$$C_+ = \omega B_1^+ a_2\,, \tag{16b}$$

$$C_- = C_+^\dagger = \omega B_1^- a_2^\dagger\,. \tag{16c}$$

It is easy to see that these quantities are conserved. Together with the Hamiltonian H and the identity operator I, these operators (16) generate the symmetry algebra of the anisotropic oscillator. The stucture relations are straightfowardly determined to be:

$$[H,D] = [H,C_+] = [H,C_-] = [H,I] = 0\,, \tag{17a}$$

$$[D,I] = [C_+,I] = [C_-,I] = 0\,, \tag{17b}$$

$$[D,C_\pm] = \pm 4\omega C_\pm\,, \tag{17c}$$

$$[C_+,C_-] = \frac{3}{2}D^2 + DH - \frac{1}{2}H^2 + \left(2 - \frac{\alpha^2}{2}\right)\omega^2 I\,. \tag{17d}$$

We note in (17d) that the commutator of C_+ and C_- is given as a quadratic combination of D and H. The operators H, D, C_+, C_- and I do not therefore define a Lie algebra. In a representation on an energy eigenstate, H is constant; it may then not be taken as a generator inequivalent to I. Since we are dealing with differential operators, the Jacobi identity is satisfied. It is also not difficult to see that this algebra possesses a Casimir operator K commuting with all the generators. Explicitly,

$$K = \frac{\omega}{2}\{C_+,C_-\} + \frac{1}{8}D^3 + \frac{1}{8}HD^2 - \left[\frac{1}{8}H^2 + (\frac{\alpha^2}{8} - \frac{3}{2})\omega^2\right]D\,. \tag{18}$$

Of course since H is central, we might equivalently take as Casimir operator

$$\tilde{K} = K - \frac{1}{8}H[H^2 - (4-\alpha^2)\omega^2]\,. \tag{19}$$

It is checked that $\tilde{K} = 0$ in our specific realization. Clearly the operators $C_\pm$ transform the degenerate eigenstates of H among themselves. In fact, the representations of the quadratic algebra (17) that we need, can be easily worked out; they will be seen to determine the spectrum of H and to completly account for its degeneracies.

Define the basis vectors $|E,d>$ as the simultaneous eigenstates of H and D:

$$H|E,d>= E|E,d>\,, \qquad D|E,d>= d|E,d>\,. \tag{20}$$

As far as the Schrödinger equation is concerned, separation of variables in Cartesian coordinates is associated to this choice: using still $\psi(x_1,x_2) = \phi(x_1)\chi(x_2)$, $D\psi(x_1,x_2) = d\psi(x_1,x_2)$ with $d = E_1 - E_2$.

In view of (17c) and the fact that $C_+ = C_-^\dagger$, we must have

$$C_+|E,d>= t_{d+4\omega}|E,d+4\omega>\,, \tag{21a}$$

$$C_-|E,d>= t_d|E,d-4\omega>\,. \tag{21b}$$

It follows of course that $C_+C_-|E,d>= t_d^2|E,d>$. Now with the help of (17d), (18) and (19), we find that

$$C_+C_- = \frac{1}{\omega}\tilde{K} + \frac{1}{8\omega}(H+D-2\omega+\omega|\alpha|)(H+D-2\omega-\omega|\alpha|)(H-D+2\omega)\,. \tag{22}$$

We have already pointed out that in our realization $\tilde{K} = 0$, we thus get here

$$t_d^2 = \frac{1}{8\omega}(d-d_+)(d-d_-)(\tilde{d}-d)\,, \tag{23}$$

where

$$d_{\pm} = -E + \omega(2 \pm |\alpha|)\,, \tag{24a}$$

$$\tilde{d} = E + 2\omega\,. \tag{24b}$$

(Naturally, this result could have been obtained by working directly with the expressions (16) and using (7b) and (10).) At this point, two requirements on the representations of the symmetry algebra allow to determine the spectrum of D and H. First, these representations should be unitary and hence we should have $t_d^2 \geq 0$. Second, they should be finite-dimensional as there are only finitely many degenerate energy eigenstates. This last condition implies that there should be D-eigenvalues d_{min} and d_{max}, such that $C_-|E, d_{min} >= 0$, $C_+|E, d_{max} >= 0$ or equivalently such that $t_{d_{min}} = 0$, $t_{d_{max}+4\omega} = 0$. The eigenvalues of D would thus run from d_{min} to d_{max} in steps of 4ω. A first series meeting the above two critera has

$$d = d_+, d_+ + 4\omega, \ldots, \tilde{d} - 4\omega\,. \tag{25}$$

If we denote by N $(= 1, 2, \ldots)$ the number of degenerate H-eigenstates and hence, the dimension of the representations, we have

$$(\tilde{d} - 4\omega) - d_+ = 4(N-1)\omega\,. \tag{26}$$

Using the definitions (24), we see that (26) entails the energy spectrum formula:

$$E = (2N + \frac{|\alpha|}{2})\omega \qquad N = 1, 2, \ldots \tag{27}$$

If $0 \leq |\alpha| < 2$, another series is present:

$$d = d_-, d_- + 4\omega, \ldots, \tilde{d} - 4\omega\,. \tag{28}$$

Note that t_d^2 is also non-negative here because the gap $-2|\alpha|\omega$ between d_- and d_+ does not dominate the increment 4ω. As before, if there are N eigenstates of D with the eigenvalues (28), they are seen to have energy

$$E = \left(2N - \frac{|\alpha|}{2}\right)\omega \qquad N = 1, 2, \ldots \tag{29}$$

This is how the quadratic symmetry algebra (17) enables one to obtain the spectrum of H - compare (15) with (27) and (29).

SEPARATION OF VARIABLES AND THE QUASI-EXACTLY SOLVABLE ANHARMONIC OSCILLATOR

For most known systems with accidental degeneracies and hidden symmetries, the Schrödinger equation admits seperation in more than one coordinate systems. The present anisotropic oscillator is no exception. In two dimensions, there is a correspondance between coordinate systems in which variables separate in the Schrödinger equation and the constants of motions that are quadratic in momenta[3]. So far with our problem, we have only exploited the fact that $H\psi = E\psi$ separates in Cartesian coordinates. We have indicated that this coordinate system is singled out when the constant of motion $D = \frac{1}{2}(p_1^2 - p_2^2) + \frac{1}{2}\omega^2 x_1^2 + \frac{1}{8}(\alpha^2 - 1)\frac{1}{x_1^2} - 2\omega^2(x_2 - \beta)^2$ is diagonalized. Consider now the following hermitian combination of the symmetry generators of H:

$$R = \sqrt{2\omega}(C_+ + C_-) - 2\beta(H + D)\,. \tag{30}$$

This conserved quantity is also quadratic in the momenta; in fact, in Cartesian coordinates, it takes the form

$$R = p_1 L + L p_1 + 2\omega^2 x_1(x_2 - 2\beta) + \frac{1}{2}(1 - \alpha^2)\frac{x_2}{x_1^2}\,, \tag{31}$$

where $L = x_1 p_2 - x_2 p_1$. Diagonalizing R instead of D leads to separation in the parabolic coordinates (ξ_+, ξ_-) that are related to the Cartesian ones as follows

$$x_1 = \xi_+ \xi_- \qquad x_2 = \frac{1}{2}\left(\xi_+^2 - \xi_-^2\right)\,. \tag{32}$$

From (30) we see that R is tridiagonal in the bases $|E, d_n^{\pm} >$, $d_n^{\pm} = d_{\pm} + 4n\omega$, $n = 0, 1, \ldots, N-1$:

$$R|E, d_n^{\pm} >= \sqrt{2\omega}(t_{d_{n+1}^{\pm}}|E, d_{n+1}^{\pm} > + t_{d_n^{\pm}}|E, d_{n-1}^{\pm} >) - 2\beta(E + d_n^{\pm})|E, d_n^{\pm} >\,. \tag{33}$$

Its eigenvalues $\omega\Lambda$ are given by the zeroes of the characteristic polynomial of Nth order associated to this matrix. Let $|E, \Lambda >$ be the corresponding eigenstates

$$R|E, \Lambda >= \omega\Lambda|E, \Lambda >\,. \tag{34}$$

A three-term recurrence relation for the overlap function $< E, \Lambda|E, d_n^{\pm} >$ is simply obtained in our algebraic framework. Take

$$< E, \Lambda|E, d_n^{\pm} >=< E, \Lambda|E, d^{\pm} > P_n^{\pm}(\Lambda)\,. \tag{35}$$

Clearly $P_0^{\pm}(\Lambda) = 1$. From (33) and (34), the coefficients $P_n^{\pm}(\Lambda)$ are seen to satisfy

$$\omega\Lambda P_n^{\pm}(\Lambda) = \sqrt{2\omega}(t_{d_{n+1}^{\pm}} P_{n+1}^{\pm}(\Lambda) + (t_{d_n^{\pm}} P_{n-1}^{\pm}(\Lambda)) - 2\beta(E + d)P_n^{\pm}(\Lambda)\,. \tag{36}$$

If we now set

$$P_n^{\pm}(\Lambda) = (2\omega)^{n/2}(\prod_{i=1}^{n} t_{d_i^{\pm}})^{-1} Q_n^{\pm}(\Lambda) \qquad n = 1, \ldots, N-1 \tag{37}$$

and use the definitions (24) as well as $E = (2N \pm \frac{|\alpha|}{2})\omega$, we find that the factors $Q_n^{\pm}(\Lambda)$ obey the three-term recurrence relation

$$Q_{n+1}^{\pm}(\Lambda) = [\frac{\Lambda}{2} + \beta(4n + 2 \pm \alpha)]Q_n^{\pm}(\Lambda) - 4\omega n(n \pm \frac{|\alpha|}{2})(N - n)Q_{n-1}^{\pm}(\Lambda) \tag{38}$$

$$n = 0, 1, \ldots, N-1\,; \qquad Q_0^{\pm}(\Lambda) = 1\,.$$

These functions of Λ, thus define orthogonal polynomial sets. (This is true for the $Q_n^-(\Lambda)$ as long as $|\alpha| < 2$.) Let us now make explicit our claim that a quasi-exactly

solvable anharmonic oscillator descents from the 2-d anisotropic oscillator. In the parabolic coordinates (32), H and R take the form

$$H = \frac{1}{2}(\xi_+^2 + \xi_-^2)^{-1} \times [\pi_+^2 + \pi_-^2 + \omega^2(\xi_+^6 + \xi_-^6) - 4\beta\omega^2(\xi_+^4 - \xi_-^4) + 4\beta^2\omega^2(\xi_+^2 + \xi_-^2) + \frac{1}{4}(\alpha^2 - 1)(\xi_+^{-2} + \xi_-^{-2})], \tag{39a}$$

$$R = (\xi_+^{-2} + \xi_-^{-2})^{-1} \times [\xi_+^{-2}\pi_+^2 - \xi_-^{-2}\pi_-^2 + \omega^2(\xi_+^4 - \xi_-^4) - 4\beta\omega^2(\xi_+^2 + \xi_-^2) + \frac{1}{4}(\alpha^2 - 1)(\xi_+^{-4} - \xi_-^{-4})], \tag{39b}$$

where $\pi_\pm = -i\frac{\partial}{\partial\xi_\pm}$. If we set

$$< \xi_+, \xi_- | E, \Lambda > = \psi_{E,\Lambda}(\xi_+, \xi_-) = u_\Lambda^{(+)}(\xi_+) u_\Lambda^{(-)}(\xi_-), \tag{40}$$

we readily see that the eigenvalue equations $H\psi_{E,\Lambda} = E\psi_{E,\Lambda}$ and $R\psi_{E,\Lambda} = \omega\Lambda\psi_{E,\Lambda}$ amount to the following two separated equations for the functions $u_\Lambda^{(\pm)}(\xi_\pm)$:

$$\left[\frac{1}{2}\pi_\pm^2 + V_E(\xi_\pm)\right] u_\Lambda^{(\pm)} = \pm\omega\Lambda u_\Lambda^{(\pm)}, \tag{41}$$

where

$$V_E(\xi_\pm) = \frac{1}{2}\omega^2\xi_\pm^6 + \frac{1}{8}(\alpha^2 - 1)\frac{1}{\xi_\pm^2} - 2\beta\omega^2\xi_\pm^4 + (2\beta^2\omega^2 - E)\xi_\pm^2 \tag{42}.$$

We recognize that the equations (41) have the form of one-dimensional Schrödinger equations in a (generalized) oscillator potential with anharmonicity terms in ξ^4 and ξ^6. (Note that $E = (2N \pm \frac{|\alpha|}{2})\omega$ enters as a parameter in the potential.) Remarkably these equations coincide with the Schrödinger equation associated to the simplest example of a quasi-exactly solvable system. To see this, return to the realization given in (1) of the $su(2)$ algebra. If we make the change of variable $z = \xi^2$ and take

$$H = e^{-a}\tilde{H}e^{a}, \tag{43}$$

with

$$\tilde{H} = -2T^0T^- - (N + 1 \pm |\alpha|)T^- - 4\beta\omega T^0 - 2\omega T^+ \tag{44a}$$

and

$$a = \omega\frac{\xi^4}{4} - \beta\omega\xi^2 - \frac{1}{2}(1 \pm |\alpha|)\ln\xi, \tag{44b}$$

we find after a short computation that

$$H = \frac{1}{2}\pi^2 + V_E(\xi) + \beta\omega\left(E \pm \frac{|\alpha|}{2}\right) \tag{45}$$

for $\pi = -i\frac{\partial}{\partial\xi}$. Up to a constant this agrees with (41) – (42). So, if one is considering the one-dimensional Schrödinger equation

$$[\frac{1}{2}\pi^2 + V_E(\xi)]u(\xi) = \epsilon u(\xi) \tag{46}$$

with $E = (2N \pm \frac{|\alpha|}{2})\omega$, the eigenvalues and eigenfunctions of the N lowest levels can be obtained exactly. This is done as follows using the connection between (46) and the 2-d anisotropic oscillator. On the one hand, the first N energies ϵ are given by the eigenvalues of R. On the other hand, the two-dimensional wave functions $\psi_{E,d_n^\pm}(x_1, x_2) = < x_1, x_2 | E, d_n^\pm >$ are very easily determined in Cartesian coordinates. In terms of the overlap coefficients $< E, \Lambda | E, d_n^\pm >$ for which we have a recursion relation, the eigenfunctions of R are given by

$$< x_1, x_2 | E, \Lambda > = \sum_{n=0}^{N-1} < E, \Lambda | E, d_n^\pm > < x_1, x_2 | E, d_n^\pm > . \tag{47}$$

One then passes to the parabolic coordinate system (32), in which we known that separation occurs and write $< x_1, x_2 | E, \Lambda > = u^{(+)}(\xi_+)u^{(-)}(\xi_-)$ to identify $u^{(+)}(\xi)$ as the wave function associated to the energy level $\epsilon = \omega\Lambda$. For an exposition of how these same eigenvalues and eigenfunctions are found by exploiting the realization (43) – (44) of H see Ref. [7].

CONCLUSIONS

Summarizing, we have provided here a novel algebraic analysis of the much studied[2,3,9] 2-dimensional anisotropic harmonic oscillator(3)-(4). We have illustrated with this very simple example that quadratic algebras do arise naturally in the description of systems with accidental degeneracies. We hope to have shown that the prejudice according to which symmetry algebras should be Lie algebras or superalgebras is not well founded. We have also explained how quasi-exactly solvable systems could be obtained through dimensional reduction by exploiting the fact that (solvable) systems with higher symmetries usually separate in more than one coordinate system.

For two dimensions, there exists a classification of potentials with accidental degeneracies for which the Schrödinger equation separates in two coordinate systems[3]. In addition to the one that we have considered in detail, the list consists of

1. $V(x_1, x_2) = \alpha(x_1^2 + x_2^2) + \frac{\beta}{x_1^2} + \frac{\gamma}{x_2^2}$
2. $V(\xi_+, \xi_-) = \frac{\alpha + \beta/\xi_+^2 + \gamma/\xi_-^2}{\xi_+^2 + \xi_-^2}$
3. $V(\xi_+, \xi_-) = \frac{\alpha + \beta\xi_+ + \gamma\xi_-}{\xi_+^2 + \xi_-^2}$

In all of these cases, we have shown that the symmetries responsible for the accidental degeneracies are again naturally described by quadratic algebras. These results will be reported in full in a forthcoming publication[10].

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REFERENCES

1. *Dynamical Groups and Spectrum generating Algebras*, vol. 1 and 2, Bohm, A., Barut, A. and Ne'eman, Y., eds. (World Scientific, Singapore, 1988).

2. Demkov, Yu., The definition of the symmetry group of a quantum system. The anisotropic oscillator, Soviet Phys. JETP **17**, 1349-1351 (1963).

3. Winternitz, P., Smorodinskii.Ya. A., Uhlir, M. and Fris, I., Symmetry groups in classical and quantum mechanics, Soviet J. Nucl. Phys. **4**, 444-450 (1967).

4. Gal'bert, O. F., Granovskii, Ya. I. and Zhedanov, A. S., Dynamical symmetry of anisotropic singular oscillator, Phys. Lett. **A 153**, 177-180 (1991); Granovskii, Ya. I., Zhedanov A. S. and Lutzenko, I. M., Quadratic algebra as a "hidden" symmetry of the Hartmann potential, J. Phys. **A 24**, 3887-3894 (1991).

5. Higgs, P. W., Dynamical symmetries in a spherical geometry I, J. Phys. **A 12**, 309-323 (1979); Zhedanov, A. S., The "Higgs algebra" as a quantum deformation of su(2), Mod. Phys. Lett **A 7**, 507-512 (1992).

6. Granovskii, Ya. I., Lutzenko, I. M., and Zhedanov A. S., Mutual integrability, quadratic algebras and dynamical symmetry, Ann. Phys. **217**, 1-20 (1992).

7. Shifman, M. A., New findings in quantum mechanics (partial algebraization of the spectral problem), Int. J. Mod. Phys. A **4**, 2897-2952 (1989).

8. Alhassid, Y., Gürsey, F. and Iachello F., Group theory approach to scattering, Ann. Phys. **148**, 346-380 (1983).

9. Duimo, F. and Zambotti, G., Dynamical group of the anisotropic harmonic oscillator, Nuovo Cimento A **43**, 1203-1207 (1966); Cisneros, A. and McIntosh, H. V., Search for universal symmetry group in two dimensions, J. Math. Phys. **11**, 870-895; Moshinsky, M., Patera, J. and Winternitz, P., Canonical transformation and accidental degeneracy. III A unified approach to the problem, J. Math. Phys. **16**, 82-92 (1975).

10. Létourneau, P. and Vinet, L., Quadratic algebras in two-dimensional quantum mechanics, in preparation.