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NONLINEAR ALGEBRAS AND SPECTRAL PROBLEMS[1]

Vyacheslav Spiridonov[2]

Laboratoire de Physique Nucléaire, Université de Montréal,
C.P. 6128, succ. A, Montréal, Québec, H3C 3J7, Canada

Abstract

An application of the particular type of nonlinear operator algebras to spectral problems is outlined. These algebras are associated with a set of one-dimensional self-similar potentials, arising due to the q-periodic closure $f_{j+N}(x) = qf_j(qx)$, $k_{j+N} = q^2k_j$ of a chain of coupled Riccati equations (dressing chain). Such closure describes q-deformation of the finite-gap and related potentials. The $N = 1$ case corresponds to the q-oscillator spectrum generating algebra. At $N = 2$ one gets a q-conformal quantum mechanics, $N = 3$ set of equations describes a deformation of the Painlevé-IV transcendent.

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[2]On leave of absence from the Institute for Nuclear Research, Moscow, Russia

Quantum algebras, or q-deformations of Lie algebras, attracted much attention during the last years. They have appeared in conformal field theories, spin-chain models, in the construction of link invariants, and so on. Another type of nonlinear algebras, known as W-algebras, also formed the subject of intensive investigations. It is quite natural to ask on the applications of these objects within the context of Sturm-Liouville type spectral problems. The answer is not so straightforward and most of the attempts done in this direction cary phenomenological character - the deformations of spectra are introduced in an *ad hoc* manner, without proper definition of arising operators. As we shall show below, the problem can be treated in a rigorous fashion on the basis of standard concepts of continuous space physics. Moreover, nonlinear algebras will appear very naturally as an inevitable consequence of the analysis of exactly solvable potentials.

The consideration will be limited to the simplest one-dimensional Schrödinger equation

$$-\psi''(x) + U(x)\psi(x) = \lambda\psi(x), \tag{1}$$

describing a particle moving in some potential $U(x)$. (The prime in (1) and below always denotes derivative with respect to real coordinate x, $x \in R$.) The quantum mechanical spectral problems, associated to (1), traditionally provide good place for probing new group-theoretical ideas. This is inspired by the general belief that any kind of regularity in spectral data is generated by some symmetry algebra of a Hamiltonian. The qualitative understanding of peculiarities of a given system is reached when an operator algebra governing the map of physical states onto each other is explicitly constructed. The most advanced approach to building of such symmetry transformations, which we are going to exhibit, is connected with the technique developed by Darboux long time ago [1]. Within quantum mechanics, it is known as factorization method [2]. In the theory of integrable nonlinear evolution equations, it was generalized and named as the dressing method [3]. Later, Darboux transformations were identified as supersymmetry transformations, mixing bosonic and fermionic degrees of freedom in specific models [4]. Some further parastatistical generalization of the latter interpretation was suggested in Ref.[5]. Using this method, we shall describe a set of self-similar one-dimensional potentials whose discrete spectra are composed from a number of geometric series. Among the corresponding nonlinear spectrum generating algebras we shall find a q-deformed Heisenberg-Weyl algebra and a quantum conformal algebra $su_q(1,1)$, where parameter q will have the meaning of interaction constant. However, we shall not discuss the Hopf algebra structure which seems to be irrelevant in the present context (perhaps it will be needed in higher dimensional models).

Quantization of the spectral parameter λ emerges due to particular boundary conditions imposed upon the wave functions $\psi(x)$. Just for an illustration we exhibit one of the possible (self-adjoint) conditions:

$$\psi'(x_1) = h_1\psi(x_1), \qquad \psi'(x_2) = h_2\psi(x_2), \tag{2}$$

where x_1 and x_2 are two different points on the line, and h_1, h_2 are arbitrary real constants. Here we would like to mention the powerful restriction on asymptotic growth of eigenvalues $\{\lambda_n\}$ for the potentials without singularities in $[x_1, x_2]$. Namely, $\{\lambda_n\}$ can not grow faster than n^2 at $n \to \infty$ [6]. This observation immediately restricts the region of applications of q-deformed commutation relations which often formally produce exponentially growing discrete spectra. Below we imply the condition of square normalizability of wave functions, $\psi(x) \in L^2[-\infty, \infty]$, for the discrete energy states. For singular potentials one still will need additional boundary conditions. The (quasi)periodic potentials with zonal structure of spectra will also appear in the consideration.

Let us first present some basic principles of the factorization method. The main object appearing after successive factorizations of the stationary one-dimensional Schrödinger operator (1) is the following chain of coupled Riccati equations (the dressing chain):

$$f_j'(x) + f_{j+1}'(x) + f_j^2(x) - f_{j+1}^2(x) = k_j, \qquad j \in Z \tag{3}$$
$$k_j \equiv \lambda_{j+1} - \lambda_j,$$

where k_j (λ_j) are some constants. The Hamiltonians L_j associated to (3) have the form,

$$L_j = p^2 + f_j^2(x) - f_j'(x) + \lambda_j = a_j^+ a_j^- + \lambda_j, \tag{4}$$
$$a_j^\pm = p \pm i f_j(x), \qquad p \equiv -i d/dx.$$

Conditional isospectrality of L_j and L_{j+1} follows from the intertwining relations

$$L_j a_j^+ = a_j^+ L_{j+1}, \qquad a_j^- L_j = L_{j+1} a_j^-.$$

Any exactly solvable spectral problem with infinite number of discrete levels can be represented in the form (3), (4) with λ_j being the Hamiltonian eigenvalues:

$$L_j \psi_n^{(j)} = E_n^{(j)} \psi_n^{(j)}, \qquad E_n^{(j)} = \lambda_{n+j}, \quad n = 0, 1, 2, \ldots$$
$$\psi_n^{(j)} \propto a_j^+ \psi_{n-1}^{(j+1)}, \qquad \psi_{n-1}^{(j+1)} \propto a_j^- \psi_n^{(j)}, \tag{5}$$
$$\psi_0^{(j)}(x) = \exp\left(-\int^x f_j(y)dy\right).$$

In this case $\lambda_{j+1} > \lambda_j$ and all wave functions $\psi_n^{(j)}$ are square integrable, with n nodes inside taken coordinate region. For Hamiltonians with finite number of bound states normalizability of $\psi_0^{(j)}$ truncates at some j. A large list of potentials, whose spectra are easily found by the factorization method, is given in the first paper of Ref.[2]. In general, however, parameters λ_j do not coincide with physical eigenvalues since for a given potential they could be chosen arbitrarily.

In order to solve underdetermined system (3) one has to impose some closure conditions. At this stage it is an art of a researcher to find such an ansatz, which allows to generate infinite number of f_j and k_j by simple recurrence relations from fewer entries. Most of the old known examples are generated by the choice $f_j(x) = a(x)j + b(x) + c(x)/j$, where a, b, c are some functions determined from setting powers of j in (3) equal to zero. Three years ago Shabat and Yamilov considered the problem from another point of view [7]. They suggested to treat (3) as some infinite-dimensional dynamical system and to find potentials corresponding to finite-dimensional truncations of the chain. In particular, it was shown that very simple periodic closure conditions:

$$f_{j+N}(x) = f_j(x), \qquad \lambda_{j+N} = \lambda_j, \tag{6}$$

for N odd lead to the potentials with zonal structure of spectra which are described by (hyper)elliptic functions and called finite-gap potentials (parameters λ_j represent physical eigenvalues when they coincide with the boundaries of gaps). First non-trivial example appears at $N = 3$ and corresponds to the one-gap Lamé equation. After rewriting corresponding potential in the Weierstrass form and demanding the wave functions to vanish at singular points one gets a different solvable spectral problem.

potential $\propto 1/x^2$. We guess that at $q < 1$ there exist such $k_{1,2}$ that series converge for arbitrarily large x. The condition $a(a+1) \geq 3/4$ guarantees that normalizable wave functions and their first derivatives vanish at zero [8]. The spectrum of such system would arise from only one geometric series (second is eliminated by boundary conditions).

For non-singular at zero solutions one has $f_{1,2} = \sum_{i=0}^{\infty} b_i^{(1,2)} x^i$, where $b_0^{(1,2)}$ are two arbitrary constants. Again, in general series diverge at $q \to 1$. Particular choice of initial conditions gives the solution which in this limit corresponds to (8) with a coordinate shift. Depending on the values of k_1 and k_2 the limit $q \to 0$ recovers either the smooth, two-level potential with bound state energies at 0 and k_1, or its analytically continued partner. Only first of these corresponds to (22) at $N = 2$, each series belonging to independent representation of $su_q(1,1)$. Moreover, it is this solution that reduces to the q-oscillator one (12)-(16) (with q replaced by $q^{1/2}$) after restrictions $f_2(x) = q^{1/2} f_1(q^{1/2}x)$, $k_2 = qk_1$. At $q \to 1$ spectral series become equidistant which means that potentials start to be unbounded at space infinities. Because of the nice connection with ordinary conformal model [8], we suggest to call the $N = 2$ system as q-deformed conformal quantum mechanics.

For the $N = 3$ system of equations:

$$\begin{aligned} f_1'(x) + f_2'(x) + f_1^2(x) - f_2^2(x) &= k_1, \\ f_2'(x) + f_3'(x) + f_2^2(x) - f_3^2(x) &= k_2, \\ f_3'(x) + q f_1'(qx) + f_3^2(x) - q^2 f_1^2(qx) &= k_3, \end{aligned} \tag{27}$$

one can exclude $f_{2,3}(x)$ and get

$$f_{2,3}(x) = -\frac{1}{2} f(x) \mp \frac{1}{2f(x)} (f'(x) + k_2), \tag{28}$$

$$f(x) = \frac{1 + \sqrt{q} T_q}{2} f_1(x) + \frac{1 - \sqrt{q} T_q}{2} \int_0^x f_1^2(y) dy - \frac{\omega}{2}(x - x_0), \tag{29}$$

$$f''(x) = \frac{f'^2(x)}{f(x)} + 2f(x)(f_1^2(x) + f_1'(x) - \frac{1}{4} f^2(x) - f'(x) - k_1 - \frac{1}{2} k_2) - \frac{k_2^2}{2f(x)}, \tag{30}$$

where x_0 is a constant of integration. At $q = 1$ one has $f_1(x) = f(x) + \omega(x - x_0)/2$ and (30) becomes the PIV equation (10). So, the system (29), (30) describes a q-deformation of the PIV transcendent.[3] In fact, all functions $f_i(x)$ satisfy one combersome equation with different choices of the parameters k_i. As a result, the relations (28) give new solutions of the q-PIV system in terms of a known one: $f_2(x; k_1, k_2, k_3) = f_1(x; k_2, k_3, q^2 k_1)$, $f_3(x; k_1, k_2, k_3) = f_1(x; k_3, q^2 k_1, q^2 k_2)$. Existence of these nonlinear maps is a result of the hidden self-similarity.[4]

The notion of q-periodicity (17) and corresponding algebraic relations (19), (20) are central in this paper. However, above we just outlined some of the properties of self-similar potentials. It is quite interesting to know what kind of potentials one gets as a result of deformation of the finite-gap potentials, i.e. when for some j one has $k_j + k_{j+1} + \ldots + k_{j+N-1} = 0$. There are other possibilities in addition to the mentioned ones. For example, the coordinate x and parameter q were taken to be real and nothing prevents from the consideration of complex values as well. An

[3] For a different approach to q-deformation of the Painlevé equations see Ref.[17].

[4] Such transformations for the solutions of standard Painlevé equations were discussed in a recent paper [18].

interesting situation is described when q is a root of unity, $q^m = 1$. From the relations (17) one easily sees that now

$$f_{j+mN}(x) = q^m f_j(q^m x) = f_j(x), \qquad k_{j+mN} = q^{2m} k_j = k_j, \tag{31}$$

which is a subcase of (6) because

$$\lambda_{j+mN} - \lambda_j = (1 - q^{2m})(k_j + k_{j+1} + \ldots + k_{j+N-1})/(1 - q^2) = 0.$$

In a short joint research with A.Shabat the $N = 1$, $q^3 = 1$ case was analyzed and appearance of the simplest Lamé equation for the equianharmonic Weierstrass function was demonstrated. Corresponding spectral problem is known to be solvable [19]. A charming property of the self-similar systems in these cases is that operationally they are naturally characterized not by the generators of the order mN polynomial algebras but rather by their m-th operator roots which are well defined and satisfy simpler (although unusual) commutation relations. Unfortunately, the general analytical structure of the q-transcendents is not known and this does not allow to analyze analytical continuation of the solutions found for the specific values of q. Also, there should exist some infinite-gap potentials which are reduced to the self-similar ones in the limit of zero widths of the gaps.

Different problems appear when the scaling operator T_q (15) in the definition of $A^{\pm}$ is replaced by the translation operator, $T_a f(x) = f(x + a)$. Instead of (13) we then have

$$f'(x) + f'(x + a) + f^2(x) - f^2(x + a) = constant. \tag{32}$$

This equation may be considered also as a special limiting case of (13). Since $q = 1$, we have a realization of the ordinary Heisenberg-Weyl algebra. The full effect of non-zero parameter a in (32) is not known to the author. For monotonically growing at $x \to \pm\infty$ functions the leading asymptotic term $f^2(x) - f^2(x + a)$ has different signs at space infinities which means that growing solutions of (32) (if any) should oscillate. Higher dimensional generalizations of the presented construction are unknown to the author (except of the simple cases when variables separate and the problem becomes effectively one-dimensional). Two final remarks are in order. First, the factorization method allows to replace superpotentials $f_j(x)$ by hermitian matrix functions [20], in which case right hand sides of Eqs. (3), (14) are proportional to unit matrices. Second, there may exist an interesting interrelation between the described self-similar potentials and the wavelet analysis [21] where affine transformations (dilatations and translations) generate orthonormal bases of the Hilbert space.

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Just slight modification of (6):

$$f_{j+N}(x) = f_j(x), \qquad k_{j+N} = k_j, \tag{7}$$

describes essentially more complicated potentials, although one has only one additional free parameter $\omega = k_j + k_{j+1} + \ldots + k_{j+N-1} = \lambda_{j+N} - \lambda_j \neq 0$. It is easy to see that (7) at $N = 1$ gives the harmonic oscillator problem, whereas (6) is degenerate. The $N = 2$ system coincides with the general conformal quantum mechanical model [8],

$$f_{1,2}(x) = \frac{1}{2}(\pm\frac{k_1 - k_2}{k_1 + k_2}\frac{1}{x} + \frac{k_1 + k_2}{2}x). \tag{8}$$

Already $N = 3$ case leads to transcendental potentials [9], namely, $f_j(x)$ depend now on solutions of the Painlevé-IV equation [1]:

$$f_1(x) = \frac{1}{2}\omega x + f(x), \qquad f_{2,3}(x) = -\frac{1}{2}f \mp \frac{1}{2f}(f' + k_2), \quad \omega = k_1 + k_2 + k_3, \tag{9}$$

$$f'' = \frac{f'^2}{2f} + \frac{3}{2}f^3 + 2\omega x f^2 + (\tfrac{1}{2}\omega^2 x^2 + k_3 - k_1)f - \frac{k_2^2}{2f}. \tag{10}$$

To the author's knowledge this is the first example, when Painlevé transcendent appears in a quantum mechanical context.

Let us take two unconstrained Hamiltonians L_j and L_{j+N} from the chain (4) and assume that both are self-adjoint. The map from unnormalized wave functions of L_{j+N} to those of L_j is performed by successive action of the operators a_l^+,

$$\psi_{n'}^{(j)}(x) = K_j^+ \psi_n^{(j+N)}(x), \qquad K_j^+ = a_j^+ a_{j+1}^+ \ldots a_{j+N-1}^+, \quad K_j^- \equiv (K_j^+)^\dagger$$

$$K_j^+ L_{j+N} = L_j K_j^+, \qquad L_{j+N} K_j^- = K_j^- L_j. \tag{11}$$

For a discrete spectrum the labelling of levels differs by some integer, $n' = n + m$, $|m| \leq N$. Qualitatively, the situation is depicted in Fig.1, where dashed lines are used to denote non-physical (unnormalizable) levels of the intermediate Hamiltonians. The latter exist if superpotentials f_j have strong enough singularities, in which case the L_j's are not isospectral and operators $a_l^\pm$ are not well defined on some physical states.

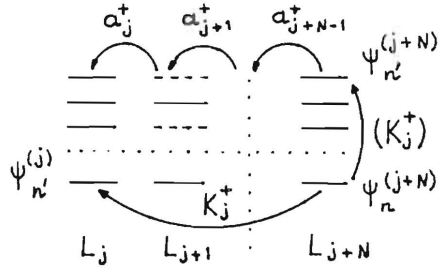

Fig.1 The action of K_j-operator on the L_{j+N}-states in arbitrary case (below) and for $L_{j+N} = L_j + const$ (on the right).

Now we restrict potential of L_{j+N} to be equal to that of L_j up to some simple transformation. Then $K_j^\pm$ map eigenstates of L_j onto each other, $\psi_{n'}^{(j)} \leftrightarrow \psi_n^{(j)}$ (there may be zero modes). The simplest case is realized when all states are mapped onto themselves, i.e. when $n' = n$. This is equivalent to $L_{j+N} = L_j$, or (6). From (11) operators $K_j^\pm$ are seen to be integrals of motion [7]. For $n' \neq n$ a spectrum generating situation arises. Taking $\lambda_{j+N} \neq \lambda_j$ and substituting $L_{j+N} = L_j + \lambda_{j+N} - \lambda_j$ into (11) one gets the standard ladder relations. Integrability of these finite-dimensional truncations of the chain (3) is analyzed in Refs.[7, 9]. The next step is to classify intrinsically infinite-dimensional solutions of (3). Let us show that q-deformations naturally appear in a consideration of the related exactly solvable spectral problems.

A peculiar potential was found in Ref.[10] by the following self-similarity constraint:

$$f_j(x) = q^j f(q^j x), \qquad k_j = q^{2j}k, \qquad 0 < q < 1, \quad k > 0, \tag{12}$$

which gives a solution of (3) provided $f(x)$ satisfies the equation

$$f'(x) + qf'(qx) + f^2(x) - q^2 f^2(qx) = k. \tag{13}$$

Quantum algebraic content of this model was uncovered in Ref.[11], where it was shown that q-deformed Heisenberg-Weyl algebra [12],

$$A^- A^+ - q^2 A^+ A^- = k, \tag{14}$$

is implemented by the choice

$$A^+ = (p + if(x))T_q, \quad A^- = T_q^{-1}(p - if(x)) = (A^+)^\dagger,$$

$$T_q f(x) = \sqrt{q} f(qx). \tag{15}$$

Intertwining relations between $A^\pm$ and Hamiltonian $H = A^+A^- - k/(1-q^2)$ easily generate the spectrum

$$HA^\pm = q^{\pm 2}A^\pm H, \quad \Rightarrow \quad E_n = -kq^{2n}/(1-q^2). \tag{16}$$

A deformation of supersymmetric quantum mechanics, inspired by this model, was suggested in Ref.[13]. The main idea is very simple - one has to replace superpartner Hamiltonian by that obtained after affine transformation (i.e. dilatation and translation) and adjust kinetic term to the standard form. Degeneracies of levels are removed and energy split is proportional to $1 - q^2$, where q is the scaling parameter. Within this scheme, Eq.(13) is a condition of homegeneity of magnetic field alond third axis for a spin-1/2 particle moving on the line. This construction is easily generalized to the particular parasupersymmetric model defined by unification of sequential members of the chain (4) into diagonal $(N+1) \times (N+1)$ matrix. Acting on each subhamiltonian by different affine transformation group elements and rearranging kinetic terms one would get multiparameter deformation of parasupersymmetric algebraic relations. Following the consideration of Ref.[5], one may impose various physical restrictions on the matrix Hamiltonian and look for explicit form of potentials accepting those constraints. Analyzing such possibilities the author have found [14] the following q-periodic closure of the chain (3),

$$f_{j+N}(x) = qf_j(qx), \qquad k_{j+N} = q^2 k_j. \tag{17}$$

It leads to a set of mixed finite-difference-differential equations which describes q-deformation of the finite-gap and related potentials discussed in Refs.[7, 9]. Shabat's self-similar system is generated at $N = 1$. Because of the highly transcendental character of self-similarity and connections with the Painlevé equations all corresponding potentials may be called as q-transcendental ones.

Let us find a symmetry algebra behind (17). First we rewrite q-periodicity at the Hamiltonian level:

$$L_{j+N} = q^2 T_q L_j T_q^{-1} + \sigma_j, \qquad \sigma_j = \lambda_{j+N} - q^2\lambda_j, \tag{18}$$

where we normalize $\sigma_j > 0$. Substituting this into (11) we get

$$L_j A_j^+ - q^2 A_j^+ L_j = \sigma_j A_j^+, \qquad A_j^- L_j - q^2 L_j A_j^- = \sigma_j A_j^-, \tag{19}$$

$$A_j^+ \equiv K_j^+ T_q, \qquad A_j^- = (A_j^+)^\dagger.$$

Obtained formulae represent a first part of the quantum algebraic relations determining structure of the system. Second part is fixed by a particular q-commutator of $A_j^\pm$ following from the identities:

$$A_j^+ A_j^- = \prod_{i=0}^{N-1}(L_j - \lambda_{j+i}), \qquad A_j^- A_j^+ = \prod_{i=0}^{N-1}(q^2 L_j + \sigma_j - \lambda_{j+i}). \tag{20}$$

As an example, we write one possible equality:

$$A_j^+ A_j^- - q^{2N} A_j^- A_j^+ = \mathcal{P}_{N-1}(L_j), \tag{21}$$

where $\mathcal{P}_{N-1}$ is a polynomial of the degree $N-1$. Formulae (19), (21) define a particular class of nonlinear algebras which may be interpreted as a q-deformation of the polynomial algebra of ordinary differential operators discussed in Ref.[9] (the latter in turn may be considered as the simplest W-algebras). Corresponding ladder relations determine the spectrum provided $A_j^\pm$ respect boundary conditions of a problem. The peculiarities of representations of general nonlinear algebra $[J_0, J_\pm] = \pm J_\pm$, $[J_+, J_-] = g(J_0)$, where $J_{\pm,0}$ are some formal operators, were discussed in Ref.[15].

For simplicity, we restrict our consideration to the whole line spectral problems with nonsingular potentials, in which case all $a_j^\pm$'s are well defined. Then the index j can always be chosen in such a way that A_j^+ will not have zero modes (this will be assumed below). Normalizability of physical states is not spoiled by $A_j^\pm$-operators, which thus raise and lower energy. As a result, the equation $A_j^- \psi(x) = 0$ determines lowest energy state. Suppose that all N independent solutions of this equation are normalizable, which corresponds to the ordering $\lambda_j < \lambda_{j+1} < \ldots < \lambda_{j+N-1}$ and normalizable $\psi_0^{(l)}$'s in (5). Energies of all bound states are easily found:

$$E_n^{(j)} = \frac{\sigma_j}{1-q^2} + \begin{cases} (\lambda_j - \frac{\sigma_j}{1-q^2})q^{2r}, & \text{for } n = Nr \\ (\lambda_{j+1} - \frac{\sigma_j}{1-q^2})q^{2r}, & \text{for } n = Nr+1 \\ \ldots & \ldots \\ (\lambda_{j+N-1} - \frac{\sigma_j}{1-q^2})q^{2r}, & \text{for } n = Nr+N-1 \end{cases} \tag{22}$$

By definition $E_n^{(j)} < E_{n+1}^{(j)}$, and since $q < 1$, one has $E_\infty^{(j)} = \sigma_j/(1-q^2)$, $f_\infty(x) = 0$, i.e. the potentials are reflectionless, with spectra comprising N geometric series. For the continuous spectrum the roles of $A_j^\pm$ are interchanged, action of A_j^- creates a series of states with exponentially

growing eigenvalues. Note that at $q \to 0$ there will remain only first N levels corresponding to zero value of r in (22). The smoothness condition allows to put $qf_j(qx) = 0$ directly in the set of equations defining $f_j(x)$ and obtain N-level dressing problem for zero potential. It is known to lead to a potential describing fixed time N-soliton solution of the KdV equation.

The formula (22) does not work at $q > 1$. In fact, this region of q may be reached by analytical continuation of $q < 1$ solutions and permutations:

$$f_1(x, q^{-1}; k_1, k_2, \ldots, k_N) = iqf_1(qx/i, q; k_N, k_{N-1}, \ldots, k_1),$$

$$f_2(x, q^{-1}) = if_N(x/i, q),\ f_3(x, q^{-1}) = if_{N-1}(x/i, q),\ \ldots,\ f_N(x, q^{-1}) = if_2(x/i, q),$$

where k_j-dependence in the last relations is the same as in the first formula. In Ref.[11] the $N = 1$ case was analyzed at $q > 1$ and the presence of singularities was demonstrated. Analogous situation takes place in general - analytic continuation of smooth potentials to imaginary axis creates singularities. The latter are moving under the scaling transformation and this breaks needed isospectrality of L_j's. Note that we do not q-deform Heisenberg equations of motion, so that quantum mechanical time evolution coincides with the standard one. However, the time evolution of the potentials $U(x)$ as infinite-number soliton solutions of the KdV equation requires special consideration.

In order to elucidate the construction, we consider $N = 2$ q-periodicity in more detail. Basic equations

$$\begin{aligned} f_1'(x) + f_2'(x) + f_1^2(x) - f_2^2(x) &= k_1, \\ f_2'(x) + qf_1'(qx) + f_2^2(x) - q^2 f_1^2(qx) &= k_2, \end{aligned} \tag{23}$$

and operators $A^+ = (p + if_1)(p + if_2)T_q$ and $H = a_1^+ a_1^-$ generate the algebra

$$HA^+ - q^2 A^+ H = (k_1 + k_2)A^+, \qquad A^- H - q^2 H A^- = (k_1 + k_2)A^-, \tag{24}$$

$$A^- A^+ - q^4 A^+ A^- = q^2(k_1(1+q^2) + 2k_2)H + k_2(k_1 + k_2). \tag{25}$$

By adding to H some constant one may rewrite equations (24) in the form (16). Relations (24), (25) at $q = 1$ define conformal algebra $su(1,1)$ and at $q \neq 1$ they represent a particular "quantization" of this Lie algebra (see, e.g., Ref.[16]).

Let us find $f_{1,2}(x)$ as formal series near $x = 0$. Consider first the singular solutions. Permitted singularity has a pole character:

$$f_1(x) = \frac{a}{x} + \sum_{i=1}^{\infty} b_i x^{2i-1}, \qquad f_2(x) = -\frac{a}{x} + \sum_{i=1}^{\infty} c_i x^{2i-1},$$

$$b_i + c_i = \sum_{j=1}^{i-1} \frac{c_j c_{i-j} - b_j b_{i-j}}{2i - 1 + 2a}, \quad q^{2i} b_i + c_i = \sum_{j=1}^{i-1} \frac{q^{2i} b_j b_{i-j} - c_j c_{i-j}}{2i - 1 - 2a}, \tag{26}$$

$$b_1 = \frac{1}{1-q^2}\left(\frac{k_1}{1+2a} - \frac{k_2}{1-2a}\right), \quad c_1 = \frac{1}{1-q^2}\left(\frac{k_2}{1-2a} - \frac{q^2 k_1}{1+2a}\right),$$

where a is an arbitrary parameter. In general, the series diverge at $q \to 1$, proper choice of a, however, provides the truncated solution (8). In the limit $q \to 0$, the function $qf_1(qx)$ does not vanish, $qf_1(qx) \to a/x$. Substituting this into (23) one gets two-level dressing problem for the