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# Ground State of the Massive Schwinger Model

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We study the massive Schwinger model, quantum electrodynamics of massive, Dirac fermions, in 1+1 dimensions; with space compactified to a circle. In the limit that transitions to fermion–anti-fermion pairs can be neglected, we study the full ground-state. We focus on the effect of instantons which mediate tunnelling transitions in the induced potential for the dynamical degree of freedom in the gauge field.

The massive Schwinger model is governed by the Lagrangean density

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}(x,t)F^{\mu\nu}(x,t) + \Psi^\dagger(x,t)(i\partial_t - h(x,t))\Psi(x,t),$$

where $F_{\mu\nu}(x,t)$ is the electromagnetic field strength and $h(x,t)$ is the Hamiltonian of the massive Dirac fermion with a minimal electromagnetic interaction. We fix the gauge by taking $A_0(x,t) = 0$. This leaves one gauge field $A_1(x,t)$, and hence

$$F_{01} = \partial_t A_1(x,t) = \dot{v}(x,t) = -F_{10}, \quad \text{and} \quad h(x,t) = -i\gamma_5(\partial_x + iev(x,t)) + m\gamma^0,$$

where $\gamma_5 = i\gamma^0\gamma^1$, and we take the representation $\gamma_5 = \sigma^3$, $\gamma^0 = \sigma^1$ in terms of the Pauli matrices. The equations of motion resulting from the Lagrangean are Ampère's Law,

$$\ddot{v}(x,t) = e\Psi^\dagger(x,t)\gamma_5\Psi(x,t),$$

and the Dirac equation,

$$i\partial_t\Psi(x,t) = h(x,t)\Psi(x,t) = \left(-i\gamma_5(\partial_x + iev(x,t)) + m\gamma^0\right)\Psi(x,t),$$

Gauss' Law, however, is absent

$$\partial_x\dot{v}(x,t) = e\Psi^\dagger(x,t)\Psi(x,t).$$

Invariance of the Lagrangean under static, local gauge transformations yields a local conserved charge $\mathcal{G}(x) = \partial_x\dot{v}(x,t) - e\Psi^\dagger(x,t)\Psi(x,t)$ and $\partial_t\mathcal{G}(x) = 0$, ie., the time derivative of Gauss' Law is zero.
The canonical formalism is straightforward, yeilding the Hamiltonian density

$$\mathcal{H} = \frac{1}{2}(\dot{v}(x))^2 + \Psi^\dagger(x,t)h(x,t)\Psi(x,t).$$

For the quantum theory we work in the Schrödinger picture with time independent operators but time dependent states. The classical Poisson brackets are replaced with commutators and/or anti-commutators,

$$[v(x,t_x),\dot{v}(y,t_y)]_{t_x=t_y} = i\delta(x-y), \quad \{\Psi(x,t_x),\Psi^\dagger(y,t_y)\} = \delta(x-y).$$

The quantization proceeds essentially without subtlety, other than operator ordering ambiguities. This yields only normal ordering infinities, which, once subtracted, gives finite well defined Hamiltonian and Gauss operator requiring no regularization. The normal ordering constants are fixed by gauge invariance and locality. The Hamiltonian for the fermions can be written as

$$H_F = \mathcal{U}^\dagger \left(H_0 + evQ_5 + Le^2v^2\right)\mathcal{U} = \mathcal{U}^\dagger \bar{H}_F \mathcal{U}$$

where

$$\mathcal{U} = \exp^{-e\sum_{\substack{p\in Z\\ p\neq 0}} Lv\frac{v(-p)j^0(p)}{p}}$$

implements the gauge transformation which removes $v(x) = \sum_{p\neq 0} v(p)\exp^{\frac{ipx}{L}} + v$, which in the Schrödinger representation is static. $H_0$ is the free fermionic Hamiltonian, $Q_5$ is the axial charge and $j^0(p)$ is the fourier transform of the charge density. This can all be rigorously established using the commutation relations satisfied by the current, the Hamiltonian and the axial charge[1].
The full Hamiltonian then is

$$\begin{aligned}\hat{H} &= \sum_{p\in Z} \frac{-1}{2\pi L}\frac{1}{2}\frac{d^2}{dv(p)dv(-p)} + H_F \\ &= \mathcal{U}^\dagger \left(\frac{-1}{2\pi L}\sum_{\substack{p\in Z\\ p\neq 0}}\frac{1}{2}\left(\frac{d}{dv(-p)} + \frac{eLj^0(p)}{p}\right)\left(\frac{d}{dv(p)} - \frac{eLj^0(-p)}{p}\right) + \frac{-1}{4\pi L}\frac{d^2}{dv^2} + \bar{H}_F\right)\mathcal{U} \\ &= \mathcal{U}^\dagger \bar{H}\mathcal{U}\end{aligned}$$

and we are looking for the ground-state of $\hat{H}$ or equivalently $\bar{H}$.
We must not forget Gauss' Law, which actually simplifies matters. At the quantum level we impose Gauss' Law as a constraint on physical states. Physical states are those which are annihilated by the Gauss operator:

$$\hat{\mathcal{G}}(x)|\text{physical} >= 0.$$

We can write $\hat{\mathcal{G}}(x)$ by fourier decomposition as

$$\hat{\mathcal{G}}(x) = \sum_{p\in Z}\frac{\exp^{i\frac{px}{L}}}{2\pi L}G(p)$$

where

$$G(p) = \frac{p}{L}\frac{d}{dv(-p)} - ej^0(p) = \begin{cases}\mathcal{U}^\dagger \frac{p}{L}\frac{d}{dv(-p)}\mathcal{U} & p\neq 0\\ \mathcal{U}^\dagger Q\mathcal{U} & p = 0\end{cases}$$

with $Q$ the charge operator[1].
Thus we look for eigenstates

$$\bar{H}|\mathcal{E} >= \mathcal{E}|\mathcal{E} >$$

subject to the conditions

$$\begin{cases}\frac{p}{L}\frac{d}{dv(-p)}|\mathcal{E} >= 0 & p\neq 0\\ Q|\mathcal{E} >= 0 & p = 0\end{cases}.$$

Clearly the conditions are trivial to satisfy, $|\mathcal{E} >$ is a charge zero state, that is indepedent of $v(p)$ $\forall p$. Then the eigenvalue problem for $|\mathcal{E} >$ reduces to

$$\left(\frac{-1}{2\pi L}\sum_{\substack{p\in Z\\ p\neq 0}}\frac{1}{2}e^2L^2\frac{j^0(p)j^0(-p)}{p^2} - \frac{1}{4\pi L}\frac{d^2}{dv^2} + \bar{H}_F\right)|\mathcal{E} >= \mathcal{E}|\mathcal{E} >$$

The first term is just the Coulomb energy of the fermions while the $v$ is the only physical degree of freedom in the gauge field.
We must still deal with topologically non-trivial gauge transformations

$$g_k(x) = \exp^{-}i\frac{kx}{L} \qquad k\in Z.$$

We leave this analysis for a more detailed publication. The basic approximation that we make, which is not exact[2], is that excitations to fermion-anti-fermion pairs (corresponding tp $\bar{H}_F(v)$) are suppressed. We also leave this analysis for a more detailed publication. The expansion parameter emerges as $\frac{e}{m}$. Finally, we obtain, projecting on the fermionic ground-state,

$$(\frac{-1}{4\pi L}\frac{d^2}{dv^2} - g(v))\psi(v) = \mathcal{E}^0\psi(v),$$

a simple, one-dimensional quantum mechanics problem on a circle with a periodic potential. For $m\neq 0$, $g(v)$ is a smooth periodic potential with a single, symmetric well at $v = \frac{1}{2eL}$. It is clear what the excitation spectrum will be. For low energies, the variable will be localized in the bottom of the approximately harmonic well. The energy will be

$$\mathcal{E}_n = \hbar\omega(n + \frac{1}{2}) \quad n = 0, 1, 2, \cdots,$$

where $w$ is the curvature at the bottom of the well. Then for high energies, the variable $v$ will hardly notice the small potential $-g(v)$, but will be essentially constrained by the size of the circle upon which it must sit. The circumference is $\frac{1}{eL}$, giving rise to an energy spectrum

$$\mathcal{E}_n = \hbar\omega'(n')^2 \quad n' = N_0, N_0+1, N_0+2, \cdots.$$

The energy of the states localized in the well is corrected due to tunnelling transitions around the circle, mediated by instantons. The effects of these tends to declocalize the variable $v$. This causes the energy of the transition regime, from localized behaviour to unlocalized behaviour, to be lowered. We do not of course expect a phase transition, as this is a one dimensional problem. For $mL$ sufficiently large we can keep only the first term in the series for $g(v)$[1],

$$-g(v) \to \frac{-4mL}{\pi L}K_1(2\pi mL)\sin^2(\pi eLv)$$

and

$$\omega = 2e\sqrt{mLK_1(2\pi mL)}.$$

The corresponding instanton equation is

$$(2\pi L^2)\frac{d^2}{d\tau^2}\bar{v}(\tau) = -8mLK_1(2\pi mL)\sin(2\pi eL\bar{v}(\tau)),$$

where $\tau$ is the dimensionless Euclidean time. This equation is easily integrated to give

$$\bar{v}(\tau) = \frac{2}{\pi e L} \tan^{-1}\left(\exp^{\pm 2\pi L\omega(\tau-\tau_0)}\right) - \frac{1}{2eL},$$

the $\pm$ choosing an instanton or and anti-instanton. The action for either is given by

$$S_0 = \frac{8\sqrt{mLK_1(2\pi mL)}}{\pi e L}.$$

We proceed along the lines pioneered by Langer[3] and made accessible by Coleman[4], to evaluate the functional integral in a saddle point approximation around the instanton. Indeed we must calculate the determinant corresponding to fluctuations about the instanton, separate the zero modes, and sum over the contributions from dilute multi-instanton configurations. We note that for the multi-instanton configurations, there are no constraints on where an instanton or an anti-instanton must lie, they both mediate tunnelling to and from the same initial and final state. The calculus of instantons is by now standard, and we leave details for a future more elaborate publication.
Finally we find,

$$\begin{aligned} < v = \frac{1}{2eL}|\exp^{TH}|v = \frac{1}{2eL} > &= \exp^{T\mathcal{E}_0} < v = \frac{1}{2eL}|\mathcal{E}_0 >< \mathcal{E}_0|v = \frac{1}{2eL} > + \cdots \\ &= \left(\frac{\omega}{\pi}\right)^{\frac{1}{2}} \exp^{-\frac{T}{\pi e L}\left(\frac{1}{2}\omega + \frac{2\omega}{\pi e L}\exp^{-S_0}\right)}. \end{aligned}$$

Hence

$$\mathcal{E}_0 = \frac{1}{2\pi L}\frac{1}{2}\omega\left(1 + \frac{8}{\pi}\left(\frac{\sqrt{mLK_1(2\pi mL)}}{2eL}\right)^{\frac{1}{2}}\exp^{-S_0}\right) = \frac{1}{2\pi L}\frac{1}{2}\omega\left(1 + \frac{2}{\sqrt{\pi}}\sqrt{S_0}\exp^{-S_0}\right).$$

The correction that we have calculated is non-perturbative. Our two approximations, $\frac{e}{m} \to 0$ and $mL \to \infty$ still leave $S_0$ arbitrary (as can be seen from its expression). It can take values from 0 to $\infty$. The function $\sqrt{x}\exp -x$ is maximum at $x = \frac{1}{2}$, where it is equal to $\frac{1}{\sqrt{2e}} \cdot \frac{2}{\sqrt{\pi}}\frac{1}{\sqrt{2e}} \cong .45$. Hence as $S_0$ approaches $\frac{1}{2}$ from above or below, the effects of the instantons becomes non-negligible. Complete details and analysis will appear in a future publication.

This work supported in part by NSERC of Canada and by FCAR du Québec. We thank G. Grübl and F. Ehlotzky for making possible our visit to Innsbruck. We also thank G. Grübl, J. LeTourneux and N. Manton for useful discussion.

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